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WEIGHTED APPROXIMATION OVER TOPOLOGICAL SPACES AND THE BERNSTEIN PROBLEM OVER FINITE DIMENSIONAL VECTOR SPACES

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I. INTRODUCTION

In this note, we shall present a proof of a general result concerning the theory of weighted approximation over topological spaces. It is concerned with the formulation and solution of a problem which generalizes the classical Bernstein approximation problem, in the same sense that the Weierstrass-Stone theorem contains the classical Weierstrass approximation theorem. The subject matter that we are going to discuss engaged the interest of Arnold Shapiro and was the object of some of our mathematical conversations. A special case of it was treated by Malliavin [1]. The theorem stated below extends our previous results in this field [2, 3]. The proof indicated here follows the pattern of one already given in our Portuguese expository article [3]. The adaptation of the proof to the more general case considered now becomes simple once we look at the problem as we do at present.

Let us describe the content of this note in a sketchy manner. We shall deal with a topological space E, which we may assume to be completely regular. Let us introduce the algebra $\mathscr{C}(E)$ of all continuous real-valued-function on E. We shall also consider a set V of upper-semi-continuous positive real-valued functions on E, referred to as weights. In terms of it we define in $\mathscr{C}(E)$ a certain vector subspace $\mathscr{C}V_{\infty}(E)$ which is given a suitable weighted topology (§2). We finally consider in $\mathscr{C}(E)$ a subalgebra A containing the unit function; and in $\mathscr{C}V_{\infty}(E)$ a vector subspace W which is an A-module, that is $AW \subset W$. The approximation problem we propose to discuss, namely that of the weighted approximation for continuous real-valued functions on a topological space, consists in asking for a description of the closure of W in $\mathscr{C}V_{\infty}(E)$ under such circumstances that W is an A-module. This problem is as yet unsolved even in classical situations. We shall deal here with a more precise, hence less general, form of this problem. We shall look for a description of the closure of W by using in a natural manner the equivalence relation $E|A$ on E determined by A, namely through the notion of localisability of W under A in $\mathscr{C}V_{\infty}(E)$ (§3). We then prove a general sufficient condition for localisability (§6) which in a sense not described here is fairly close to being necessary.

Our theorem reduces the weighted approximation over topological spaces to the Bernstein problem over finite dimensional vector spaces. In this note, we deal exclusively with the real case, from which the self-adjoint complex case is an easy consequence. The general complex case is as yet beyond our reach.

§2. WEIGHTED TOPOLOGICAL VECTOR SPACES OF CONTINUOUS FUNCTIONS

Let E be a topological space, which we shall assume to be completely regular without losing generality. We shall denote by $\mathscr{C}(E)$ the algebra of all continuous real-valued functions on E. Put $\| f \|_K = \sup\{|f(x)|;\ x \in K\}$ for $f \in \mathscr{C}(E)$ and $K \subset E$ compact. We shall endow $\mathscr{C}(E)$ with the topology determined by the family of semi-norms $f \to \|f\|_K$, $K \subset E$ compact, that is the compact-open topology.

Consider a set $\mathscr{V}$ of upper-semi-continuous positive real-valued functions on E, whose elements shall be referred to as weights. Introduce the weighted topological vector space $\mathscr{C}\mathscr{V}_b(E)$ formed by all $f \in \mathscr{C}(E)$ such that vf is bounded on E, for any $v \in \mathscr{V}$. Put $\|f\| = \sup\{v(x).|f(x)|;\ x \in E\}$ for $f \in \mathscr{C}\mathscr{V}_b(E)$ and $v \in \mathscr{V}$. We shall endow $\mathscr{C}\mathscr{V}_b(E)$ with the weighted topology determined by the family of all semi-norms $f \to \|f\|_v$, $v \in \mathscr{V}$.

