Wigner Oscillators, Twisted Hopf Algebras and Second Quantization

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Abstract

By correctly identifying the role of central extension in the centrally extended Heisenberg algebra h, we show that it is indeed possible to construct a Hopf algebraic structure on the corresponding enveloping algebra $\mathcal{U}(h)$ and eventually deform it through Drinfeld twist. This Hopf algebraic structure and its deformed version $\mathcal{U}^{\mathcal{F}}(h)$ is shown to be induced from a more "fundamental" Hopf algebra obtained from the Schrödinger field/oscillator algebra and its deformed version, provided that the fields/oscillators are regarded as odd-elements of a given superalgebra. We also discuss the possible implications in the context of quantum statistics.

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1 Introduction

The initial activities in noncommutative (NC) quantum field theories [1] were plagued by the problem of violation of the Poincaré symmetry. In its simplest version, one introduces the matrix-valued noncommutative parameter $\Theta = \{\theta^{\mu\nu}\}$ through the commutation relation $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$. At the level of this commutation relation the Lorentz covariance clearly implies that this antisymmetric object $\theta^{\mu\nu}$ should transform as a second rank contravariant tensor. The presence of such a constant tensor-valued parameter, which acts like a constant background field, however, generates a certain torque on the system, thereby modifying the criterion for Lorentz invariance [2] in the form of yielding non-vanishing 4-divergence of the angular momentum tensor. With these modified criteria, it is indeed possible to verify the Lorentz invariance properties of the various actions in NC QFT's. For that one usually writes effective commutative actions using the Seiberg-Witten map. However, here one has to terminate the series to a certain order of the NC parameter. Besides, the physical equivalence of these two versions is not clear, as it has been shown explicitly earlier, in a simple quantum mechanical context, that the Seiberg-Witten flow in the non-commutative parameter is not spectrum preserving in presence of interactions [3]. Furthermore, one can show, by considering the vacuum expectation value of the above mentioned commutator $[\hat{x}^\mu, \hat{x}^\nu]$, that the presence of such a transforming Θ gives rise to a spontaneous violation of Lorentz symmetry. On the other hand, one would like to hold all the components of the Θ matrix fixed in all the Lorentz frames, so that these can be elevated to the status of some new fundamental constants of Nature like $\hbar, c, G$ etc. [4], as required by certain Gedanken experiment [5, 6] trying to probe spacetime structures at Planck level. However, in this case the Lorentz symmetry is violated explicitly. The only way which can reconcile constant and non-transforming $\theta^{\mu\nu}$ with Poincaré symmetry is by a twisted implementation of this group in a Hopf algebraic setting by using certain abelian twist *à la* Drinfeld [7]. Since then there has been an upsurge of interest in the study of various implications of non-commutativity and its experimental consequences within this framework. This was extended later to the case of Galilean symmetry [8]-relevant for the nonrelativistic systems- and it was shown that Pauli principle can be violated by the so called "twisted fermions". This indicates that the twist can have a non-trivial effect on quantum statistics as well. However, most of the studies were confined to the relativistic QFT. Besides, even in [8], the Schrödinger field was considered as $c \to \infty$ limit of the corresponding relativistic field (reminiscent of Wigner-İnönü group contraction) and an explicit Hopf algebraic deformation of the Schrödinger fields/oscillators were not considered and just the twisted anti-commutation relations were exploited. This is an important question to be addressed in order to analyse noncommutative quantum mechanical systems consistently, where one has to deal with the Heisenberg algebra h involving the composite position and momentum operators, which are known to be given by certain integrated objects of bilinears of Schrödinger field operators. A natural follow-up of this question is whether the above mentioned deformation at the level of Schrödinger oscillators can induce appropriate deformations at the level of the position and momentum operators as well. For this, one has to first understand whether one can, at all, construct a Hopf algebra structure on Heisenberg algebra or, for that matter, any Lie algebra with central extension. In fact, this issue was addressed earlier in the literature [9], where

it was shown that one cannot construct Hopf algebra structure on Heisenberg algebra. However, in this analysis the central extension was regarded as a mere multiple of the identity belonging to the corresponding Lie group, so that the central extension has the same co-product structure as that of the identity element.

It deserves being mentioned that in the usual Hopf algebraic framework of noncommutative theories, one constructs the Hopf algebra out of the universal enveloping algebra of the corresponding Lie algebra/group algebra which describes the symmetry of the system. For example, in this context of the nonrelativistic system, the symmetry is described by the Galilean algebra/group. The Hopf algebraic structure of the Heisenberg algebra is not studied per se, since the Heisenberg algebra generators are not associated with any of the symmetry generators (the momentum operator generates the translational symmetry, while the x operator does not and, at best, it can be regarded as a translational generator in the momentum space where, again, p is not a symmetry generator). Nevertheless, it has been observed (see [10]) that a noncommutative theory admits an underlying commutative structure, s.t. the commuting x's can be obtained from the noncommutative x's and p's via a linear transformation. A natural question to be addressed is whether this transformation can have a Hopf algebraic interpretation in terms of the universal Hopf algebra of the Heisenberg algebra and its deformations.

The purpose of the present paper is to first revisit the above mentioned issues. Particularly, we find that it is indeed possible to construct a Hopf algebra structure on Heisenberg algebra or, more precisely, on the corresponding enveloping algebra $\mathcal{U}(h)$, by carefully reinterpreting the role of central extension. This also enables one to deform it by using Drinfeld twist. In particular, we shall be using the abelian twist appropriate for Moyal star product. With this, we reproduce most of the existing results in the literature. At the next stage, we show how the Hopf algebraic structure and its deformed version can be similarly constructed out of the Schrödinger oscillator algebra, which in turn can induce an appropriate Hopf algebraic structure on the composite Heisenberg operators, where we observe the important roles played by the super-algebras. We also note the important consequences of these constructions in quantum statistics.

The paper is organized as follows. In section **2**, we provide a brief review of Hopf algebras, collecting all the important formulae to be used in the subsequent sections. We then apply this formalism in section **3** to construct the Hopf algebra structure on $\mathcal{U}(h)$, involving bosonic variables only. The corresponding fermionic case is taken up in section **4**, which is used subsequently in the context of super-algebra. In the following section **5** we introduce the second quantized Schrödinger fields/oscillators and show that, although these induce the correct algebraic structures on $\mathcal{U}(h)$, they fail to induce the appropriate co-algebraic structures on $\mathcal{U}(h)$. To circumvent this problem, we begin by providing a brief review of the concept of Wigner oscillators and the associated super-algebras in section **6**. With an appropriate interpretation through the super-algebra structure of the oscillator algebra we show in section **7** that the correct co-algebra structure at the level of $\mathcal{U}(h)$ is also correctly induced. We then study the physical implications of all these aspects in the context of quantum statistics in section **8**. The conclusions are contained in section **9**.

