Notas de Física - Volume XIII - Nº 9

FIVE DIMENSIONAL FIELD THEORY

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(Received 26 December, 1967)

1 - INTRODUCTION

An unitary field theory is constructed in a Riemaniann space R_5 which has cylindrical symmetry with respect to a unit vector field. The field is described by fifteen potentials $\gamma_{\mu\nu}$ which satisfy in absence of external sources the five dimensional Einstein's equations $G_{\mu\nu} = 0$.

It is shown that a four dimensional analysis of this theory includes three types of force fields, the electromagnetic field, the gravitational field as described by general relativity and a relativistic scalar field.

The study of five dimensional general covariant field theory was done before by several authors (see references). How ever, presently we give more importance to the properties of this scalar field. The usual versions of Kaluza's theory do not treat with this field since they consider the 5-5 metric component as a constant by means of an extra condition which is not essential. The first author to consider such scalar field was Thiry. Our present treatment is equivalent to Thiry's work. However, presently we give more importance to the physical point of view than to the mathematical aspects of the formalism.

The invariance groups of the theory are discussed in consistent way by dividing them in two classes of transformations. A very interesting result is shown regarding these invariant groups, namely, the gauge group of electrodynamics is here represented as a point dependent translation along the fifth direction. This gives a geometric meaning to gauge transformations and put those transformations in a similar position with the group of coordinate transformations.

The field equation for the scalar field is derived in a simple situation of empty spaces with no gravitational field. The scalar satisfies a massless differential equation of the Klein-Gordon type with a non-linear term involving the first derivatives of the field potential. Its interactions with massive particles implies in a variation of the particle mass which becomes a function of the scalar field. The gravitational interaction can be introduced by rewriting the scalar field equation in general covariant form. However, gravitational metric fields corresponding to waves are to be excluded since the scalar field does not interact with particles of vanishing rest mass.

2.- THE THEORY IN GENERAL COORDINATE SYSTEMS

Consider a five dimensional space denoted by the coordinates ξ^{α} and with a metric $\gamma_{\alpha\beta}$ solution of Einstein's field equations at five dimensions (all greek indices will fun from 1 to 5, and latin indices from 1 to 4). Into this space we define a system of parameters $x^a = x^a(\xi^{\alpha})$ [1].

Two kinds of transformation laws can be defined, the usual transformation of the coordinates, $\xi'^{\alpha} = \xi'^{\alpha}(\xi)$ and the parameter transformation $x'^a = x'^a(x^b)$ which we will call for short as the p-transformation. Later on we shall identify these parameters as the coordinates in four-dimensions where the gravitational field is defined. The role of the extra coordinates are then to be identified with the other fields which may be introduced in such unitary theory. Thus, we require that physical important quantities are to be covariant under both types of transformations. Geometric quantities transforming in such way will be called coordinate-tensors (or simply as tensors) and p-tensors. A simple example of such quantities is given by

$$\gamma^a_{\mu} = \frac{\partial x^a}{\partial \xi^{\mu}} , \qquad (1)$$

which represent five contravariant p-tensors and at the same time represent four covariant coordinate vectors.

The four parameters x^a define a set of curves x^a = constant into the five dimensional space. Along all such curves we have

$$dx^a = \gamma_\mu^a \; d\xi^\mu = 0$$

thus, the γ_μ^a represent four vectors perpendicular to the curves x^a = constant. A reciprocal field of vectors γ_a^μ is defined by,

$$\gamma_\mu^a \gamma_b^\mu = \delta_b^a \; . \qquad (2)$$

The unit vector field perpendicular to γ_μ^a at each point of the space will be called by A^μ and will deserve an important behavior in this theory,

$$\gamma_\mu^a \; A^\mu = 0 \, , \qquad (3.1)$$

$$\gamma_{\mu\nu} A^\mu \; A^\nu = 1 \; . \qquad (3.2)$$

The field A^μ posess five independent components which are determined by the five relations (3).

