PATH INTEGRALS FOR ARBITRARY CANONICAL TRANSFORMATION

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ABSTRACT

The path integral formulation of quantum mechanics is generalized to any arbitrary infinitesimal generators. It is shown that, in the case of Cartesian coordinates, the path integral formulation reproduces Weyl's quantization rule. It is also shown that, if a set of classical generators of infinitesimal canonical transformations forms a Lie algebra of a group, then the path integral amplitudes corresponding to these generators form a representation of the group.

A geometrical interpretation of the quantum process of measurement is discussed.

1. INTRODUCTION

Postulates of quantum mechanics are classified into two categories: one is concerned with dynamical principles such as definitions of dynamical variables, commutation relations and equation of motion, and the other with the statistical interpretation of the physical observations. The former one, i.e., the dynamical structure, is closely related to classical mechanics through its canonical formulation. The latter introduces the stochastic nature of the quantum process of measurement. This stochastic character of quantum mechanics differs essentially from that of classical statistical mechanics. There have been through discussions on the physical interpretation of the principles of quantum measurements. No attempt, though, has been successfull in constructing a further microscopic theory, e.g. hidden variables. On the other hand, the usual formulation of quantum mechanics in terms of operators and Hilbert space somewhat bypasses the discussion of the physical details of quantum processes of observation. On this aspect, it is always tempting to formulate quantum mechanics in a different way than the operator formalism, in spite of its firm basis.

In 1948, Feynman[1]) proposed the third formulation of quantum mechanics by means of path integrals, and showed that this formalism is equivalent to the operator formalism. One of the important aspects of path integral formulation is that it gives a geometrical insight into the quantization procedure which may help us to understand some characteristic concepts in quantum mechanics, such as the relation between spin and statistics[2]). We also expect that semiclassical phenomena are adequately described in such formalism[3]).

Today, the method of path integration has been proved to be a powerful tool in almost every field of theoretical physics[4]). In fact, areas for which path integral technic is applied cover quantum field theory, nuclear physics, solid state physica, plasma physics, statistical mechanics, hydrodynamics, diffusion equation, etc.

Sometimes the path integral method provides a beautiful solution permiting an heuristic argument of its mathematical properties, such as in renormalization in quantum field theory. However, in most cases, exact evaluation of the path integrals is extremely difficult. Mathematical efforts are now overcoming some of these difficulties[5]).

Originally, the path integral formulation of quantum mechanics was applied to the Green function of the time development of the system. However, in order to make explicit the role of path integral method as a basic procedure of quantization, it is desirable to generalize the path integral formalism to any kind of physical quantities. Such an attempt was done by Campbell et al.[6]) and they showed that the path integral method also provides an interesting way of obtaining eigenfunctions of differential operators.

In this paper, we try to formulate a generalized version of path integral quantization procedure in an explicit manner, especially for a set of generators of a Lie algebra of transformation group.

2. PATH INTEGRAL QUANTIZATION

Classical mechanics is constructed on a set of canonical variables (q_i,p_i) for which a simplectic algebraic structure, i.e, the Poisson bracket, is defined:

$$\{q_i,p_j\}_{PB} = \delta_{ij} \quad . \tag{1}$$

Any physical observables are functions of q and p.

Let $G = G(q,p)$ be a physical observable. We may define a canonical one-parameter transformation by

$$\begin{cases} \dfrac{dq_i}{d\alpha} = \{q_i, G\}_{PB} \\ \\ \dfrac{dp_i}{d\alpha} = \{p_i, G\}_{PB} \end{cases} \tag{2}$$

where α is the parameter of the transformation. The formal solution to Eq. (2) corresponding to the transformation from $\alpha = 0$ to $\alpha = \alpha_0$ is written as

$$\begin{cases} q_i' = e^{-i\alpha_0 D_{[G]}} q_i \\ \\ p_i' = e^{-i\alpha_0 D_{[G]}} p_i \end{cases} \tag{3}$$

where $D_{[G]}$ is an operator defined by[7]

$$D_{[G]} f \equiv i\{f, G\}_{PB} \tag{4}$$

for any arbitrary function f of q and p. If G is taken to be the Hamiltonian of a system, Eq. (2) describes its time development, and the parameter α is identified with the time.

In quantum mechanics, the time development of the system is described by an amplitude, or Green function, $K_H(q',q;t)$, and the probability of finding the system, whose initial configuration was $\{q\}$, in the final configuration $\{q'\}$ after a time t is given by

$$P_H(q',q;t) = |K_H(q',q;t)|^2 \quad . \tag{5}$$

The phase-space path integral formulation[4] of quantum mechanics gives the following expression to the probability amplitude:

$$K_H(q',q;t) = \int_q^{q'}\int \mathcal{D}^2\left[\frac{q.p}{2\pi\hbar}\right] e^{\frac{i}{\hbar}\int_0^t [p\dot{q}-H(q,p)]dt} \tag{6}$$

where $\iint \mathcal{D}^2\left[\frac{q.p}{2\pi\hbar}\right]$ denotes the double functional integration over the phase space.