2 Brief Review on Hopf Algebras

Here we provide a brief review of Hopf algebra and collect some of the essential formulae to be used subsequently. For this we essentially follow [6, 11].

If we have a finite Lie algebra $\mathfrak{g}$ comprising generators τ_a satisfying

$$[\tau_a, \tau_b] = i f_{ab}^c \tau_c \tag{1}$$

and having the costructures, i.e. the co-product, co-unit and the antipode, given respectively as ($g \in \mathfrak{g}$):

$$\begin{aligned} \Delta(g) &= g \otimes \mathbf{1} + \mathbf{1} \otimes g & (2)\\ \varepsilon(g) &= 0 & (3)\\ S(g) &= -g, & (4) \end{aligned}$$

we can construct its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ which, by definition, contains the identity $\mathbf{1}$ and polynomials of the generators τ_a modulo the commutation relations (1). Its algebraic structures are given by a linear map μ

$$\begin{aligned} \mu : \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) &\rightarrow \mathcal{U}(\mathfrak{g}) \\ \mu(a \otimes b) &= a \cdot b & (5) \end{aligned}$$

so that $a \cdot \mathbf{1} = \mathbf{1} \cdot a = a \ \forall a \in \mathcal{U}(\mathfrak{g})$, whereas its costructures are determined by the following homomorphisms,

$$\begin{aligned} \Delta : \mathcal{U}(\mathfrak{g}) &\rightarrow \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) & (6)\\ \varepsilon : \mathcal{U}(\mathfrak{g}) &\rightarrow \mathbb{C} & (7)\\ S : \mathcal{U}(\mathfrak{g}) &\rightarrow \mathcal{U}(\mathfrak{g}) & (8) \end{aligned}$$

which satisfy linearity for Δ and ε and anti-multiplicativity for S ($\xi, \zeta \in \mathcal{U}(\mathfrak{g})$):

$$\begin{aligned} \Delta(\xi\zeta) &= \Delta(\xi)\Delta(\zeta) & (9)\\ \varepsilon(\xi\zeta) &= \varepsilon(\xi)\varepsilon(\zeta) & (10)\\ S(\xi\zeta) &= S(\zeta)S(\xi). & (11) \end{aligned}$$

Additionally, they satisfy

$$\begin{aligned} (\Delta \otimes id)\Delta(\xi) &= (id \otimes \Delta)\Delta(\xi) \text{ (co-associativity)} & (12)\\ (\varepsilon \otimes id)\Delta(\xi) &= (id \otimes \varepsilon)\Delta(\xi) = \xi & (13)\\ \mu(S \otimes id)\Delta(\xi) &= \mu(id \otimes S)\Delta(\xi) = \varepsilon(\xi)\mathbf{1}, & (14) \end{aligned}$$

whereas for the identity $\mathbf{1} \in \mathcal{U}(\mathfrak{g})$ one defines

$$\begin{aligned} \Delta(\mathbf{1}) &= \mathbf{1} \otimes \mathbf{1} & (15)\\ \varepsilon(\mathbf{1}) &= \mathbf{1} & (16)\\ S(\mathbf{1}) &= \mathbf{1}. & (17) \end{aligned}$$

With this, the universal enveloping algebra has the structure of a Hopf algebra.

The Sweedler notation shall be used (a sum over ξ_1 and ξ_2 is understood):

$$\Delta(\xi) = \sum_i \xi_1^i \otimes \xi_2^i \equiv \xi_1 \otimes \xi_2 \tag{18}$$

Now we can deform the Hopf algebra $\mathcal{U}(\mathfrak{g})$ into the Hopf algebra $\mathcal{U}^{\mathcal{F}}(\mathfrak{g})$ by means of a twist $\mathcal{F} \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ that is invertible and satisfies the cocycle condition

$$(\mathcal{F} \otimes \mathbf{1})(\Delta \otimes id)\mathcal{F} = (\mathbf{1} \otimes \mathcal{F})(id \otimes \Delta)\mathcal{F}. \tag{19}$$

The costructures will be deformed as follows

$$\Delta^{\mathcal{F}}(g) = \mathcal{F}\Delta(g)\mathcal{F}^{-1} \tag{20}$$
$$\varepsilon^{\mathcal{F}}(g) = \varepsilon(g) \tag{21}$$
$$S^{\mathcal{F}}(g) = \chi S(g)\chi^{-1}, \tag{22}$$

where

$$\chi = f^\alpha S(f_\alpha) \in \mathcal{U}(\mathfrak{g}). \tag{23}$$

We are denoting $\mathcal{F} = f^\alpha \otimes f_\alpha$ and $\mathcal{F}^{-1} = \bar{f}^\alpha \otimes \bar{f}_\alpha$. As an algebra $\mathcal{U}^{\mathcal{F}}(\mathfrak{g})$ is identical to $\mathcal{U}(\mathfrak{g})$ [6].

The generators of $\mathcal{U}^{\mathcal{F}}(\mathfrak{g})$ are given as

$$g^{\mathcal{F}} = \bar{f}^\alpha(g)\bar{f}_\alpha, \tag{24}$$

while their coproduct is

$$\Delta^{\mathcal{F}}(g^{\mathcal{F}}) = g^{\mathcal{F}} \otimes \mathbf{1} + \bar{R}^\alpha \otimes \bar{R}_\alpha(g^{\mathcal{F}}), \tag{25}$$

with $\bar{R}^\alpha \otimes \bar{R}_\alpha = \mathcal{R}^{-1}$, where $\mathcal{R} = (f_\alpha \otimes f^\alpha)\mathcal{F}^{-1}$ is the universal $\mathcal{R}$-matrix.

Note that it is no longer co-commutative in general. These generators $g^{\mathcal{F}}$ form a linear subspace $\mathfrak{g}^{\mathcal{F}}$ of $\mathcal{U}^{\mathcal{F}}(\mathfrak{g})$, the counterpart of $\mathfrak{g} \subset \mathcal{U}(\mathfrak{g})$.

Finally, the deformed brackets in $\mathcal{U}^{\mathcal{F}}(\mathfrak{g})$ are

$$[\xi^{\mathcal{F}}, \zeta^{\mathcal{F}}]_{\mathcal{F}} = (\xi^{\mathcal{F}})_1 \zeta^{\mathcal{F}} S(\xi^{\mathcal{F}})_2, \tag{26}$$

where, of course, $(\xi^{\mathcal{F}})_1 \otimes (\xi^{\mathcal{F}})_2 = \Delta^{\mathcal{F}}(\xi^{\mathcal{F}})$ (Sweedler notation).