Actually we shall be more interested in the weighted topological vector subspace $\mathscr{C}\mathscr{V}_\infty(E)$ of $\mathscr{C}\mathscr{V}_b(E)$ formed of all $f \in \mathscr{C}(E)$ such that, for any $v \in \mathscr{V}$ and any $\varepsilon > 0$, the set of all $x \in E$ where $v(x) \cdot |f(x)| \geqslant \varepsilon$ is compact (this set being a priori only closed). $\mathscr{C}\mathscr{V}_\infty(E)$ is a closed subspace of $\mathscr{C}\mathscr{V}_b(E)$.

Without loss of generality, we shall assume $\mathscr{V}$ to be directed, in the sense that, given $v_1, v_2 \in \mathscr{V}$, there are $v \in \mathscr{V}$ and $\lambda \geqslant 0$ such that $v_1, v_2 \leqslant \lambda v$.

When $\mathscr{V}$ is reduced to a single function v, we shall write $\mathscr{C}v_b(E)$ and $\mathscr{C}v_\infty(E)$ in place of $\mathscr{C}\mathscr{V}_b(E)$ and $\mathscr{C}\mathscr{V}_\infty(E)$ respectively.

Remark. We notice that, if we wanted, we could also assume $\mathscr{V}$ to satisfy the following more stringent conditions: that if $v \in \mathscr{V}$ and v' is an upper-semi-continuous positive real-valued function on E such that $v' \leqslant v$ then $v' \in \mathscr{V}$; that if $v \in \mathscr{V}$ and $\lambda \geqslant 0$ then $\lambda v \in \mathscr{V}$; and that if $v_1, v_2 \in \mathscr{V}$ then $v_1 + v_2 \in \mathscr{V}$. In fact, if we replace $\mathscr{V}$ by the smallest set containing $\mathscr{V}$ and satisfying these conditions, then the topological vector spaces $\mathscr{C}\mathscr{V}_b(E)$ and $\mathscr{C}\mathscr{V}_\infty(E)$ will not change.

§3. LOCALISABILITY IN THE WEIGHTED APPROXIMATION

Let E, $\mathscr{C}(E)$, $\mathscr{V}$ and $\mathscr{C}\mathscr{V}_\infty(E)$ be as indicated in §2. Consider a subalgebra $\mathscr{A} \subset \mathscr{C}(E)$ containing the unit function 1. Consider also a vector subspace $\mathscr{W} \subset \mathscr{C}\mathscr{V}_\infty(E)$ which we shall assume to be an $\mathscr{A}$-module, that is $\mathscr{A}\mathscr{W} \subset \mathscr{W}$. Notice that $\mathscr{A}$ defines an equivalence relation $E|\mathscr{A}$ on E, if we consider $x_1, x_2 \in E$ as being equivalent modulo $E|\mathscr{A}$ when $f(x_1) = f(x_2)$ for any $f \in \mathscr{A}$. We shall say that $\mathscr{W}$ is localisable under $\mathscr{A}$ in $\mathscr{C}\mathscr{V}_\infty(E)$, if, for any given $f \in \mathscr{C}\mathscr{V}_\infty(E)$, the following condition holds: a sufficient (and always necessary) condition for f to belong to the closure of $\mathcal{W}$ in $\mathcal{C}\mathcal{V}_\infty(E)$ is that, corresponding to any $v \in \mathcal{V}$, any $\varepsilon > 0$ and any equivalence class $X \subset E$ modulo $E|\mathcal{A}$, there is some $w \in \mathcal{W}$ such that $v(x).|w(x) - f(x)| < \varepsilon$ for $x \in X$.

Notice that, with this terminology, the content of the Weierstrass-Stone theorem may be phrased precisely as follows: if $\mathcal{V}$ is the set of characteristic functions of all compact subsets of E and $\mathcal{A} = \mathcal{W}$, then localisability holds good. More generally, if $\mathcal{V}$ is the set of such characteristic functions it can be shown, as a consequence of the Weierstrass-Stone theorem, that we still have localisability, regardless of the circumstance that $\mathcal{A}$ and $\mathcal{W}$ are equal or not. This is the motivation of the above definition.