The above formulation of the quantization procedure can immediately be generalized to any arbitrary continuous canonical transformation given by Eq. (2). Let us define an amplitude for a generator G of an arbitrary classical transformation by

$$K_G(q',q;\alpha) = \int_q^{q'}\int \mathcal{D}^2\left[\frac{q.p}{2\pi\hbar}\right] e^{\frac{i}{\hbar}\int_0^\alpha \left[p\frac{dq}{d\alpha} - G(q,p)\right] d\alpha} \quad . \tag{7}$$

Now we demand the probability of finding the system in the final configuration {q'} *after the transformation generated by* G *from the initial configuration* {q} *to be*

$$P_G(q',q;\alpha) = |K_G(q',q;\alpha)|^2 \quad . \tag{8}$$

One of the interesting applications of the above generalization was first pointed out by Campbell et al.[6]) to obtain eigenfunctions of differential operators. For simplicity, let us consider 2-dimensional phase space. If we find a classical generator $G(q,p)$ that transforms $(q,p) \to (q',p') = \{Q(q,p),P(q,p)\}$, then the path integral expression for G is an eigenfunction of the quantum mechanical operator $\hat{Q}$ whose classical correspondent is $Q(q,p)$. This result is heuristically plausible since our path integral gives the transformation function from the basis $|q\rangle$ to $|q'\rangle$ where $|q'\rangle$ is an eigenfunction of the operator $\hat{Q}$ with eigenvalue q'. When the operator $\hat{Q}$ has a discrete spectrum, the path integral exhibits poles at eigenvalues of $\hat{Q}$, and its residues are eigenfunctions.

Another interesting application is found in the following. The eigenvalues of a physical observable G in the path integral formulation may be defined as poles of the Fourier transform of the amplitude $K_G(q',q;\alpha)$,

$$\mathcal{F}_G(q',q;\omega) = \int d\alpha \; K_G(q',q;\alpha) \; e^{i\frac{\omega\alpha}{\hbar}} \quad . \tag{9}$$

This definition of the eigenvalues enables us to calculate them, for some cases, without any knowledge of group theoretical methods.

For example let us consider one of the components of angular momentum $\mathbf{L}$, say $G = L_z = xP_y - yP_x$. The corresponding amplitude is

$$\begin{aligned} K_{L_z}(\mathbf{r}',\mathbf{r};\alpha) &= \int_{\mathbf{r}}^{\mathbf{r}'} \mathcal{D}[\mathbf{r}] \int \mathcal{D}\left[\frac{\mathbf{P}}{2\pi\hbar}\right] e^{\frac{i}{\hbar}\int_0^\alpha \left[\mathbf{P}\cdot\frac{d\mathbf{r}}{d\alpha} - L_z\right] d\alpha} \\ &= \int_{\mathbf{r}}^{\mathbf{r}'} \mathcal{D}[\mathbf{r}] \cdot \delta\left[\frac{dz}{d\alpha}\right] \delta\left[\frac{dx}{d\alpha} + y\right] \delta\left[\frac{dy}{d\alpha} - x\right] \end{aligned} \tag{10}$$

where we have introduced the functional δ defined by

$$\delta[f] \equiv \int \mathcal{D}\left[\frac{x}{2\pi}\right] e^{i\int_0^\alpha xf \, d\alpha} \quad . \tag{11}$$

This functional δ has the property

$$\int_{x_0}^{x_1} \mathcal{D}^{(n)}[x] \prod_{i=1}^{n} \delta\left[\frac{dx^{(i)}}{d\alpha} - f^{(i)}\right] \Psi[x] =$$

$$= e^{\frac{1}{2}\int_0^{\alpha_0} \sum \frac{\partial f^{(i)}}{\partial x^{(i)}} d\alpha} \Psi[x]\Big|_{x=F} \delta^{(n)}\left(x_1 - F(\alpha_0)\right) \qquad (12)$$

where Ψ is an arbitrary functional of $x = x(\alpha)$, $x = (x^{(1)}, x^{(2)}, \ldots, x^{(n)})$ and $F(\alpha)$ is the solution of a coupled differential equation

$$\frac{dF^{(i)}}{d\alpha} = f^{(i)}(F,\alpha) \ , \quad i = 1, \ldots, n \ . \qquad (13)$$

with initial condition $F(\alpha=0) = x_0$.

Using Eq. (12), we get

$$K_{L_z}(\mathbf{r}',\mathbf{r};\alpha) = \delta(z'-z)\delta(x'-x\cos\alpha+y\sin\alpha)\delta(y'-x\sin\alpha-y\cos\alpha) \ . \qquad (14)$$

Eigenvalues of L_z are them obtained from the poles of

$$\mathrm{Tr}\left[\mathcal{F}_{L_z}(\mathbf{r}',\mathbf{r};\omega)\right] \equiv \mathrm{Tr}\left\{\int d\alpha\ K_{L_z}(\mathbf{r}',\mathbf{r};\alpha)\ e^{\frac{i\alpha\omega}{\hbar}}\ d\alpha\right\} \ .$$

As a function of α, $\mathrm{Tr}[K_{L_z}(\mathbf{r},\mathbf{r}';\alpha)]$ behaves as

$$\det\begin{vmatrix} 1-\cos\alpha & \sin\alpha \\ -\sin\alpha & 1-\cos\alpha \end{vmatrix}^{-1} = \frac{1}{2}\,(1-\cos\alpha)^{-1} \ ,$$

so the Fourier transform of such a function diverges. However, as we are interested in the behaviour of the Fourier transform only as a function of ω, only the $\alpha = 2n\pi$ pole contributions are important. Thus, apart from an infinite constant factor, we get

$$\mathrm{Tr}\left[F_{L_z}(\mathbf{r}',\mathbf{r};\omega)\right] \propto \sum_{n=-\infty}^{\infty} e^{\frac{i2n\pi\omega}{\hbar}}$$

which has poles at $\omega = 0, \pm\hbar, \pm 2\hbar, \ldots$. The observed values of z-component of angular momentum are restricted to integer multiple of $\hbar$. Of course the same result is obtained using polar coordinates[8]).