At each point ξ^α of the five-dimensional space the system of five vectors γ_μ^a, A^μ define uniquely the five independent directions, given any vector V^μ we may therefore write [2]

$$V^a = \gamma_\mu^a \; V^\mu \qquad (4.1)$$

$$V = A_\mu \; V^\mu \qquad (4.2)$$

or equivalently,

$$V^\mu = V^a \gamma_a^\mu + V A^\mu \qquad (5)$$

where the V^a and V are given by the relations (4). Multiplying Eq. (4.1) by γ_a^ρ and using (5) we obtain

$$V^\rho - V A^\rho = \gamma_a^\rho \, \gamma_\mu^a \; V^\mu$$

is a p-tensor of the same order than $t^{a\cdots}_{b\cdots}$. This operation is called the A-derivative of $t^{a\cdots}_{b\cdots}$. A given p-tensor is said to be cylindrical with respect to the A-field (or for short A-cylindrical) if its A-derivative vanishes. For the A-derivative of the p-metric we obtain,

$$g_{ab,\mu} A^{\mu} = \gamma^{\alpha}_{a}\gamma^{\beta}_{b}\left(A_{\alpha;\beta} + A_{\beta;\alpha}\right) \qquad (12)$$

One of the possibilities of g_{ab} to be A-cylindrical is when A_{α} is a Killing vector field in the five-dimensional space. It may happen that g_{ab} is A-cylindrical even if A_{α} does not posess such symmetry (the whole right hand side of (12) vanishes).

The p-derivative of a given function of the coordinates is defined similarly as the previous case by

$$V_{|a} = V_{,\alpha}\gamma^{\alpha}_{a} = \frac{\partial V}{\partial \xi^{\alpha}}\ \gamma^{\alpha}_{a} \qquad (13)$$

In general we may write

$$V_{,\alpha} = V_{|a}\gamma^{a}_{\alpha} + V_{,\rho}\ A^{\rho}\ A_{\alpha}\ .$$

For the definition of the covariant derivatives, Christoffel symbols and curvature tensors the reader is reffered to the work given in the reference (1).

3 - THE SPECIAL COORDINATE SYSTEM

The theory as presented in the previous section in spite of its generallity as the background of an unitary field theory is nevertheless too much complicated for practical applications. However, its is possible to select a set of simple coordinate systems and the covariance group of transformations relating them. Is in this special coordinates that Kaluza's [3] theory is usually presented. We shall restrict our discussions in all what follows to those coordinates. They are defined by imposing that the first four coordinates ξ^{μ} equal the parameters x^a,

$$x^a = x^a(\xi^{\mu}) = \xi^a .$$

The group of coordinate transformations which transform a given special coordinate system into another special coordinate system will be called special coordinate transformations, and for short will be indicated by s.c.t. They will be the unique transformations which will be considered in this paper.

For a better presentation we shall divide the set of s.c.t. into two parts. As will be clear later on, such division will allow us to interpret in a obvious fashion two different symmetry properties of the formalism. Such division of the invariance group of this theory was not stressed before, the author believes that it represents one of the more interesting features of this formalism. We call the two parts by s.c.t. type 1 and type 2. They are defined as,

$$\text{type 1} \begin{cases} \xi'^i = \xi'^i(\xi^j), \quad x'^i = x'^i(x^j), \\ \xi'^5 = \xi^5 , \end{cases}$$

$$\text{type 2} \begin{cases} \xi'^i = \xi^i, \; x'^i = x^i \\ \xi'^5 = \xi^5 + f^5(\xi^i) . \end{cases}$$

The product of these two transformations is another s.c.t., and is the transformation used in the presentation of Kaluza's theory [4].

We have, in a given special coordinate system,

$$\gamma^a_b = \delta^a_b \,, \qquad (14.1)$$

$$\gamma^a_5 = 0 \,. \qquad (14.2)$$

which may be represented as,

$$\gamma^a_\mu = \left(\frac{1}{0 \quad 0 \quad 0 \quad 0} \right),$$

where 1 is the unit four by four matrix. From Eqs. (3.1) and (14.2) we get,

$$A^a = 0 \,. \qquad (15)$$

The relation (3.2) here gives simply $A^5 A_5 = 1$, which together with the relation (15) gives

$$A_5 = \sqrt{\gamma_{55}} \,, \qquad (16)$$

$$A_a = \frac{\gamma_{a5}}{\sqrt{\gamma_{55}}} \,. \qquad (17)$$

In compact notation we may therefore write

$$A^\mu = \frac{\delta^\mu_5}{\sqrt{\gamma_{55}}} \,, \quad A_\mu = \frac{\gamma_{\mu 5}}{\sqrt{\gamma_{55}}} \,.$$

The inverse γ-field is here given by

$$\gamma_a^{\ \mu} = \left(1 \;\middle|\; - \frac{A_a}{\sqrt{\gamma_{55}}} \right).$$

The equation (9) giving the metric $\gamma_{\mu\nu}$ now reads as

$$\gamma_{\mu\nu} = \left(\begin{array}{c|c} g_{ab} + A_a A_b & A_5 \; A_a \\ \hline A_5 \; A_a & A_5^2 \end{array} \right)$$

and its inverse has the form,

$$\gamma^{\mu\nu} = \left(\begin{array}{c|c} g^{ab} & - g^{ab} \dfrac{A_b}{A_5} \\ \hline -g^{ab} \dfrac{A_b}{A_5} & \dfrac{1}{A_5^2} \left(1 + g^{ab} \; A_b A_a \right) \end{array} \right).$$

In following we give a table of the transformation properties of the several geometric quantities which deserve importance.