The deformed brackets satisfy the Jacobi identity

$$[f, [f', f'']_{\mathcal{F}}]_{\mathcal{F}} + [f'', [f, f']_{\mathcal{F}}]_{\mathcal{F}} + [f', [f'', f]_{\mathcal{F}}]_{\mathcal{F}} = 0, \tag{27}$$

for all $f, f', f'' \in \mathfrak{g}^{\mathcal{F}}$.

The multiplication map m in the module * also gets deformed as

*This multiplication map m should not be confused with the linear map μ introduced earlier (5). The former acts on a module, which is just an algebra under m, and furnishes a representation of $\mathfrak{g}$ or, for that matter, of $\mathcal{U}(\mathfrak{g})$ [6].

$$m^{\mathcal{F}} = m \circ \mathcal{F}^{-1} \tag{28}$$

to maintain its compatibility with the deformed coproduct $\Delta^{\mathcal{F}}(g^{\mathcal{F}})$ (20). On the other hand, the multiplication map μ undergoes no deformation, because as an algebra $\mathcal{U}^{\mathcal{F}}(\mathfrak{g})$ is the same as $\mathcal{U}(\mathfrak{g})$, as mentioned earlier.

3 Hopf algebra structure of the (twisted) bosonic Heisenberg algebra

In the previous section we have provided a general outline of the Hopf algebra $\mathcal{U}(\mathfrak{g})$ and its deformed counterpart $\mathcal{U}^{\mathcal{F}}(\mathfrak{g})$. We now apply it to quantum Heisenberg algebra, involving commutators of bosonic variables only.

We start with the algebra $h(\mathcal{N}),\ i,j = 1,...,\mathcal{N}$ (in the following we also consider the limit $\mathcal{N} \to \infty$). We have

$$\begin{aligned} [x_i, x_j] &= [p_i, p_j] = 0 && (29)\\ [x_i, p_j] &= i\delta_{ij}\hat{\hbar} && (30)\\ \left[\hat{\hbar}, x_i\right] &= \left[\hat{\hbar}, p_j\right] = 0 && (31) \end{aligned}$$

and apply the twist

$$\mathcal{F} = \exp\left(\frac{i}{2}\frac{\theta_{ij}}{\hbar^2} p_i \otimes p_j\right), \tag{32}$$

where θ_{ij} is a skew-symmetric matrix. Note that here we do not consider any spacetime noncommutativity and therefore set $\theta_{0i} = 0$. As the twist involves only commuting momentum generators, it trivially satisfies the cocycle condition (19).

Before proceeding further, we have to remember that the central extension of a Lie algebra has to be treated at par with other generators of the Lie algebra and not as a multiple of the identity, as done in [9]. This is because the identity belongs to the universal enveloping algebra, whereas the central extension belongs to the Lie algebra.

In our situation it is therefore important to note that, although they represent the same constant, $\hbar$ in the denominator of the twist element (32) is a c-number and is introduced for dimensional reasons, while a hat has been put on $\hat{\hbar}$ occurring in the Lie algebra (29-31) to make it explicit that the latter plays the role of a central extension, having the coproduct $\Delta(\hat{\hbar}) = \hat{\hbar} \otimes \mathbf{1} + \mathbf{1} \otimes \hat{\hbar}$ and the antipode $S(\hat{\hbar}) = -\hat{\hbar}$.

By using the Baker-Campbell-Hausdorff formula, the deformed coproduct of x_k can be calculated, by using (20), as

$$\begin{aligned} \Delta^{\mathcal{F}}(x_k) &= \exp\left(\frac{i}{2}\frac{\theta_{ij}}{\hbar^2} p_i \otimes p_j\right) \Delta(x_k) \exp\left(-\frac{i}{2}\frac{\theta_{ij}}{\hbar^2} p_i \otimes p_j\right) = \\ &= x_k \otimes \mathbf{1} + \mathbf{1} \otimes x_k + \frac{\theta_{kj}}{2\hbar^2}\left(\hat{\hbar} \otimes p_j - p_j \otimes \hat{\hbar}\right). && (33) \end{aligned}$$

Since p_i's and $\hat{\hbar}$ commute with the p_j's, their coproducts go undeformed:

$$\Delta^{\mathcal{F}}(p_k) = \Delta(p_k) \tag{34}$$
$$\Delta^{\mathcal{F}}(\hat{\hbar}) = \Delta(\hat{\hbar}). \tag{35}$$

Now let us show that the antipode does not get deformed. It amounts to computing the element (23)

$$\chi \equiv f^{\alpha} S(f_{\alpha}) = \exp\left(-\frac{i}{2}\frac{\theta_{ij}}{\hbar^2} p_i p_j\right) = \mathbf{1}, \tag{36}$$

so that $S^{\mathcal{F}} = \chi S \chi^{-1} = S$.

Now, the deformed x_k is

$$x_k^{\mathcal{F}} = \bar{f}^{\alpha}(x_k)\bar{f}_{\alpha} = x_k - \frac{1}{2\hbar^2}\theta_{kj}\hat{\hbar}p_j, \tag{37}$$

while p_i and $\hat{\hbar}$ do not undergo deformation.

In this case, the universal $\mathcal{R}$-matrix is just $\mathcal{R} = \mathcal{F}^{-2}$, so that the deformed coproduct of $x_k^{\mathcal{F}}$ is obtained by using (25) as

$$\Delta^{\mathcal{F}}(x_k^{\mathcal{F}}) = x_k^{\mathcal{F}} \otimes \mathbf{1} + \mathbf{1} \otimes x_k^{\mathcal{F}} + \frac{1}{\hbar^2}\theta_{ik} p_i \otimes \hat{\hbar}. \tag{38}$$

It is important to note that the central charge $\hat{\hbar}$ gives a vital contribution to the coproduct.