In a similar manner, we may calculate eigenvalues of $\mathbb{L}^2$. Path integration necessary for this problem was carried out by Marinov and Terentyev[9]) and we see that the Fourier transform of the expression

$$K_{\mathbb{L}^2}(\mathbf{r}',\mathbf{r};\alpha) = \int_{\mathbf{r}}^{\mathbf{r}'} D\left[\mathbf{r}\right] \int D\left[\frac{\mathbf{P}}{2\pi\hbar}\right] e^{\frac{i}{\hbar}\int_0^\alpha (\mathbf{P}\cdot\dot{\mathbf{r}}-\mathbb{L}^2)d\alpha}$$

has poles at $\omega = \hbar^2\ell(\ell+1)$, ℓ: positive integer.

The above scheme may also be applied to the theory of fields. For simplicity let us consider a scalar field Ψ. We assume that the dynamics of the system is described by the field variable $\Psi(x)$ and its canonical conjugate $\pi(x)$. The charge of the system may be defined as

$$Q = \frac{ie}{2\hbar} \int \left[\pi^*\Psi^* - \pi\Psi\right] d^3x \qquad (15)$$

where e is the unit of charge.

Then the path integral for charge as a generator is[(*)]

(*) When the dynamical variables assume complex values, we use their real and imaginary parts as the path integral variables. Namely,

$$q = \frac{1}{\sqrt{2}}(q_1+iq_2), \quad P = \frac{1}{\sqrt{2}}(p_1-ip_2), \quad q_i, p_i\text{: real.}$$

$$K_Q\left[\Psi',\Psi;\alpha\right] = \int_{\Psi}^{\Psi'} \int \mathcal{D}[\Psi]\,\mathcal{D}[\Psi^*] \iint \mathcal{D}\left[\frac{\pi}{2\pi\hbar}\right] \mathcal{D}\left[\frac{\pi^*}{2\pi\hbar}\right] \times$$

$$\times \exp\left\{\frac{i}{\hbar}\int_0^{\alpha} d\alpha.\int d^3x \left\{\pi\frac{\partial\Psi}{\partial\alpha} + \pi^*\frac{\partial\Psi^*}{\partial\alpha} - \frac{ie}{2\hbar}\left[\pi^*\Psi^* - \pi\Psi\right]\right\}\right\}$$

$$= \delta\left[\Psi' - e^{ie\alpha}\Psi\right] \qquad (16)$$

Eq. (16) shows clearly that the charge Q defined in Eq. (15) is precisely the generator of the gauge transformation of the field, $\Psi \to \Psi' = e^{ie\alpha}\Psi$. Eigenvalues of charge are calculated in a manner analogous to the L_z case, and we get

$$Q/e = 0, \pm 1, \pm 2, \ldots . \qquad (17)$$

Now there arises a very interesting question: Is it possible to interprete the process of quantum measurement in terms of a geometrical point of view ? Eqs. (15)–(17) are very suggestive that the quantized value of electrical charge is closely related to the gauge transformation. Unfortunately, we don't have any clear answer to the question. However it may be worthwhile to note that the process of quantum measurement can be expressed in the following manner. Let G be an observable and {q} be the initial configuration of a given system. Assume that we get a value g_o for G by a measurement; then the probability amplitude of finding the system after the measurement in the configuration {q'} is given by

$$A_{g_o}(q';q) = \frac{1}{2\pi i}\oint_{g_o} d\omega \int d\alpha\, e^{\frac{i\omega\alpha}{\hbar}} K_G(q',q;\alpha) \qquad (18)$$

where the complex integration on ω is carried out around the observed value g_o.

It is more convenient to express such a complex integral in terms of a real integral as

$$\oint_{g_o} d\omega\, f(\omega) = 2\pi i \iint_{\Omega} d(\mathrm{Re}\,\omega)\, d(\mathrm{Im}\,\omega)\, \partial f(\omega) \tag{19}$$

where Ω is a domain which contains g_o, and we define the "Cauchy derivative" ∂ by

$$\partial f(\omega) = \frac{1}{2\pi}\left(\frac{\partial f_1}{\partial \omega_1} - \frac{\partial f_2}{\partial \omega_2}\right) + \frac{i}{2\pi}\left(\frac{\partial f_2}{\partial \omega_1} + \frac{\partial f_1}{\partial \omega_2}\right) \tag{20}$$

where $f = f_1 + if_2$ and $\omega = \omega_1 + i\omega_2$. If $f(\omega)$ is analytic, $\partial f(\omega) = 0$. At poles of $f(\omega)$, $\partial f(\omega)$ behaves as a δ-function. More precisely,

$$\partial f(\omega) = \sum_{\text{poles}} \delta(\omega_1 - \omega_{01})\delta(\omega_2 - \omega_{02})\ \mathrm{Res}\ f(\omega_o) \quad . \tag{21}$$