§4. THE BERNSTEIN PROBLEM OVER FINITE DIMENSIONAL VECTOR SPACES

Let E be a real vector space of finite dimension n. We shall denote by $\mathcal{P}(E)$ the algebra of all real polynomials on E, that is the subalgebra of the algebra of all real-valued functions on E generated by the constant functions and the linear forms. Given as upper-semi-continuous positive real-valued function ω on E, we introduce the vector space $\mathcal{C}\omega_b(E)$ semi-normed by $f \to \|f\|_\omega$ and its semi-normed subspace $\mathcal{C}\omega_\infty(E)$ (in the notation set at §2). The weight ω is said to be rapidly decreasing at infinity if $\mathcal{P}(E) \subset \mathcal{C}\omega_b(E)$ or equivalently $\mathcal{P}(E) \subset \mathcal{C}\omega_\infty(E)$. In such a case, ω is called a fundamental weight function in the sense of Bernstein provided $\mathcal{P}(E)$ is dense in the semi-normed space $\mathcal{C}\omega_\infty(E)$.

Notice, for future reference, that if $\mathcal{C}_b(E)$ represents the algebra of all bounded continuous real-valued functions on E, then $\mathcal{C}_b(E) \subset \mathcal{C}\omega_\infty(E)$ provided that ω tends to zero at infinity, which is the case if ω is rapidly decreasing at infinity, hence if ω is a fundamental weight function.

We shall denote by $\Omega(E)$ the set of all upper-semi-continuous positive real-valued functions on E which are fundamental weight functions in the sense of Bernstein.

When $E = R^n$, where R is the real number system, we shall write $\mathcal{P}_n = \mathcal{P}(R^n)$ and $\Omega_n = \Omega(R^n)$ for short.

§5. A TOPOLOGICAL LEMMA

In the following, given a set E and an integer $m \geqslant 1$, then E^m shall denote the cartesian m-power $E \times \ldots \times E$ (m times) and $\Delta(E^m)$ shall be the diagonal of E^m. If $f: E \to F$ is a mapping, then $f^m: E^m \to F^m$ shall be the mapping given by $(x_1, \ldots, x_m) \to (f(x_1), \ldots, f(x_m))$.

LEMMA (1). Let $f_i: E \to E_i$ ($i \in I$) be a family of continuous mappings from a topological space E into Hausdorff spaces E_i. Let $\{f_i\}_{i \in I}$ be separating on E, that is, if $x_1, x_2 \in E$, $x_1 \neq x_2$, there is some $i \in I$ such that $f_i(x_1) \neq f_i(x_2)$. Then, if $\mathcal{K}$ is a collection of compact subsets of E with empty intersection, there exist $i_1, \ldots, i_n \in I$ such that, if we denote by $\phi: E \to E_{i_1} \times \ldots \times E_{i_n}$ the mapping given by $x \to (f_{i_1}(x), \ldots f_{i_n}(x))$, the collection $\phi(\mathcal{K})$ will also have an empty intersection.