The antipode of $x_k^{\mathcal{F}}$ is obtained by using the anti-multiplicative property of the antipode

$$S(x_k^{\mathcal{F}}) = -x_k - \frac{1}{2\hbar^2}\theta_{kj}\hat{\hbar}p_j = -x_k^{\mathcal{F}} - \frac{1}{\hbar^2}\theta_{kj}\hat{\hbar}p_j. \tag{39}$$

With these expressions in hand, we can calculate the deformed brackets using (26)

$$\left[x_i^{\mathcal{F}}, p_j^{\mathcal{F}}\right]_{\mathcal{F}} = i\delta_{ij}\hat{\hbar} \tag{40}$$
$$\left[x_i^{\mathcal{F}}, x_j^{\mathcal{F}}\right]_{\mathcal{F}} = 0 \tag{41}$$
$$\left[p_i^{\mathcal{F}}, p_j^{\mathcal{F}}\right]_{\mathcal{F}} = 0 \tag{42}$$
$$\left[\hat{\hbar}^{\mathcal{F}}, x_i^{\mathcal{F}}\right]_{\mathcal{F}} = \left[\hat{\hbar}^{\mathcal{F}}, p_i^{\mathcal{F}}\right]_{\mathcal{F}} = 0. \tag{43}$$

Note that the deformed brackets of the deformed quantities have the same structure constants as the undeformed brackets of the undeformed quantities. The same feature was observed in the case of the deformed universal enveloping algebra $\mathcal{U}(iso(1,3))$ of the Poincaré algebra [6].

We can calculate the ordinary brackets of the deformed quantities:

$$\left[x_i^{\mathcal{F}}, p_j^{\mathcal{F}}\right] = i\delta_{ij}\hat{\hbar} \tag{44}$$
$$\left[x_i^{\mathcal{F}}, x_j^{\mathcal{F}}\right] = \frac{i}{\hbar^2}\theta_{ij}\hat{\hbar}^2 = i\theta_{ij} \tag{45}$$
$$\left[p_i^{\mathcal{F}}, p_j^{\mathcal{F}}\right] = 0 \tag{46}$$
$$\left[\hat{\hbar}^{\mathcal{F}}, x_i^{\mathcal{F}}\right] = \left[\hat{\hbar}^{\mathcal{F}}, p_i^{\mathcal{F}}\right] = 0. \tag{47}$$

At this stage we observe that the deformed $x_i^{\mathcal{F}}$'s become noncommutative in nature, while the original x_i's were commutative (29). The inverse transformation of (37), $x_i = x_i^{\mathcal{F}} + \frac{\theta_{ij}}{2\hbar}p_j$, is an algebra morphism and is known as Bopp shift in the literature [12]†. The x_i's have also been identified as the commuting coordinates which are obtained by taking "average" of left and right action of non-commuting $x_i^{\mathcal{F}}$'s [10]. They have also been identified as "dipole coordinates", as they represent certain non-local position operators which grow with increasing center-of-mass momentum transverse to their extension, due to their dipole momentum (see for example, [14]). Interestingly, here we find that they have another deep mathematical attribute viz. they represent the linear subspaces of $\mathcal{U}(h(\mathcal{N}))$ and $\mathcal{U}^{\mathcal{F}}(h(\mathcal{N}))$.

In this context let us mention that the $SO(D)$ vectors x_i and p_i transform as

$$x_i \to x_i' = U(R)x_iU(R)^\dagger = R_{ij}x_j \tag{48}$$
$$p_i \to p_i' = U(R)p_iU(R)^\dagger = R_{ij}p_j. \tag{49}$$

$U(R)$ represents some unitary transformation in an appropriate Hilbert space, corresponding to the rotation $R \in SO(D)$, which induces the following transformation on $x_i^{\mathcal{F}}$:

$$x_i^{\mathcal{F}} \to x_i^{\mathcal{F}'} = R_{ij}x_j^{\mathcal{F}} + \frac{1}{2\hbar}[R,\theta]_{ij}p_j. \tag{50}$$

The presence of the θ-dependent inhomogeneous term, which vanishes only for $D = 2$, indicates that $x_i^{\mathcal{F}}$ does not transform as a vector under $SO(D)$ for $D > 2$. Nevertheless, it can be easily checked that the commutators (45) transform as scalar:

$$[x_i^{\mathcal{F}}, x_j^{\mathcal{F}}] \to [x_i^{\mathcal{F}'}, x_j^{\mathcal{F}'}] = [x_i^{\mathcal{F}}, x_j^{\mathcal{F}}] = i\theta_{ij}, \tag{51}$$

as required by the constancy of θ_{ij}.‡ They are regarded as elements of a constant matrix and not as components of a transforming tensor, as if representing some new constants of Nature like $\hbar$, c, G, etc. [6].

Finally, the deformed multiplication on the module is

$$\begin{aligned} a \star b &\equiv m^{\mathcal{F}}(a \otimes b) = (m \circ \mathcal{F}^{-1})(a \otimes b) = \bar{f}^\alpha(a)\bar{f}_\alpha(b) = \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-i}{2\hbar^2}\right)^n \theta_{i_1j_1}...\theta_{i_nj_n}\left[p_{i_1},...[p_{i_n},a]\right]\left[p_{j_1},...[p_{j_n},b]\right]. \end{aligned} \tag{52}$$

The module is the space of functions where x_i acts by ordinary multiplication and p_i acts by differentiation.

†In the context of non-linear integrable systems its analog is known as a dressing transformation [13].

‡One of us, B.C., thanks Sachindeo Vaidya for pointing this out to him.

Now defining the Moyal bracket, $[a,b]_\star \equiv (a\star b - b\star a)$, we have

$$
\begin{aligned}
[x_i,p_j]_\star &= i\delta_{ij}\hat{\hbar} && (53)\\
[x_i,x_j]_\star &= \frac{i}{\hbar^2}\theta_{ij}\hat{\hbar}^2 && (54)\\
[p_i,p_j]_\star &= 0 && (55)\\
\left[\hat{\hbar},x_i\right]_\star &= \left[\hat{\hbar},p_i\right]_\star = 0. && (56)
\end{aligned}
$$

This shows that the undeformed brackets of the deformed quantities have the same structure constants as the Moyal-brackets of the undeformed quantities.

Having studied the bosonic Heisenberg algebra, we now take up the fermionic Heisenberg algebra in the next section.

4 Hopf algebra structure of the (twisted) fermionic Heisenberg algebra

Let us start with the fermionic algebra $h_F(\mathcal{N}),\ \alpha,\beta = 1,...,\mathcal{N}$.

$$
\begin{aligned}
\{\theta_\alpha,\theta_\beta\} &= \{\partial_\alpha,\partial_\beta\} = 0 && (57)\\
\{\partial_\alpha,\theta_\beta\} &= \delta_{\alpha\beta}c && (58)\\
[c,\partial_\alpha] &= [c,\theta_\alpha] = 0. && (59)
\end{aligned}
$$

θ_α and ∂_α are odd generators, while the central charge c is an even generator.

Analogously, c is the central extension and is treated at par with the other generators θ_α and ∂_α, so its coproduct is $\Delta(c) = c\otimes\mathbf{1}+\mathbf{1}\otimes c$ and the antipode is $S(c) = -c$.