Using the above definition, we may rewrite Eq. (18) as

$$A_{g_o}(q',q) = \iint_{\Omega} d\omega_1 d\omega_2\, \partial\left[\int d\alpha\, e^{\frac{i\omega\alpha}{\hbar}}\, K_G(q',q;\alpha)\right] \quad . \tag{22}$$

Now consider two successive measurements of a single observable G, the first around the value g_o and the second around the value g_1. The final amplitude for such a process is given by

$$A_{g_o \to g_1}(q',q) = \int dq''\, A_{g_1}(q',q'')\, A_{g_o}(q'',q) =$$

$$= \iint_{\Omega \ni g_1} d\omega_1 d\omega_2 \, \partial \cdot \iint_{\Omega' \ni g_0} d\omega_1' d\omega_2' \, \partial' \,.$$

$$\times \int d\alpha' \, e^{i \frac{\alpha'\omega'}{\hbar}} \int d\alpha \, e^{i \frac{\alpha\omega}{\hbar}} K_G(q',q;\alpha+\alpha')$$

$$= \iint_{\Omega \ni g_1} d\omega_1 d\omega_2 \, \partial \int e^{\frac{i}{\hbar}(\omega - g_0)\alpha'} d\alpha' \, A_{g_0}(q',q) \qquad (23)$$

where we have used the property

$$\int K_G(q',q'';\alpha') \, K_G(q'',q;\alpha) dq'' = K_G(q',q;\alpha+\alpha') \quad .$$

It is easy to see that

$$\int_{\Omega \ni g_1} d\omega_1 d\omega_2 \, \partial \cdot \int e^{i \frac{(\omega - g_0)\alpha'}{\hbar}} d\alpha' = \begin{cases} 1 \;, & g_0 \in \Omega \\ 0 \;, & g_0 \notin \Omega \end{cases}$$

so that

$$A_{g_0 \to g}(q',q) = A_{g_0}(q',q) \qquad \text{if} \quad g_0 \in \Omega$$

$$= 0 \qquad \text{if} \quad g_0 \notin \Omega \quad . \qquad (24)$$

This implies that a consecutive measurement of a physical observable gives the same result as the former one. This fact will be discussed further in Sec. 3.

3. SUCCESSIVE PATH INTEGRAL AND GROUP PROPERTY

What would be the consequences if we applied two different transformations successively ? This is particularly of importance when we consider a set of generators $\{G_\mu(q,p);\ \mu = 1, \ldots, m\}$ whose group properties are given. Namely, suppose we are given the Lie algebra

$$\{G_\mu, G_\nu\}_{PB} = \sum_{\lambda=1}^{m} C^\lambda_{\mu\nu} G_\lambda \tag{25}$$

where $C^\lambda_{\mu\nu}$ is the structure constant of the group. For each $G_\mu(q,p)$, we may define an amplitude

$$K_{G_\mu}(q',q;\alpha) = \int_q^{q'} \mathcal{D}[q] \int \mathcal{D}\left[\frac{p}{2\pi\hbar}\right] e^{\frac{i}{\hbar}\int_0^\alpha [p\dot{q}-G_\mu]\, d\alpha}$$

$$= K_{\alpha G_\mu}(q',q;1) \quad . \tag{26}$$

To discuss the group property of the set of amplitudes $\{K_{G_\mu}\}$, it is convenient to define infinitesimal path integrals

$$\lim_{\alpha\to\varepsilon} K_{G_\mu}(q',q;\alpha) \to \delta(q'-q) - \frac{i}{\hbar}\,\varepsilon\, M_\mu(q',q) \tag{27}$$

where ε is an infinitesimal positive number. From Eq. (26) we get

$$M_\mu(q',q) = \int_q^{q'} \mathcal{D}[q] \int \mathcal{D}\left[\frac{p}{2\pi\hbar}\right] \left\{ e^{\frac{i}{\hbar}\int_0^1 p\dot{q}\,d\alpha} \cdot \int_0^1 G_\mu(q,p)\, d\beta \right\} \tag{28}$$

.

In the case of Cartesian coordinates, Eq. (28) can also be written, using the mid-point rule, as

$$M_\mu(q',q) = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar}\, e^{\frac{i}{\hbar} p(q'-q)}\, G_\mu\left(\frac{q'+q}{2},p\right) \quad . \qquad (29)$$

The quantity $M_\mu(q',q)$ can be identified with the matrix element of the quantum mechanical operator $\hat{G}_\mu$ corresponding to $G_\mu(q,p)$, namely

$$M_\mu(q',q) \equiv \langle q'|\hat{G}_\mu|q\rangle \quad . \qquad (30)$$

Such an explicit relationship between the classical quantity $G_\mu(q,p)$ and its associated quantum mechanical operator $\hat{G}_\mu$ can be regarded as a special type of the general quantization scheme[7,10]. In fact, Eq. (29) (consequence of Cartesian coordinates and the mid-point rule) coincides with Weyl's quantization rule[11],

$$q^m p^n \rightarrow \frac{1}{2^m} \sum_{k=0}^{m} \binom{m}{k} \hat{q}^{m-k}\, \hat{p}^n\, \hat{q}^k \qquad (31)$$

with

$$\langle q'|\hat{p}|q\rangle = \frac{\hbar}{i}\, \delta(q-q')\, \frac{d}{dq'} \quad .$$