Quantity	S.C.T. Type 1	S.C.T. Type 2	General Coord.System General transformation
A_5, T_5	invariant	invariant	fifth component of a covariant vector
A^5	invariant	invariant	fifth component of a contravariant vector
$T_{5\ldots5}$	invariant	invariant	fifth component of covariant tensor
A^a	vector with vanishing components	invariant equal to zero	first four components of a contravariant vector

A_a	vector	changes as $A'_a = A_a - f_{,a} A_5$	first four components of a covariant vector
$\varphi_a = \frac{A_a}{A_5}$	vector	changes by a gauge, $\varphi'_a = \varphi_a - f^5_{,a}$	relation between components of a vector
$\varphi_{ab} = \varphi_{[a,b]}$	skew symmetric tensor	invariant	"spatial" curl of φ_a
$g^{ab} = \gamma^{ab}$	symmetric tensor	invariant	first 4×4 contravariant metric tensor
g_{ab}	symmetric tensor	invariant	$\gamma_{ab} - A_a A_b$
γ_{ab}	symmetric tensor	changes as $\delta\gamma_{ab} = - f^5_{,(a}\gamma_{5b)} + f^5_{,a} f^5_{,b}\gamma_{55}$	first 4×4 covariant metric tensor
$T^{a \dots b}$	tensor	invariant	components of a tensor
T^a_5	vector	invariant	mixed components of a tensor
$T_a - \frac{\gamma_{a5}}{\gamma_{55}} T_5$	vector	invariant	changes as is indicated by the indices of this quantity
T^5	invariant	changes as $T^5 = f^5_{,i} T^i$	fifth component of a contravariant vector

In this table we have used the usual notation

$$v_{[a,b]} = v_{a,b} - v_{b,a}$$

$$v_{(a,b)} = v_{a,b} + v_{b,a}$$

The unique quantity constructed with the A_a (but not involving its derivatives) which displays invariance under transformation of the type 2 is given in the previous table as,

$$A_a - \frac{\gamma_{a5}}{\gamma_{55}} A_5$$

However, by consequence of (17) this quantity vanishes in all special coordinate systems. Thus, in special coordinate systems there is no algebraic combination involving the A_a which keeps itself invariant under the s.c.t. of the type 2.

As it is already obvious, the transformations of the type 1 represent usual coordinate transformations in the four space and those of the type 2 represent gauge transformations. Here they are represented as point dependent translations along the fifth direction.

4 - <u>THE EQUATION OF MOTION OF A TEST PARTICLE</u>

So far we have considered the geometric properties of the several quantities which can be defined in the theory. Presently we proceed to introduce the physical meaning of some of these quantities. This will be done by considering the equation of a geodesic at five dimensions and comparing this equation with the path of a test particle under action of gravitation, electromagnetism and also under the action of a given external scalar field.

We begin by writing the equation of motion for a particle under the action of electromagnetism and of a scalar field ϕ.

$$M \frac{d^2x^a}{dt^3} = \frac{q}{c} F_{bc} \frac{dx^c}{dt} \overset{\circ}{g}^{ba} - M \frac{\partial \phi}{\partial x^b} \overset{\circ}{g}^{ba} \left(\frac{d\tau}{dt}\right)^2 . \qquad (18)$$

We mention here an important property of the scalar interaction, in equation (18) the term $(d\tau/dt)^2$ was included in order to maintain the Lorentz covariance of the equation, nevertheless, it posess a significance very important from the physical point of view: The scalar force acting on the particle weakens as the particle speed increases, and it changes according to the factor $(1 - v^2/c^2)\,\frac{1}{2}$, since the quantities M and $(d\tau/dt)$ appearing in (18) have the values,

$$M = M_o \frac{dt}{d\tau} ,$$

$$\frac{d\tau}{dt} = (1 - v^2/c^2)^{\frac{1}{2}} .$$

Besides this, the scalar foce goes to zero if the particle speed approaches the velocity of light. Thus, the scalar field does not interact with particles moving with the velocity of light.