The graded version of the formulae (9-11) is

$$
\begin{aligned}
(\xi\otimes\zeta)(\xi'\otimes\zeta') &= (-1)^{|\zeta||\xi'|}(\xi\xi'\otimes\zeta\zeta') && (60)\\
S(\xi\zeta) &= (-1)^{|\xi||\zeta|}S(\zeta)S(\xi) && (61)
\end{aligned}
$$

where $|\xi|$ is the degree of ξ.

Introducing the twist, in terms of the constant symmetric matrix $C_{\alpha\beta}$,

$$
\mathcal{F} = \exp\left(C_{\alpha\beta}\partial_\alpha\otimes\partial_\beta\right) \qquad (62)
$$

and the graded expressions

$$
\begin{aligned}
\theta^{\mathcal{F}}_\beta &= \sum_\alpha (-1)^{|\bar{f}_\alpha||\theta_\beta|}\bar{f}^\alpha(\theta_\beta)\bar{f}_\alpha && (63)\\
\mathcal{R} &= \sum_{\alpha,\beta}(-1)^{|\bar{f}^\beta||f^\alpha|}(f_\alpha\bar{f}^\beta\otimes f^\alpha\bar{f}_\beta) && (64)\\
\left[u^{\mathcal{F}},v^{\mathcal{F}}\right\}_{\mathcal{F}} &= \sum_k (u^{\mathcal{F}})^k_1 v^{\mathcal{F}}(-1)^{|v^{\mathcal{F}}||(u^{\mathcal{F}})^k_2|}S(u^{\mathcal{F}})^k_2, && (65)
\end{aligned}
$$

one can calculate the deformed coproduct of θ_α:

$$\Delta^{\mathcal{F}}(\theta_\alpha) = \Delta(\theta_\alpha) + C_{\alpha\beta}(\partial_\beta \otimes c - c \otimes \partial_\beta), \tag{66}$$

the others undergoing no deformation.

Since

$$\chi = f^\alpha S(f_\alpha) = \exp\left(C_{\alpha\beta}\partial_\alpha\partial_\beta\right) = \mathbf{1}, \tag{67}$$

the antipode is also undeformed.

It can be easily seen that only the generator θ_α gets deformed as

$$\theta_\alpha^{\mathcal{F}} = \theta_\alpha + C_{\alpha\beta}\partial_\beta c. \tag{68}$$

Its antipode is

$$S(\theta_\beta^{\mathcal{F}}) = -\theta_\alpha + C_{\alpha\beta}\partial_\beta c = -\theta_\alpha^{\mathcal{F}} + 2C_{\alpha\beta}\partial_\beta c. \tag{69}$$

The universal $\mathcal{R}$-matrix is simply $\mathcal{F}^{-2}$, so that

$$\Delta^{\mathcal{F}}(\theta_\alpha^{\mathcal{F}}) = \theta_\alpha^{\mathcal{F}} \otimes \mathbf{1} + \mathbf{1} \otimes \theta_\alpha^{\mathcal{F}} + 2C_{\alpha\beta}\partial_\beta \otimes c. \tag{70}$$

The deformed brackets are now

$$\begin{aligned}
\left\{\theta_\alpha^{\mathcal{F}}, \partial_\beta^{\mathcal{F}}\right\}_{\mathcal{F}} &= \delta_{\alpha\beta} c^{\mathcal{F}} && (71)\\
\left\{\theta_\alpha^{\mathcal{F}}, \theta_\beta^{\mathcal{F}}\right\}_{\mathcal{F}} &= 0 && (72)\\
\left\{\partial_\alpha^{\mathcal{F}}, \partial_\beta^{\mathcal{F}}\right\}_{\mathcal{F}} &= 0 && (73)\\
\left[\partial_\alpha^{\mathcal{F}}, c^{\mathcal{F}}\right]_{\mathcal{F}} &= \left[\theta_\alpha^{\mathcal{F}}, c^{\mathcal{F}}\right]_{\mathcal{F}} = 0. && (74)
\end{aligned}$$

The ordinary brackets of the deformed quantities are

$$\begin{aligned}
\left\{\theta_\alpha^{\mathcal{F}}, \partial_\beta^{\mathcal{F}}\right\} &= \delta_{\alpha\beta} c && (75)\\
\left\{\theta_\alpha^{\mathcal{F}}, \theta_\beta^{\mathcal{F}}\right\} &= 2C_{\alpha\beta} c^2 && (76)\\
\left\{\partial_\alpha^{\mathcal{F}}, \partial_\beta^{\mathcal{F}}\right\} &= 0 && (77)\\
\left[c^{\mathcal{F}}, \theta_\alpha^{\mathcal{F}}\right] &= \left[c^{\mathcal{F}}, \partial_\alpha^{\mathcal{F}}\right] = 0. && (78)
\end{aligned}$$

It should be observed once more that the ordinary brackets of the deformed quantities give rise to the above non-anticommutative structure. This is just analogous to what happens in the bosonic case.

The inverse of (68), connecting non-anticommutative variables $\theta_\alpha^{\mathcal{F}}$ with the anticommuting θ_α: $\theta_\alpha = \theta_\alpha^{\mathcal{F}} - C_{\alpha\beta}\partial_\beta c$, is the fermionic counterpart of the Bopp shift.

Finally, the deformed multiplication is

$$a \star b \equiv m^{\mathcal{F}}(a \otimes b) = (m \circ \mathcal{F}^{-1})(a \otimes b) = \sum_\alpha (-1)^{|\bar{f}_\alpha||a|} \bar{f}^\alpha(a)\bar{f}_\alpha(b). \tag{79}$$

Additionally, defining $[a,b\}_\star \equiv a\star b + (-1)^{|a||b|} b\star a$, we have

$$\begin{aligned}
\{\theta_\alpha,\theta_\beta\}_\star &= 2C_{\alpha\beta}c^2 && (80)\\
\{\partial_\alpha,\theta_\beta\}_\star &= \delta_{\alpha\beta}c && (81)\\
\{\partial_\alpha,\partial_\beta\}_\star &= 0 && (82)\\
[c,\theta_\alpha]_\star &= [c,\partial_\alpha]_\star = 0. && (83)
\end{aligned}$$

Clearly, as in the bosonic case, the fermionic Moyal-brackets are isomorphic to the ordinary brackets of the deformed quantities (75-78).

5 On second quantized operators

Having studied the Hopf algebra and the deformed Hopf algebra arising from the universal enveloping algebra of bosonic and fermionic algebras, here we demonstrate how this Hopf algebra structure can be induced from the more fundamental Hopf algebra structure of second-quantized (Schrödinger) field-operators or the basic oscillators. This is important, since elements of the Heisenberg algebra can be expressed in terms of certain integrated objects of bilinears of those field operators.