From this direct relationship, we can readily verify that a "commutator" of two generators is calculated to be

$$\langle q'|\left[\hat{G}_\mu,\hat{G}_\nu\right]|q\rangle \overset{\text{def}}{\equiv} \int dq''\, \{M_\mu(q',q'')M_\nu(q'',q)-M_\nu(q',q'')M_\mu(q'',q)\}$$

$$= i\hbar \int_q^{q'} \mathcal{D}[q] \int \mathcal{D}\left[\frac{p}{2\pi\hbar}\right] e^{\frac{i}{\hbar}\int_0^1 p\dot{q}d\alpha} \int_0^1 \{G_\mu,G_\nu\}_{PB}\, d\beta \quad . \qquad (32)$$

In other words, the path integral formalism establishes the correspondence between the quantum mechanical commutator and the classical Poisson bracket, $[\hat{A},\hat{B}] \longleftrightarrow i\hbar \{A,B\}_{PB}$.

Let us introduce the Feynman graph technique to symbolize the somewhat cumbersome notations of path integral. First, we assign a graph

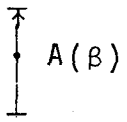
to the quantity

$$\int_q^{q'} \int D^2 \left[\frac{q.p}{2\pi\hbar}\right] e^{\frac{i}{\hbar} \int_0^1 p\dot{q}d\alpha} .$$

The quantity

$$\int_q^{q'} \int D^2 \left[\frac{q.p}{2\pi\hbar}\right] e^{\frac{i}{\hbar} \int_0^1 p\dot{q}d\alpha} \int_0^1 A(\beta)d\beta$$

is represented by a graph



Note that the quantity

$$\int_q^{q'} \int D^2 \left[\frac{q.p}{2\pi\hbar}\right] e^{\frac{i}{\hbar} \int_0^x p\dot{q}d\alpha} \int_0^x A(\beta)d\beta$$

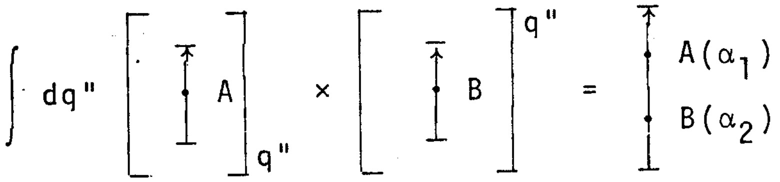
is proportional to x. The product

$$\int dq'' \left\{ \int_{q''}^{q'} \int \mathcal{D}^2 \left[\frac{q.p}{2\pi\hbar}\right] e^{\frac{i}{\hbar}\int p\dot{q}d\alpha} \int_0^1 Ad\beta . \int_q^{q''} \int \mathcal{D}^2 \left[\frac{q.p}{2\pi\hbar}\right] e^{\frac{i}{\hbar}\int q\dot{p}d\alpha} \int_0^1 Bd\beta \right\}$$

is written also as

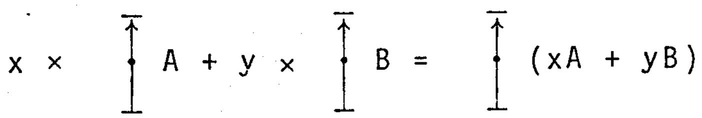
$$\int_q^{q'} \int \mathcal{D}^2 \left[\frac{q.p}{2\pi\hbar}\right] e^{\frac{i}{\hbar}\int p\dot{q}d\alpha} \int_0^1 d\alpha_1 \int_0^{\alpha_1} d\alpha_2 \; A(\alpha_1)B(\alpha_2) \quad ,$$

so that its graphical representation is

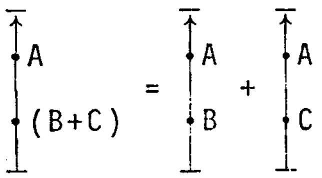


We may also confirm the following rules hold:

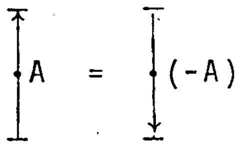
i)



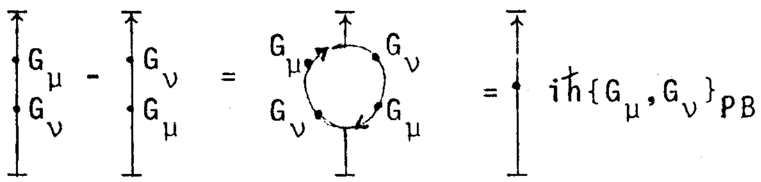
ii)



iii)