Another very important property of the scalar interaction is the following: The mass of the particle interacting with the scalar field becomes a function of the field,

$$M_o = M_o(\phi) .$$

This result can be obtained by studying the structure of the relativistic scalar interaction [5]. One of the more interesting consequences of this result is: The gravitational coupling constant also becomes a function of the scalar field. Indeed, the gravitational constant given as a dimensionless number is,

$$GM_p^2/\hbar c \approx 10^{-40} ,$$

where M_p is the mass of the proton. If this later is a function of the scalar field, the above ratio is also function of the scalar [6].

The first suggestion for a varying gravitational constant was based on Milne's ideas of cosmology, and was treated by Dyrac [7], and more recently was put in field theoretic form by introducing a scalar field [8]. In the framework of an unitary field theory this scalar field may be introduced very naturally, so that we obtain again a theory with varying gravitational constant [9]. This later case is just what we will obtain in this paper.

Going back to the Eq. (18), we mention that no gravitational action is still involved since this relation bears only the Lorentz invariance. Thus, the $\overset{\circ}{g}{}^{ab}$ denotes the metric of special relativity.

Introducing the four-velocity

$$U^a = \frac{dx^a}{d\tau}$$

we write Eq. (18) as,

$$\frac{\partial u^a}{\partial x^b} u^b = \frac{q}{M_o c} F_{bc} u^c \overset{o}{g}{}^{ab} - \frac{\partial \Phi}{\partial x^b} \overset{o}{g}{}^{ba} . \tag{20}$$

Gravitation is now introduced by writing this relation in general covariant **form**,

$$u^a_{;b} u^b = \frac{q}{M_o c} F_{bc} u^c g^{ab} - \frac{\partial \Phi}{\partial x^b} g^{ab} \tag{21}$$

here g^{ab} is the gravitational potential. The semi-colon indicates the covariant derivative constructed with the Christoffel symbols associated to g_{ab}.

$$u^a_{;b} = \frac{\partial u^a}{\partial x^b} + \Gamma^a_{bc} u^c , \tag{22}$$

$$\Gamma^a_{bc} = \frac{1}{2} . g^{ar} \left(\frac{\partial g_{rb}}{\partial x^c} + \frac{\partial g_{rc}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^r} \right) .$$

After some calculations, we can write Eq. (21) under the form

$$\frac{d}{d\tau} (g_{ab} u^b) - \frac{1}{2} \frac{\partial g_{bc}}{\partial x^a} u^b u^c = \frac{q}{M_o c} F_{ab} u^b - \frac{\partial \Phi}{\partial x^a} . \tag{23}$$

Consider now the equation of a geodesic at five dimensions,

$$\frac{d^2 \xi^\mu}{d\sigma^2} = - \Delta^\mu_{\rho\sigma} \dot{\xi}^\rho \dot{\xi}^\sigma , \tag{24}$$

where $d\sigma$ is the differential of "proper time" at five dimensions, and the dot means differentiation with respect to σ.

$$ds^2 = \gamma_{\alpha\beta} \, d\xi^\alpha \, d\xi^\beta = - c^2 d\sigma^2 ,$$

$$\dot{\xi}^\rho = \frac{d\xi^\rho}{d\sigma} .$$

Since we want to compare Eqs. (23) and (24), we make the following change of parameters in (24),

$$\dot{\xi}^{\rho} = \frac{d\xi^{\rho}}{d\tau}\frac{d\tau}{d\sigma} = \lambda^{\rho}\frac{d\tau}{d\sigma} ,$$

where, as before, τ is the proper time at four dimensions. A simple calculation gives

$$\frac{d\tau}{d\sigma} = \left(1 + \frac{1}{c^2\,\gamma_{55}}(\gamma_{5\alpha}\dot{\xi}^{\alpha})^2\right)^{\frac{1}{2}} .$$

In this new notation it is possible to write the equation (24) as

$$\frac{d}{d\tau}(\gamma_{\rho\alpha}\lambda^{\alpha}) = \frac{1}{2}\frac{\partial\gamma_{\mu\nu}}{\partial\xi^{\rho}}\lambda^{\mu}\lambda^{\nu}, \qquad (25)$$

these equations contain more information than those contained in (23) since here we have an extra relation when $\rho = 5$. This extra relation will be reduced to a simple result by imposing that the metric $\gamma_{\mu\nu}$ is A-cylindrical.

$$A^{\alpha}\frac{\partial\gamma_{\mu\nu}}{\partial\xi^{\alpha}} = A^5\,\frac{\partial\gamma_{\mu\nu}}{\partial\xi^5} = 0 , \qquad (26)$$

since A^5 is left arbitrary, the requirement of $\gamma_{\mu\nu}$ being A-cylindrical reduces to the imposition that ξ^5 is not present in the metric. This result is also true for any other quantity which is A-cylindrical (we recall that our presentation is restricted entirely to special coordinate systems).