To this end, consider the Schrödinger action

$$S = \int dtL, \tag{84}$$

where

$$L = \int d^Dx \left(\frac{i\hbar}{2}\psi^* \overleftrightarrow{\partial_o}\, \psi - \frac{\hbar^2}{2m}|\vec{\nabla}\psi|^2 \right) \tag{85}$$

is the Lagrangian in D-dimensional space. It can be easily checked that it yields the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi \tag{86}$$

as the Euler-Lagrange equation of motion. One can also check that the system is subject to the pair of second-class constraints $\chi \approx 0$ and $\chi^* \approx 0$, where χ is given by

$$\chi = \pi_\psi - \frac{i\hbar}{2}\psi^* \tag{87}$$

and π_ψ represents the canonically conjugate momentum to ψ.

Strong imposition of this pair of constraints results in the following Dirac brackets [15]

$$\{\psi(\vec{x},t),\psi^*(\vec{y},t)\}_{DB} = \frac{1}{i\hbar}\delta^D(\vec{x}-\vec{y}). \tag{88}$$

They can be obtained more simply by using the Faddeev-Jackiw symplectic technique [16], as already in first-order form.

They can now be upgraded to the level of quantum commutators for bosonic systems. Anticommutators should be used for fermions.

$$\left[\psi(\vec{x}), \psi^\dagger(\vec{y})\right] = \delta^D(\vec{x}-\vec{y}) \tag{89}$$

$$\left[\psi(\vec{x}), \psi(\vec{y})\right] = \left[\psi^\dagger(\vec{x}), \psi^\dagger(\vec{y})\right] = 0 \tag{90}$$

We now define the following integrated objects involving bilinears in fields:

$$X_i = \int d^D y \; y_i \psi^\dagger(\vec{y})\psi(\vec{y}) \tag{91}$$

$$P_i = -\frac{i\hbar}{2}\int d^D y \; \psi^\dagger(\vec{y}) \overleftrightarrow{\partial_i} \psi(\vec{y}) \tag{92}$$

This expression of momentum is obtained from Noether's theorem and by making use of the pair of (by now) strong constraints χ (87) and χ^*.

It should be mentioned at this stage that these integrated objects involving field bilinears provide a mapping from second-quantization formalism to first-quantization formalism.

Indeed, it can be easily seen, by using (89, 90), that X_i can really be identified with the position operator

$$X_i|\vec{y}\rangle = y_i|\vec{y}\rangle, \tag{93}$$

where $|\vec{y}\rangle = \hat{\psi}^\dagger(\vec{y})|0\rangle$.

Furthermore, it can now be shown that

$$[X_i, P_j] = i\hbar\delta_{ij}N, \tag{94}$$

where $N = \int d^D y \; \psi^\dagger(\vec{y})\psi(\vec{y})$ is the number operator. The rest of commutators vanish, so that $\hbar N$ is identified as the central charge in the N-particle sector. One can also see that

$$[P_i, \psi(\vec{x})] = i\hbar\partial_i\psi(\vec{x}) \tag{95}$$

$$\left[P_i, \psi^\dagger(\vec{x})\right] = i\hbar\partial_i\psi^\dagger(\vec{x}), \tag{96}$$

thus P_i generates the appropriate translations.

It can be recalled that in second quantization the fields are the operators acting on an appropriate Fock space, which is nothing but the infinite direct sum of all possible Hilbert spaces containing all possible number of particles, i.e., $\mathcal{H} = \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)} \oplus ... \oplus \mathcal{H}^{(n)} \oplus ...$, where $\mathcal{H}^{(n)}$ is the n-particle Hilbert space and the coordinate variables play the role of 'labels' for the infinite number of degrees of freedom. On the other hand, in the first-quantization formalism, the coordinate variables are themselves the operators acting on an appropriate Hilbert space and satisfy the Heisenberg algebra along with the conjugate momenta.

It turns out that in non-relativistic quantum mechanics these two formalisms are completely equivalent, as a generic N-particle state $|\psi_N\rangle$ can be obtained by superposing

states, in terms of first-quantized N-particle wavefunctions $\Phi(\vec{x}_1, ..., \vec{x}_N)$, obtained by N-fold action of the creation operators $\hat{\psi}^\dagger(\vec{x})$ on the vacuum state $|0\rangle$ defined as $\hat{\psi}(\vec{x})|0\rangle = 0$:

$$|\psi_N\rangle = \int d^D x_1 ... d^D x_N \; \Phi(\vec{x}_1, ..., \vec{x}_N) \hat{\psi}^\dagger(\vec{x}_1) ... \hat{\psi}^\dagger(\vec{x}_N)|0\rangle \tag{97}$$

Now the fields can be expanded in their Fourier modes

$$\psi(\vec{x}) = \frac{1}{(2\pi\hbar)^D} \int d^D p \; e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}} a_{\vec{p}} \tag{98}$$

$$\psi^\dagger(\vec{x}) = \frac{1}{(2\pi\hbar)^D} \int d^D p \; e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{x}} a^\dagger_{\vec{p}} \tag{99}$$

and conversely

$$a_{\vec{p}} = \int d^D x \; e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{x}} \psi(\vec{x}) \tag{100}$$

$$a^\dagger_{\vec{p}} = \int d^D x \; e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}} \psi^\dagger(\vec{x}). \tag{101}$$

The algebra of $a_{\vec{p}}$, $a^\dagger_{\vec{p}}$ is

$$\left[a_{\vec{p}}, a^\dagger_{\vec{p}'}\right] = (2\pi\hbar)^D \delta^D(\vec{p} - \vec{p}') \tag{102}$$

$$[a_{\vec{p}}, a_{\vec{p}'}] = \left[a^\dagger_{\vec{p}}, a^\dagger_{\vec{p}'}\right] = 0. \tag{103}$$