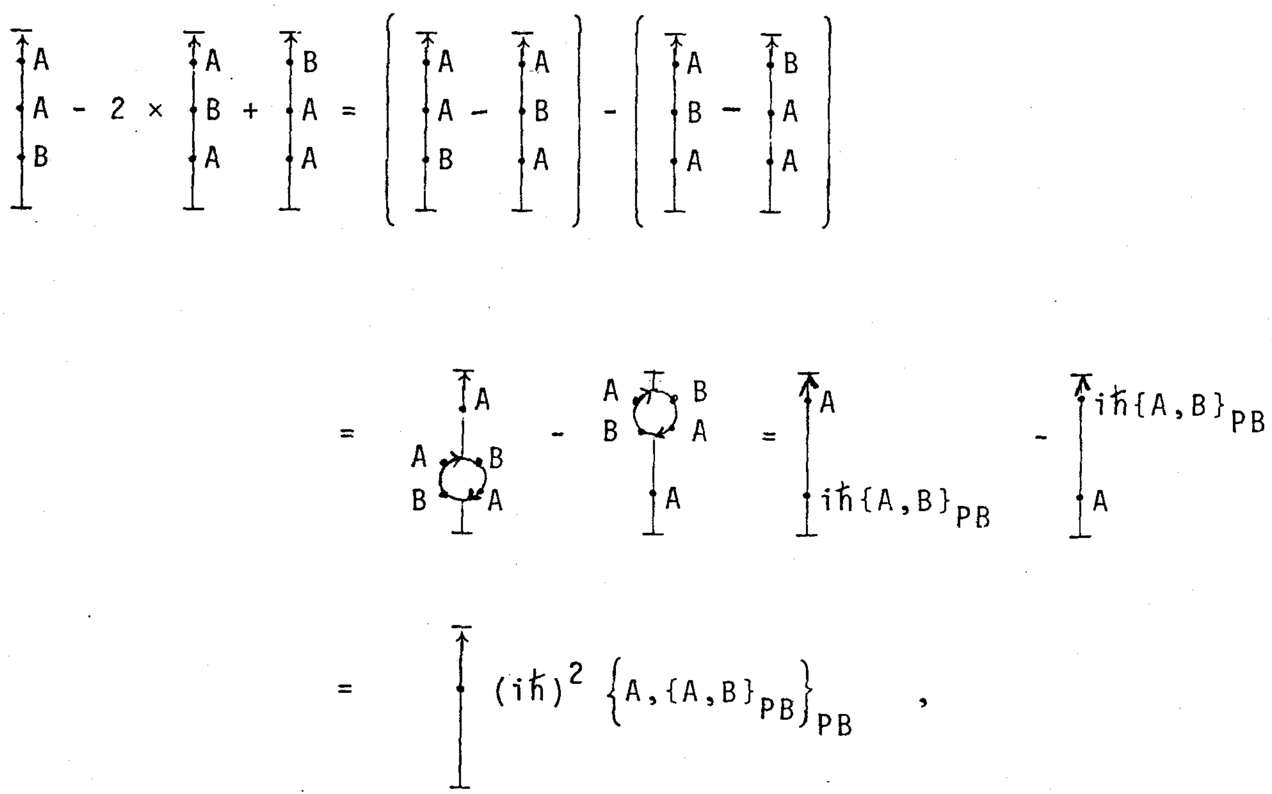
Using these diagrams, we express Eq. (32) as



We note that the rules for these diagrams are exactly the same as those of the algebra of non-commutative numbers. For example, consider a multiple commutator of two non-commutative numbers $\hat{A}$ and $\hat{B}$, say,

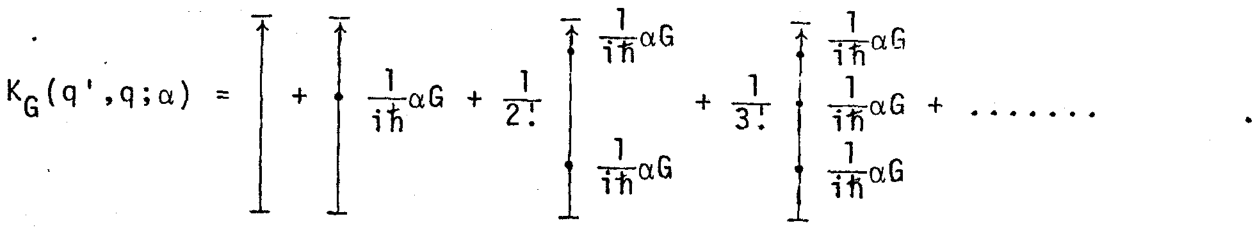
$$\left[\hat{A},\left[\hat{A},\hat{B}\right]\right] = \hat{A}^2\hat{B} - 2\hat{A}\hat{B}\hat{A} + \hat{B}\hat{A}^2 \quad . \tag{33}$$

The right-hand side of this expression may correspond to a graphic representation;



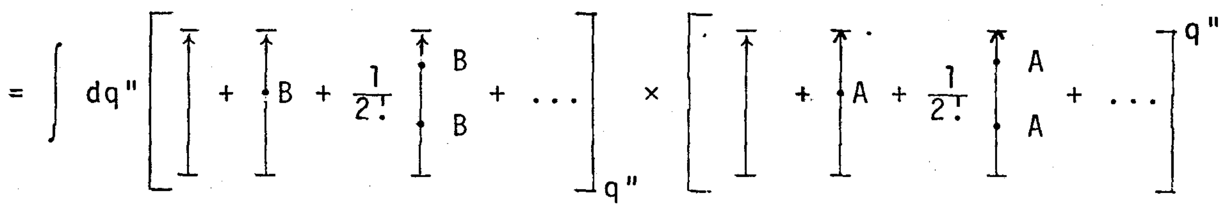
which corresponds exactly to the left-hand side of Eq. (33).

