On the Banach-Stone Theorem and the Manifold Topological Classification

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Abstract:

We present a simple set-theoretic proof of the Banach-Stone Theorem. We thus apply this Topological Classification theorem to the still-unsolved problem of topological classification of Euclidean Manifolds through two conjectures and additionally we give a straightforward proof of the famous Brower theorem for manifolds topologically classified by their Euclidean dimensions.

We start our comment by announcing the:

**Banach-Stone Theorem** ([1]). Let $X$ and $Y$ be compact Hausdorff spaces, such that the associated function algebras of continuous functions $C(X, R)$ and $C(Y, R)$ separate points in $X$ and $Y$ respectively. We have thus

a) $X$ and $Y$ are homeomorphic ⇔

b) $C(X, R)$ and $C(Y, R)$ are isomorphic.

**Proof** (Elementary).

Let us first prove that b) $\Rightarrow$ a). We thus consider the explicitly given isomorphism:

$$I : C(X, R) \rightarrow C(Y, R)$$

$$f \rightarrow I(f) \quad (1)$$

Associate to it, we consider the multiplicative linear continuous functional on $C(Y, R)$ defined below for each point $x \in X$ (fixed)

$$\mathcal{L}(g) \equiv I^{-1}(g)(x). \quad (2)$$

Since $\mathcal{L}(g)$ is multiplicative linear function. We have by a simple application of the Riesz-Markov theorem, that there is a Dirac ($\mu$-point supported) measure on $Y$, such that ([2]) for
any continuous function on $Y$

\[ \mathcal{L}(g) = \int_Y g(y) d\mu(y - \bar{y}) = g(\bar{y}). \quad (3) \]

We have thus constructed our obvious candidate for our homeomorphism between $X$ and $Y$, namely:

\[ i: X \rightarrow Y \]
\[ x \rightarrow \bar{y}. \quad (4) \]

Now we can see that $i$ is a function with domain $X$ and range $Y$, since in the case of existence of two distincts points $\bar{y}_1$ and $\bar{y}_2$ supposedly image of an unique point $x \in X$, certainly for all function $g \in C(Y, R)$, we would have the result

\[ I^{-1}(g)(x) = g(\bar{y}_1) = g(\bar{y}_2) \quad (5) \]

which is in clear contradiction with the hypothesis that the function algebras $C(X, R)$ and $C(Y, R)$ separate points. The function $i$ defined by eq.(4) is thus clearly injective since if there is points $x_1$ and $x_2$, with $x_1 \neq x_2$ such that $i(x_1) = i(x_2) = \bar{y}$, naturally this leads again to the contradictory result on point separation of the algebra $C(X, R)$. Namely

\[ I^{-1}(g)(x_1) = I^{-1}(g)(x_2) = \bar{y} \quad (6) \]

valid for all functions $I^{-1}(g)$ in $C(X, R)$ since $J$ is supposed to be a isomorphism between the functions algebras $C(X, R)$ and $C(Y, R)$ and both are supposed to separate points (as in the case of Regular Topological spaces for instance!).

By just considering the inverse isomorphism

\[ I^{-1}: C(Y, R) \rightarrow C(X, R) \quad (7) \]

we obtains that $i$ is an onto application between $X$ and $Y$.

To check the continuity of the application $i$ one can easily apply ultra-filters arguments (in the case of $X$ and $Y$ possessing on enumerable basis it is straightforward to apply the sequential criterium for continuity!).

As a result $i: X \rightarrow Y$ as defined by eq.(4) defines a homeomorphism between $X$ and $Y$, that a) $\Rightarrow$ b) is trivial. ■

Let us use the above Topological Classification Theorem to show that if two continuous manifolds $M$ and $N$ with dimensions $m$ and $n$ respectively are homeomorphic, then their
dimensions are equal $m = n$. By using charts (simplexes) one reduces the above claimed result to show that if $C([0, 1]^n, R)$ is isomorphic to $C([0, 1]^m, R)$. [Here $[0, 1]^n$ and $[0, 1]^m$ are $n$ and $m$ dimensional cubes (simplexes)], then $n = m$ (the Brower dimension theorem), since the set of coordinate projections make closed sub-algebras on the above considered continuous functions $C([0, 1]^n, R)$ and $C([0, 1]^m, R)$, by a straightforward application of Weierstrass-Stone theorem ([3]), we have the famous Brower Topological Dimension Theorem as a scolium of the combined Stone-Banach theorem.

Let us conjecture the following generalization of the Banach-Stone of ours to the case of manifolds possessing a $C^k$-differentiable structure.

**Conjecture 1.** $M$ and $N$ are $C^k$-diffeomorphics if and only if $C^k(M)$ and $C^k(N)$ are isomorphics as functions algebras.

Another dimensional reduction procedure to classify topologically-differentiable $C^k$-manifolds is the following (for $k \geq 1$).

Firstly we introduce some “surgery-tomographic operations” on the given manifold. Let $T_x(M)$ be the tangent space of $M$ at the point $x \in M$. We consider thus the Whitney-Nash manifold “ambient” immersion euclidean space $R^{2m+1}$ ([3]), namely $M \hookrightarrow R^{2m+1}$. We thus “rotate” the tangent plane $T_x(M)$ in all directions “$O$” of the ambient immersions euclidean space $R^{2m+1}$: $R_O T_x(M)$. These are obviously $m$-dimensional affim euclidean spaces. We further suppose that $\{R_O T_x(M) \cap M\} \equiv M^{(m-1)}(x)$ are all $C^k$-manifolds of dimension less than $m$ (for definiteness $m - 1$). We make the additional hypothesis that there is a $1 \rightarrow 1$, onto and continuous function of $M$ into $N$.

**Conjecture 2.** Then if $M$ is homeomorphic topologically to $N$. $M^{(m-1)}(x)$ is homeomorphic topologically to $N^{(m-1)}(f(x))$ and the converse is true for all “directions” $\theta$ and all $x \in M$.

**Disclaimer:** This author Luiz C.L. Botelho although Full Professor at the Mathematical Institute of Fluminense Federal University, DOES NOT PARTICIPATE OR BELONGS TO ITS POST-GRADUATE RESEARCH GROUP.

**References**