Now, expressing $\vec{P}$ (92) in momentum space as

$$\vec{P} = \frac{1}{(2\pi\hbar)^D} \int d^D p \; \vec{p} a^\dagger_{\vec{p}} a_{\vec{p}}, \tag{104}$$

we obtain

$$[P_i, a_{\vec{p}}] = -p_i a_{\vec{p}} \tag{105}$$

$$\left[P_i, a^\dagger_{\vec{p}}\right] = p_i a^\dagger_{\vec{p}}. \tag{106}$$

One can now construct the Hopf algebra through the universal enveloping algebra of the centrally extended algebra formed by $a_{\vec{p}}, a^\dagger_{\vec{p}}$ [§] and deform it through the same twist element $\mathcal{F}$ (32), just as before. The linear subspace of this universal enveloping algebra $\mathcal{U}^{\mathcal{F}}(a, a^\dagger)$ contains $a^{\mathcal{F}}_{\vec{p}}$ and $a^{\dagger\mathcal{F}}_{\vec{p}}$, the deformed version of $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$. These are easily obtained as before to get

$$a^{\mathcal{F}}_{\vec{p}} = \bar{f}^\alpha(a_{\vec{p}}) \bar{f}_\alpha = a_{\vec{p}} \, e^{\frac{i}{2\hbar^2}\theta_{ij} p_i P_j} \tag{107}$$

$$a^{\dagger\mathcal{F}}_{\vec{p}} = \bar{f}^\alpha(a^\dagger_{\vec{p}}) \bar{f}_\alpha = a^\dagger_{\vec{p}} \, e^{-\frac{i}{2\hbar^2}\theta_{ij} p_i P_j}, \tag{108}$$

[§]Note that this algebra is isomorphic to the Heisenberg algebra in the basis of creation and annihilation operators.

while the deformation of $\psi(\vec{x})$ and $\psi^\dagger(\vec{x})$ is:

$$\psi^{\mathcal{F}}(\vec{x}) = \psi(\vec{x})\, e^{\frac{1}{2\hbar}\theta_{ij}\overleftarrow{\partial_i}P_j} \tag{109}$$

$$\psi^{\dagger\mathcal{F}}(\vec{x}) = \psi^\dagger(\vec{x})\, e^{\frac{1}{2\hbar}\theta_{ij}\overleftarrow{\partial_i}P_j}. \tag{110}$$

Likewise, they belong to $\mathcal{U}^{\mathcal{F}}(\psi,\psi^\dagger)$.

It is interesting to note that (107,108), relating deformed and undeformed oscillators, are exactly the same found in [17].

The deformation of $\psi(\vec{x})$ and $\psi^\dagger(\vec{x})$ is consistent with that of $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$ because

$$a^{\mathcal{F}}_{\vec{p}} = \int d^Dx\; e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{x}}\psi^{\mathcal{F}}(\vec{x}) \tag{111}$$

$$a^{\dagger\mathcal{F}}_{\vec{p}} = \int d^Dx\; e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}}\psi^{\dagger\mathcal{F}}(\vec{x}). \tag{112}$$

The above expressions can be easily proven taking into account that $\vec{p}\cdot\vec{x}$ is invariant under the twist $\mathcal{F}$:

$$p^{\mathcal{F}}_i x^{\mathcal{F}}_i = p_i x_i = \vec{p}\cdot\vec{x} \tag{113}$$

We can therefore identify $\mathcal{U}^{\mathcal{F}}(a,a^\dagger)$ with $\mathcal{U}^{\mathcal{F}}(\psi,\psi^\dagger)$, as they are basically obtained by deforming $\mathcal{U}(a,a^\dagger)$ and $\mathcal{U}(\psi,\psi^\dagger)$ (the universal enveloping algebra obtained from $\psi(\vec{x})$ or their Fourier modes). They therefore correspond to different bases for the primitive elements.

We can observe at this stage that the algebra satisfied by the field variables ψ (89, 90) or their Fourier amplitudes (102, 103) is actually the same as that of the original Heisenberg algebra $h(N)$ with $N\to\infty$. Indeed, they are infinite in number, with $\psi^\dagger(\vec{x})$ and $a^\dagger_{\vec{p}}$ playing the role of conjugate momentum respectively of $\psi(\vec{x})$ and $a_{\vec{p}}$, while $\delta^D(\vec{x}-\vec{y})$ and $(2\pi\hbar)^D\delta^D(\vec{p}-\vec{p}')$ play the role of central charge instead of $i\hbar\delta_{ij}$. They also have the nice feature that the deformation induced on one induces a compatible deformation on the other (111,112).

At the level of Lie-algebra, the basic brackets (89,90) induce the appropriate brackets of X_i and P_j (94).

Thus, although the algebraic structures of the $\mathcal{N}$-Heisenberg algebra (94) are obtained from those of $\psi,\psi^\dagger$ or $a,a^\dagger$, the costructures are not. To see this, let us try to naively construct the Hopf algebra structure corresponding to the universal enveloping algebra of the second-quantized field algebra (89, 90) by applying the rules of coproduct homomorphism (9) valid for bosonic variables. This yields, using (91),

$$\begin{aligned}\Delta(X_i) &= \int d^Dy\; y_i\Delta(\psi^\dagger(\vec{y}))\Delta(\psi(\vec{y})) = \\ &= \int d^Dy\; y_i(\psi^\dagger(\vec{y})\otimes\mathbf{1}+\mathbf{1}\otimes\psi^\dagger(\vec{y}))(\psi(\vec{y})\otimes\mathbf{1}+\mathbf{1}\otimes\psi(\vec{y})) = \\ &= X_i\otimes\mathbf{1}+\mathbf{1}\otimes X_i + \int d^Dy\; y_i(\psi^\dagger(\vec{y})\otimes\psi(\vec{y})+\psi(\vec{y})\otimes\psi^\dagger(\vec{y})).\end{aligned} \tag{114}$$

Clearly, the presence of the integral involving cross-terms spoils the expected coproduct of X_i, which is expected to be given by

$$\Delta(X_i) = X_i \otimes \mathbf{1} + \mathbf{1} \otimes X_i. \tag{115}$$

We solve this problem in Section **7**, using the concepts of Wigner oscillators and superalgebras. We therefore begin by providing a brief review in the next section.

6 Wigner Oscillators and Superalgebras

The notion of Wigner's oscillators derives from [18] (see also [19], which started a series of related works; updated references can be found in [20]). In that work Wigner realized that the Hamiltonian equations can be identical to the Heisenberg equations for position and momentum operators, without necessarily realizing the canonical commutation relations. For oscillators, the Wigner's consistency conditions induce non-linear relations, involving commutators and anticommutators of the position and momentum operators, which can be recast as superalgebras.