Now, consider an amplitude $K_G(q',q;\alpha)$. This can be represented graphically as



By the graphic technic, we can calculate the product of two amplitudes corresponding to two successive transformations.

$$K_{G_1 \to G_2}(q',q;\alpha,\beta) = \int dq'' \, K_{G_2}(q',q'';\beta) K_{G_1}(q'',q;\alpha)$$



(34)

where $B = \frac{\beta}{i\hbar} G_2$ and $A = \frac{\alpha}{i\hbar} G_1$. The above expansion is completely analogous to the expansion of two exponentials of non-commutative numbers[2])

$$e^{\hat{B}} e^{\hat{A}} = \left[1 + \hat{B} + \frac{1}{2!}\hat{B}^2 + \cdots\right] \times \left[1 + \hat{A} + \frac{1}{2!}\hat{A}^2 + \cdots\right] ,$$

and if the series converges, we get

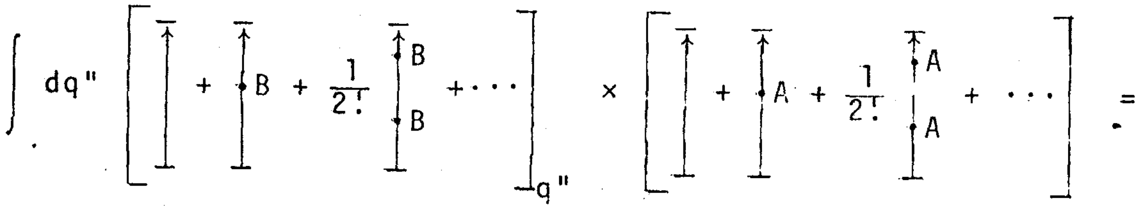
$$e^{\hat{B}} e^{\hat{A}} = e^{\hat{C}} \tag{35}$$

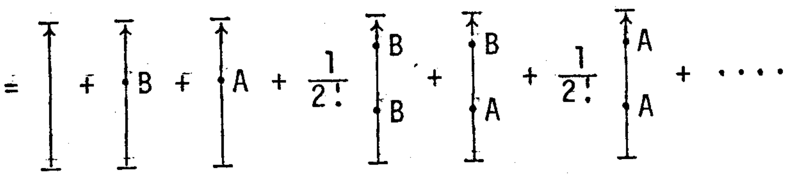
where

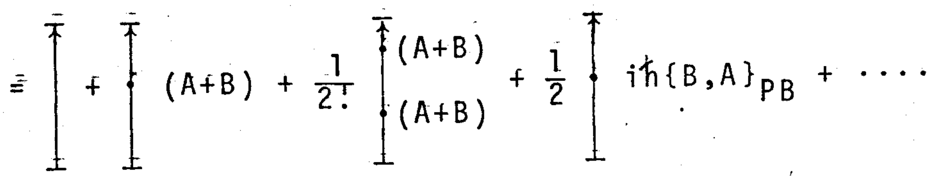
$$\hat{C} = \hat{B} + \hat{A} + \frac{1}{2}\left[\hat{B},\hat{A}\right] + \frac{1}{12}\left[\hat{B}-\hat{A},\left[\hat{B},\hat{A}\right]\right] + \cdots \tag{36}$$

(Baker-Campbell-Hausdorff formula[13])).

In our case the product Eq. (34) is expanded as







We verify the collection of these diagrams to be exactly the same as the quantity

$$\int_q^{q'}\int \mathcal{D}^2\left[\frac{q.p}{2\pi\hbar}\right] e^{\frac{i}{\hbar}\int_0^1 p\dot{q}d\alpha}\, e^{-\frac{i}{\hbar}\int_0^1 \bar{G}d\alpha}$$

$$\doteq \int_q^{q'}\int \mathcal{D}^2\left[\frac{q.p}{2\pi\hbar}\right] e^{\frac{i}{\hbar}\int_0^1 (p\dot{q}-\bar{G})d\alpha} \equiv K_{\bar{G}}(q',q;1)$$

where

$$\bar{\bar{G}} \doteq i\hbar\left[\bar{B}+\bar{A}\right] + \frac{1}{2}(i\hbar)^2\{B,A\}_{PB} + \frac{(i\hbar)^3}{12}\left\{B-A,\{B,A\}_{PB}\right\}_{PB} + \cdots$$

$$\doteq \beta G_2 + \alpha G_1 + \frac{1}{2}\{\beta G_2,\alpha G_1\}_{PB} + \frac{1}{12}\left\{(\beta G_2-\alpha G_1),\{\beta G_2,\alpha G_1\}_{PB}\right\}_{PB} + \cdots \quad (37)$$

Thus we have established the relation

$$\int dq''\int_{q''}^{q'}\int \mathcal{D}^2\left[\frac{q.p}{2\pi\hbar}\right] e^{\frac{i}{\hbar}\int_0^\beta (p\dot{q}-G_2)d\beta}\cdot\int_q^{q''}\int \mathcal{D}^2\left[\frac{q.p}{2\pi\hbar}\right] e^{\frac{i}{\hbar}\int_0^\alpha (p\dot{q}-G_1)d\alpha}$$

$$\doteq \int_q^{q'}\int \mathcal{D}^2\left[\frac{q.p}{2\pi\hbar}\right] e^{\frac{i}{\hbar}\int_0^1 (pq-\bar{G})d\alpha}. \quad (38)$$