It is important to verify that as consequence of the previous symmetry of $\gamma_{\mu\nu}$, the four-dimensional metric g_{ab} is also A-cylindrical. This follows directly from the fact that the field A_a by itself is A-cylindrical since this field is just an algebraic combination of the components of the five-dimensional metric.

Thus, for $\rho = 5$ we get from (25)

$$\gamma_{5\alpha}\lambda^{\alpha} = a = \text{constant} . \tag{27}$$

Taking $\rho = r$ in equation (25), we find after some calculations,

$$\frac{d}{d\tau}(g_{rs}\, u^s) - \frac{1}{2}\,\frac{\partial g_{mn}}{\partial x^r}\, u^m\, u^n = a u^i \left(\frac{\partial \varphi_i}{\partial x^r} - \frac{\partial \varphi_r}{\partial x^i}\right) - \frac{a^2}{2}\,\frac{\partial}{\partial x^r}\,\frac{1}{\gamma_{55}} \tag{28}$$

(we have also used the relation (27)). A comparison of the relations (23) and (28) gives

$$a\left(\frac{\partial \varphi_i}{\partial x^r} - \frac{\partial \varphi_r}{\partial x^i}\right) = \frac{q}{M_o c}\, F_{ir} , \tag{29}$$

$$\frac{a^2}{2}\left(\frac{1}{\gamma_{55}}\right) = \phi . \tag{30}$$

We can identify φ_i directly with the electromagnetic potentials since this four-vector changes under the s.c.t. of type 2 similarly as a electromagnetic potential changes under a gauge transformation, and its curl remains invariant. With this choice we get the following value for the constant a,

$$a = \frac{q}{M_o c}$$

Then, the scalar field is described by $\left(\frac{1}{\gamma_{55}}\right)$, which is an invariant quantity under both types of transformation of the s.c.t. group. Its couple with the particle is here given by the above constant (the square of it). The dimension of a^2 is that of a distance divided by a mass. We make this constant a dimensionless quantity by multiplying it by m/r_o, where we take m as the mass of the electron and r_o the first Bohr radius. We also consider M_o as equal to m, this gives,

$$\frac{ma^2}{r_o} = \alpha^2$$

where α is the fine structure constant. Thus, the order of magnitude for the coupling of the particle with the scalar field is here given by the square of the electromagnetic coupling, and therefore is smaller than this coupling by approximately a factor of 10^{-2}. Since a long range atractive scalar interaction was never observed there is a strong reason for supposing that such interaction (if it exists) must be very weak, of the order of the gravitational interaction [10]. This possibility is not ruled out here since we still have the freedom to incorporate any constant as a multiplicative factor of ϕ in the equation (30). Thus, the above order of magnitude for the scalar coupling is not uniquely determined, it may be smaller than the above value and eventually of the order of magnitude of the gravitational coupling.

For finishing this section we add the following important comment: A five dimensional general covariant field theory has given us a well prescribed method for introducing in a similar way three fields describing long range interactions, and for the first time it also gave us a similar interpretation for the two function groups of theoretical physics, the coordinate transformation group of general relativity and the gauge group of electrodynamics. Both groups are here interpreted as associated to change of the coordinates in the five dimensional space.

5 - THE FIELD EQUATIONS

According to our previous results, we can interpret g_{ab} as the gravitational potential, $\varphi_a = \frac{\gamma_{a5}}{\gamma_{55}}$ as the electromagnetic potential and $\phi = \frac{1}{\gamma_{55}}$ as the scalar potential. In five dimensions they are put together as the several components of the metric.

The field equations in five dimensions follow from a variational principle similar to that of general relativity, with a Lagrangian density

$$\mathcal{L} = R(|\gamma_{\mu\nu}|)^{\frac{1}{2}},$$

where R is the five dimensional scalar curvature. This variational principle is subjected to the conditions

$$(\delta g^{ab})_{,5} = 0, \ (\delta \varphi_a)_{,5} = 0, \ (\delta\phi)_{,5} = 0 .$$

The resulting field equations are those of general relativity together with the Maxwell equations and the equation for the scalar.