Proof. We may assume that $\mathcal{K} = \{K_1, \ldots, K_m\}$ is finite. By assumption $K \cap \Delta(E^m) = \emptyset$, where $K = K_1 \times \ldots \times K_m \subset E^m$. If $x = (x_1, \ldots, x_m) \in E^m$ is outside $\Delta(E^m)$, we may find an open subset $U \subset E^m$ containing x and some $i \in I$ such that $f_i^m(U) \cap \Delta(E_i^m) = \emptyset$. In fact, there are r, s such that $x_r \neq x_s$. Choose $i \in I$ so that $f_i(x_r) \neq f_i(x_s)$ and then select open subsets $V, W \subset E$ such that $x_r \in V$, $x_s \in W$, and such that $v \in V$, $w \in W$ imply $f_i(v) \neq f_i(w)$. The open subset $U = A_1 \times \ldots \times A_m \subset E^m$, where $A_p = E$ for $p \neq r, s$ and $A_r = V$, $A_s = W$, and the chosen i have the asserted properties. Once these remarks are made, we can cover K by a finite number of such open subjects $U_1, \ldots, U_n \subset E^m$ to which there are associated suitable indices $i_1, \ldots, i_n \in I$. Let us introduce the mapping $\Psi: E^m \to E_{i_1}^m \times \ldots \times E_{i_n}^m$ which is defined by $t \to (f_{i_1}^m(t), \ldots, f_{i_n}^m(t))$. Then the image $\Psi(K)$ will be disjoint from $\Delta(E_{i_1}^m) \times \ldots \times \Delta(E_{i_n}^m)$. If Φ is the mapping referred to in the statement of the lemma, we have $\Phi^m = \Lambda \circ \Psi$, where Λ: $E_{i_1}^m \times \ldots \times E_{i_n}^m \to (E_{i_1} \times \ldots \times E_{i_n})^m$ is one of the finitely many natural identifications between the two spaces in question. It follows that $\Phi^m(K) = \Phi(K_1) \times \ldots \times \Phi(K_m)$ is disjoint from $\Delta((E_{i_1} \times \ldots \times E_{i_n})^m)$, that is $\Phi(K_1) \cap \ldots \cap \Phi(K_m) = \emptyset$, as we wanted.

Corollary. Let $E = \prod_{i \in I} E_i$ be a cartesian product of Hausdorff spaces. Then if a collection $\mathcal{K}$ of compact subsets of E has an empty intersection, there is a finite subset $J \subset I$ such that, letting π_J denote the natural projection from E onto the cartesian subproduct $\prod_{i \in J} E_i$, the collection $\pi_J(\mathcal{K})$ will also have an empty intersection.

Notice that the lemma and its corollary imply each other.

§6. AN APPROXIMATION LEMMA

Let E, $\mathcal{C}(E)$, $\mathcal{V}$, $\mathcal{C}\mathcal{V}_\infty(E)$, $\mathcal{A}$ and $\mathcal{W}$ be as described in §3. We shall now introduce a subset $A \subset \mathcal{A}$ which topologically generates $\mathcal{A}$ as an algebra with unit, that is such that the subalgebra of $\mathcal{A}$ generated by A and 1 is dense in $\mathcal{A}$ for the topology of $\mathcal{C}(E)$. Let us also consider a subset $W \subset \mathcal{W}$ which topologically generates $\mathcal{W}$ as an $\mathcal{A}$-module, that is the $\mathcal{A}$-submodule of $\mathcal{W}$ generated by W is dense in $\mathcal{W}$ for the topology of $\mathcal{C}\mathcal{V}_\infty(E)$. Let also $\mathcal{C}_b(R^n)$ be as defined in §4.

LEMMA (2). Let $f \in \mathcal{C}\mathcal{V}_\infty(E)$, $v \in \mathcal{V}$ and $\varepsilon > 0$ be given. Assume that, for every equivalence class $X \subset E$ modulo $E|\mathcal{A}$, there exists some $w \in \mathcal{W}$ such that $v(x).|w(x) - f(x)| < \varepsilon$ for $x \in X$. Then there exist $a_1,\dots,a_n \in A$, $w_1,\dots,w_m \in W$ and $\alpha_1,\dots,\alpha_m \in \mathcal{C}_b(R^n)$ such that

$$v(x)\left|\sum_{i=1}^{m} \alpha_i\left[a_1(x),\dots,a_n(x)\right]w_i(x) - f(x)\right| < \varepsilon \text{ for } x \in E.$$