The generator $\overline{G}$ is precisely the classical generator of the two successive transformations generated by G_1 and G_2. This can be seen as follows: The first transformation $(q,p) \overset{G_1}{\rightarrow} (q",p")$ generated by G_1 is written as (Eq. (3)),

$$q" = e^{{}^{D}[\frac{\alpha}{i}G_1]}q \quad , \quad p" = e^{{}^{D}[\frac{\alpha}{i}G_1]}p$$

The second transformation $(q",p") \overset{G_2}{\rightarrow} (q',p')$ is then

$$\begin{cases} q' = e^{{}^{D}[\frac{1}{i}\beta G_2]}q" = e^{{}^{D}[\frac{1}{i}\beta G_2]}\, e^{{}^{D}[\frac{1}{i}\alpha G_1]}q \\ p' = e^{{}^{D}[\frac{1}{i}\beta G_2]}\, e^{{}^{D}[\frac{1}{i}\alpha G_1]}p \end{cases} .$$

Now applying the Baker-Campbell-Hausdorff formula to the operators ${}^{D}[\frac{\alpha}{i}G_1]$ and ${}^{D}[\frac{\beta}{i}G_2]$, the product

$$e^{{}^{D}[\frac{1}{i}\beta G_2]}\, e^{{}^{D}[\frac{1}{i}\alpha G_1]}$$

is expressed as

$$e^{{}^{D}[\frac{\beta G_2}{i}]}\, e^{{}^{D}[\frac{\alpha G_1}{i}]} = \exp\left\{ {}^{D}[\frac{\beta G_1}{i}] + {}^{D}[\frac{\alpha G_1}{i}] + \frac{1}{2}\left[{}^{D}[\frac{\beta G_2}{i}], {}^{D}[\frac{\alpha G_1}{i}] \right] + \ldots \right. \tag{39}$$

Since the commutator of the two operators ${}^{D}[A]$ and ${}^{D}[B]$ is reproduced to[7])

$$\left[{}^{D}[A], {}^{D}[B] \right] = i\,{}^{D}[\{A,B\}_{PB}]$$

as a direct consequence of the Jacobi identity for the Poisson bracket, and since the Baker-Campbell-Hausdorff formula is written as a sum of successive commutators, we conclude that

$$e^{-iD[\beta G_1]}\, e^{-iD[\alpha G_1]} = e^{-iD[\overline{G}]} \tag{40}$$

where $\overline{G}$ is again given by Eq. (37). Therefore we get

$$\begin{cases} q' = e^{-iD[\overline{G}]}q \\ p' = e^{-iD[\overline{G}]}p \end{cases} \quad . \tag{41}$$

Eqs. (38) and (41) suffice to see that the association

$$G \text{ (classical generator)} \longleftrightarrow K_G(q',q;\alpha)$$

forms a representation of the group whose Lie algebra is determined by the Poisson bracket

$$\{G_\mu, G_\nu\}_{PB} = \sum_\lambda C^\lambda_{\mu\nu} G_\lambda \quad .$$

Also as a straightforward consequence of Eq. (38) we have

$$\int dq''\, K_{G_1}(q',q'';\alpha) K_{G_2}(q'',q;\beta)$$
$$= \int dq''\, K_{G_2}(q',q'';\beta) K_{G_1}(q'',q';\alpha), \text{ if } \{G_1,G_2\}_{PB} = 0 \quad . \tag{42}$$

The above property is essential to discuss the relation between symmetries and conserved quantities in terms of path integral formalism. For example, let us consider an observable G whose Poisson bracket with the Hamiltonian vanishes:

$$\{G,H\}_{PB} = 0 \quad . \tag{43}$$

In other words, G is a conserved quantity in a classical sense, and the system is invariant under the transformation G. At t=0, suppose that we observe the quantity G and get the value g_o. Then the amplitude after the measurement is given by Eq. (18);

$$A_{g_o}(q',q) = \frac{1}{2\pi i} \oint_{g_o} d\omega \quad e^{\frac{i\omega\alpha}{\hbar}} K_G(q',q;\alpha) \quad . \tag{18}$$

At time t = t, the amplitude develops with the time and we have

$$A(q',q;t) = \int_{q''} K_H(q',q'';t)A_{g_o}(q',q)dq''$$

$$= \int A_{g_o}(q',q'')K_H(q'',q)dq''$$

where we have used the property Eq. (42). In an analogous manner to Eq. (24), we conclude that the observation of the quantity G after time t = t will certainly gives g_o again. Thus G is a conserved quantity also in a quantum mechanical sense.

4. CONCLUDING REMARKS AND OUTLOOK

We have discussed some interesting aspects of the generalization of path integral quantization scheme for arbitrary classical generators. Of course, these properties of quantized amplitudes are also derived, even in an easier way, by the operator formulation. However, as Feynman stated in this original article[1]), it is always interesting to look at old things from a new point of view. In this paper we made explicit how the quantization of a physical system is performed by means of the path integral formalism. Specially, we have shown that, if a set of classical generators belongs to a transformation group in classical mechanics, path integral amplitudes for these quantities form a representation of the group.

The concept of path integral quantization for an arbitrary transformation may be extended to a wider region than described in this paper. For example, it will be worthwhile to reconsider the concept of relativistic invariance in view of an amplitude associated to the Lorentz transformation.

It was also suggested the possibility of formulating quantum mechanical concepts like eigenvalues, eigenfunctions, and conserved quantities without introducing operators. It would be extremely interesting to investigate a possible physical interpretation of the quantum process of measurement in terms of such a formulation.

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