Presently we will be interested solely in this last equation, we will suppose the simple situation where there is no other field besides ϕ. In other words, we have a five dimensional empty space with the metric,

$$g_{ab} = \overset{\circ}{g}_{ab} \,,$$

$$\gamma_{a5} = 0$$

$$\gamma_{55} = \frac{1}{\phi} \,,$$

which form $\gamma_{\mu\nu}$. The inverse matrix is given by the elements,

$$g^{ab} = \overset{\circ}{g}{}^{ab}$$

$$\gamma^{a5} = 0$$

$$\gamma^{55} = \phi$$

The components of the five dimensional Ricci tensor and scalar curvature are given by

$$R_{\alpha\beta} = \overset{\circ}{g}{}^{ab} R_{\alpha a\beta b} + \phi R_{\alpha 5\beta 5} \tag{31}$$

$$R = \overset{\circ}{g}{}^{ab} R_{ab} + \phi R_{55} \tag{32}$$

The five dimensional Einstein tensor is

$$G_{\alpha\beta} = R_{\alpha\beta} - 1/2\, \gamma_{\alpha\beta} R \tag{33}$$

which vanishes in our case. Using the symmetry obtained from the fact that the metric is cylindrical with respect to the unit vector field A, we calculate the components of the five dimensional Riemann tensor. Some of these components, which are of interest presently are,

$$R_{abcd} = 0$$

$$R_{a5b5} = -\frac{1}{2}\frac{\partial^2}{\partial x^a \partial x^b}\left(\frac{1}{\phi}\right) + \frac{\phi}{4}\frac{\partial}{\partial x^a}\left(\frac{1}{\phi}\right)\frac{\partial}{\partial x^b}\left(\frac{1}{\phi}\right)$$

$$R_{5555} = 0$$

Using these results, we compute the several components of $G_{\alpha\beta}$ and the scalar R. We find as result,

$$R = 2\phi\,\theta \tag{34}$$

$$G_{ab} = \phi\,\theta_{ab} - \phi\,\overset{\circ}{g}_{ab}\,\theta \tag{35}$$

where θ and θ_{ab} stand for,

$$\theta_{ab} = -\frac{1}{2}\frac{\partial^2}{\partial x^a \partial x^b}\left(\frac{1}{\phi}\right) + \frac{\phi}{4}\frac{\partial}{\partial x^a}\left(\frac{1}{\phi}\right)\frac{\partial}{\partial x^b}\left(\frac{1}{\phi}\right)$$

$$\theta = \overset{\circ}{g}{}^{ab}\,\theta_{ab}$$

All other components of $G_{\alpha\beta}$ vanish. From Eqs. (34) and (35) we see that the scalar field ϕ satisfies the equation

$$-\overset{\circ}{g}{}^{cd}\frac{\partial^2}{\partial x^c \partial x^d}\left(\frac{1}{\phi}\right) + \frac{\phi}{2}\overset{\circ}{g}{}^{cd}\frac{\partial\left(\frac{1}{\phi}\right)}{\partial x^c}\frac{\partial}{\partial x^d}\left(\frac{1}{\phi}\right) = 0$$

which has the form of a Klein-Gordon equation for $1/\phi$ with no mass term and with a non-linear term involving the first derivatives of $1/\phi$. This differential equation may be written as,

$$\overset{\circ}{g}{}^{cd}\frac{\partial^2\phi}{\partial x^c \partial x^d} - \frac{3}{2\phi}\overset{\circ}{g}{}^{cd}\frac{\partial\phi}{\partial x^c}\frac{\partial\phi}{\partial x^d} = 0$$

Which is the best form to be considered.

Since ϕ is real, we treat with a field which does not interact with the electromagnetic field. However, gravitational inter actions may be introduced if we restrict those interactions in order to prevent gravitational waves. As we have seen before, the field ϕ do not interact with particles of vanishing mass.

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$$W_a = \gamma_a^\mu W_\mu, \quad W = A^\mu W_\mu$$

where

$$A^\mu = \gamma^{\mu\nu} A_\nu .$$

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10. By a long range atractive scalar interaction we mean a relativistic scalar interaction, the potential of it being a function (not an operator). Thus, thw Newtonian potential is ruled out since it represents a part of the 4-4 component of the metric tensor.