Proof. Consider the space of all real-valued function on A: denote it by R^A and endow it with the cartesian product topology, also called finite-open topology. Let $\pi: E \longrightarrow R^A$ be the continuous mapping which to every $x \in E$ associates the function $\pi(x) \in R^A$ such that $\pi(x)(a) = a(x)$ if $a \in A$. To every $y \in \pi(E)$ we may associate $\pi^{-1}(y) \subset E$. We thus obtain a one-to-one correspondence between $\pi(E)$ and the set of equivalence classes of E modulo $E|\mathcal{A}$ because A topologically generates $\mathcal{A}$ as an algebra with unit. By the assumption made in the lemma, for each $y \in \pi(E)$ there exists some $w_y \in \mathcal{W}$ such that $v(x).|w_y(x) - f(x)| < \varepsilon$ for $x \in \pi^{-1}(y)$. We may assume that w_y belongs to the vector subspace of $\mathcal{W}$ generated by W, because W topologically generates $\mathcal{W}$ as an $\mathcal{A}$-module and the elements of $\mathcal{A}$ are constant on $\pi^{-1}(y)$. Let us denote by K_y, the compact subset of E of all x where $v(x).|w_y(x)-f(x)| \geqslant \varepsilon$. Then $\pi(K_y)$ is a compact subset of $\pi(E)$. Since $y \notin \pi(K_y)$, the intersection of all $\pi(K_y)$, as $y \in \pi(E)$, is empty. We now apply the corollary to Lemma (1). There are $a_1, \dots, a_n \in A$ such that, if we denote by Φ: $E \longrightarrow R^n$ the mapping $t \longrightarrow \Phi(t)=(a_1(t), \dots, a_n(t))$, then the intersection of all $\Phi(K_y)$, as $y \in \pi(E)$, is empty too. By compactness, we find $y_1, \dots, y_m \in \pi(E)$ such that the intersection of all $\Phi(K_{y_i})$, for $i = 1, \dots, m$, is empty. We now use normality of R^n and the method of continuous partition of unit. We then get positive functions $\alpha_1, \dots, \alpha_m \in \mathcal{C}(R^n)$ such that $\alpha_1 + \dots + \alpha_m = 1$ and α_i vanishes on $\Phi(K_{y_i})$ for $i = 1, \dots, m$. We claim that

$$v(x).\left| \sum_{i=1}^{m} \alpha_i\left[a_1(x), \dots, a_n(x)\right] w_{y_i}(x) - f(x) \right| < \varepsilon \text{ for } x \in E. \quad (1)$$

This is a consequence of

$$v(x)\alpha_i\left[a_1(x), \dots, a_n(x)\right].|w_{y_i}(x) - f(x)| \leqslant \varepsilon\alpha_i\left[a_1(x), \dots, a_n(x)\right] \quad (2)$$

$$\text{for } x \in E,\ i = 1, \dots, m,$$

and from the fact that, once x is given, there is some i for which the inequality (2) holds true in the strict sense. In fact, to prove (2), we simply remark that, if $\Phi(x) \in \pi(K_{y_i})$, then $\alpha_i\left[\Phi(x)\right] = 0$; and if $\Phi(x) \notin \pi(K_{y_i})$, then $v(x).|w_{y_i}(x) - f(x)| < \varepsilon$. In both cases, (2) is satisfied. On the other hand, once x is given, there is some i for which $\alpha_i\left[\Phi(x)\right] > 0$. This requires that $\Phi(x) \notin \pi(K_{y_i})$, hence that $v(x).|w_{y_i}(x) - f(x)| < \varepsilon$. We then conclude that (2) is true in the strict sense. By addition, (1) follows from (2). Finally, since each w_{y_i} is an element of the vector subspace of $\mathcal{W}$ generated by W, as we already said, (1) implies the inequality in the statement of the lemma, as we wanted.

§7. REDUCTION OF THE TOPOLOGICAL CASE TO THE FINITE DIMENSIONAL VECTOR SPACE CASE

The notation will be that already introduced in §4 and §6.

THEOREM. Suppose that, for every $v \in \mathcal{V}$, every $a_1, \ldots, a_n \in A$ and every $w \in W$, there are $a_{n+1}, \ldots, a_N \in A$, where $N \geqslant n$, and $\omega \in \Omega_N$ such that

$$v(x).|w(x)| \leqslant \omega\left[a_1(x), \ldots, a_N(x)\right] \qquad \text{for } x \in E.$$

Then $\mathcal{W}$ is localisable under $\mathcal{A}$ in $\mathcal{C}\mathcal{V}_\infty(E)$.

Proof. Let us start by remarking that, if $v \in \mathcal{V}$, $a_1, \ldots, a_n \in A$, $w \in W$, $\alpha \in \mathcal{C}_b(R^n)$ and $\delta > 0$ are given, there exists some $w' \in \mathcal{W}$ for which

$$v(x).|w'(x) - \alpha\left[a_1(x), \ldots, a_n(x)\right]w(x)| < \delta \qquad \text{for } x \in E. \quad (1)$$

In fact, by the assumption, there are $a_{n+1}, \ldots, a_N \in A$, where $N \geqslant n$, and $\omega \in \Omega_N$, such that the inequality in the statement of the theorem holds true. Now $\alpha \in \mathcal{C}_b(R^n)$ determines $\alpha' \in \mathcal{C}_b(R^N)$ by the formula $\alpha'(t_1, \ldots, t_N) = \alpha(t_1, \ldots, t_n)$ for $t_1, \ldots, t_N \in R$. Since $\mathcal{C}_b(R^N) \subset \mathcal{C}\omega_\infty(R^N)$(§4), there is $p \in \mathcal{P}_N$ such that

$\omega(t_1,\dots,t_N).|p(t_1,\dots,t_N) - \alpha(t_1,\dots,t_n)| < \delta$ for $t_1,\dots,t_N \in R$.

Therefore we shall have

$$v(x).|p[a_1(x),\dots,a_N(x)]w(x) - \alpha[a_1(x),\dots,a_n(x)]w(x)| \leqslant$$
$$\omega[a_1(x),\dots,a_N(x)].|p[a_1(x),\dots,a_N(x)] - \alpha[a_1(x),\dots,a_n(x)]| < \delta$$

which proves (1) with $w' = p(a_1,\dots,a_N)w$.

We now complete the proof of the theorem. Let $f \in \mathscr{C}\mathscr{V}_\infty(E)$ be such that, corresponding to any $v \in \mathscr{V}$, any $\varepsilon > 0$ and any equivalence class $X \subset E$ modulo $E|\mathscr{A}$, there is some $w \in \mathscr{W}$ such that $v(x).|w(x) - f(x)| < \varepsilon$ for $x \in X$. By Lemma (2), once f, v and ε are given, there are $a_1,\dots,a_n \in A$, $w_1,\dots,w_m \in W$ and $\alpha_1,\dots,\alpha_m \in \mathscr{C}_b(R^n)$ such that

$$v(x).\left|\sum_{i=1}^{m} \alpha_i[a_1(x),\dots,a_n(x)]w_i(x) - f(x)\right| < \varepsilon \quad \text{for } x \in E .$$

We apply the preliminary remark made above to get $w'_1,\dots,w'_m \in \mathscr{W}$ such that

$$v(x).|w'_i(x) - \alpha_i[a_1(x),\dots,a_n(x)]w_i(x)| < \delta \quad \text{for } x \in E,\ i = 1,\dots,m,$$

from which we get $v(x).|w(x)-f(x)| < 2\varepsilon$ for $x \in E$, where $w = \sum w'_i$, provided $\delta = \varepsilon/m$. This finishes the proof.

COROLLARY. Suppose that $A = \{a_1,\dots,a_n\}$, $W = \{w_1,\dots,w_m\}$ and that, for every $v \in \mathscr{V}$ and every $i = 1,\dots,m$, there is $\omega \in \Omega_n$ such that

$$v(x).|w_i(x)| \leqslant \omega[a_1(x),\dots,a_n(x)] \quad \text{for } x \in E.$$

Then $\mathscr{W}$ is localisable under $\mathscr{A}$ in $\mathscr{C}\mathscr{V}_\infty(E)$.

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Added in proof: For applications of the main theorem established above and a different approach to its proof, the reader is referred to our article "Weighted approximation for algebras and modules of continuous functions: real and self-adjoint complex cases" (submitted for publication).

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