The simplest example, for a single bosonic oscillator, requires the hamiltonian H to be expressed as an anticommutator of the oscillators $a^{\pm}$. ¶ By defining

$$\begin{aligned} H &= \frac{1}{2}\{a, a^{\dagger}\}, \\ [H, a^{\pm}] &= \pm a^{\pm} \end{aligned} \tag{116}$$

‖ and setting

$$E^{\pm} = \{a^{\pm}, a^{\pm}\}, \tag{117}$$

one recovers that $H, a^{\pm}, E^{\pm}$ are a set of generators of the $osp(1|2)$ superalgebra (see [21] for a quick introduction to $osp(1|2)$ and the other superalgebras). H is the Cartan element, while $a^{\pm}$ are the fermionic simple roots and correspond to the odd sector of the superalgebra. It should be noticed that the superalgebra interpretation requires $a^{\pm}$ to be odd generators, and therefore being of opposite statistics w.r.t. the usual interpretation of $a^{\pm}$ as bosonic creation and annihilation operators satisfying the Heisenberg algebra. The hamiltonian H, nevertheless, being bilinear in $a^{\pm}$, keeps its bosonic character. The connection of the hamiltonian (conveniently normalized as $H' = \frac{1}{2}H$) with the ordinary harmonic oscillator hamiltonian (normalized s.t. $\omega\hbar = 1$) is made in terms of the highest weight representations of $osp(1|2)$. The energy levels of the oscillator hamiltonian are given by $E_n = \frac{1}{2} + n$, for non-negative n ($\frac{1}{2}$ is the vacuum energy). On the other hand, the highest weight representations of $osp(1|2)$, defined by the more general condition $a^{-}|0>= 0$ and $H'|0>= \lambda|0>$, imply that the energy spectrum of H is given by a set of bosonic energy levels, whose eigenvalues are $E_n = \lambda + n$, for $(E^{+})^n|0>$ eigenvectors, as well as a set of fermionic energy levels given by $E_n = \lambda + n + \frac{1}{2}$, for

¶In the following, we shall use the notation $a = a^{-}$ and $a^{\dagger} = a^{+}$.

‖This second equation, involving the commutator with H, represents the compatibility condition between the Heisenberg equation and the operator-valued Hamilton's equation.

the associated $(a^+)^{2n+1}|0>$ eigenvectors. Disregarding the fermionic sector, the bosonic sector reproduces the eigenvalues of the ordinary harmonic oscillator for $\lambda = \frac{1}{2}$.

The Wigner's approach can be extended to several sets of both bosonic and fermionic creation and annihilation operators (originally satisfying the bosonic and fermionic Heisenberg algebras). In this case the Wigner's construction induces superalgebras of $osp(m|n)$ and $sl(m|n)$ series (see e.g. [22]).

A somehow different connection between superalgebras and bosonic and fermionic oscillators can be found in [23]. There, several series of superalgebras are realized in terms of an oscillator construction requiring m fermionic oscillators $f_i, f_i^\dagger$ and n bosonic oscillators $a_j, a_j^\dagger$. An explicit construction was given for the superalgebras of the $A(m-1|n-1) = sl(m|n)$, $B(m|n) = osp(2m+1|2n)$ and $D(m|n) = osp(2m|2n)$ series. As an example, the $osp(2m|2n)$ superalgebra whose bosonic sector coincides with $so(2m) \oplus sp(2n)$ and whose total number of generators is $2m^2 + 2n^2 - m + n$ bosonic and $4mn$ fermionic, is reproduced by the whole set of bilinear combinations of m fermionic and n bosonic oscillators.

7 Hopf algebra structure of the second-quantized operators

The second quantization requires the introduction of operators constructed with bilinear combinations of (bosonic and/or fermionic) creation and annihilation operators. We are interested in such operators like the oscillator-number N, the position $\vec{X}$, the momentum $\vec{P}$, as well as the functions thereof. The hamiltonian H is given, e.g., by $H = \frac{1}{2m}P^2 + V(\vec{X})$. An interesting class of operators is given by the bilinear combinations of $\vec{P}$ and $\vec{X}$, namely P^2, X^2 and $\frac{1}{2}(\vec{X}\vec{P} + \vec{P}\vec{X})$, which will be discussed later. The bosonic $(a_i, a_i^\dagger)$ and fermionic $(b_j, b_j^\dagger)$ creation and annihilation operators satisfy the bosonic and, respectively, the fermionic Heisenberg algebra introduced in the previous sections. As discussed there, both the bosonic and the fermionic Heisenberg algebra must be replaced by a graded Lie algebra. The universal enveloping algebra of the Heisenberg algebra acquires the status of a Hopf algebra, which can be eventually deformed with a twist.

In general, for a finite Lie algebra g of dimension N, which admits a central extension c (we denote as τ_i the set of generators of g, while $\check{\tau}_j$ denotes the subset of generators which do not coincide with c), we can introduce an ordering in its generators $(\tau_1 < \tau_2 < \ldots < \tau_N)$. The universal enveloping algebra $\mathcal{U}(g)$ can be decomposed, as a vector space, into

$$\mathcal{U}(g) \;=\; g_0 \oplus g_1 \oplus g_2 \oplus \ldots \tag{118}$$

where g_0 coincides with the identity,

$$g_0 \;=\; \mathbf{1}, \tag{119}$$

g_1 coincides with the Lie algebra g itself,

$$g_1 \;=\; g, \tag{120}$$

while g_k is spanned by the ordered k-ples of the τ_i generators,

$$\tau_{i_1}\tau_{i_2}\ldots\tau_{i_k} \quad \in \quad g_k \tag{121}$$

for $\tau_{i_1} \leq \tau_{i_2} \leq \ldots \leq \tau_{i_k}$.

The g_k space can be further decomposed into its h_k^l subspaces,

$$g_k \quad = \quad h_k^0 \oplus h_k^1 \oplus \ldots \oplus \ldots h_k^k, \tag{122}$$

s.t. l denotes the power of c entering the decomposition. Therefore h_k^0 is spanned by k-ples of ordered $\check{\tau}_j$ generators, while, symbolically, $h_k^l \equiv c^l h_{k-l}^0$.

Taking into account that the only generator entering the non-vanishing r.h.s. in the (bosonic and fermionic) Heisenberg algebras is the central extension c, we can therefore conclude that

$$[h_2^0, h_2^0] \subset h_2^1. \tag{123}$$

It implies that the bilinear combinations of $\check{\tau}_j$ generators acquire a Lie algebra structure, provided that c, the central extension, would be re-interpreted as a c-number. The Lie algebra structure on h_2^0 is naturally induced by the Lie algebra structure on g. On the other hand, as a Lie algebra, h_2^0 induces a Hopf algebra structure in its universal enveloping algebra $\mathcal{U}(h_2^0)$. This Hopf algebra structure is not related with the original Hopf algebra structure defined on $\mathcal{U}(g)$ and in particular is not a sub-Hopf algebra of the $\mathcal{U}(g)$ Hopf algebra. This is due to the different role of c, entering as a central extension in g and as a c-number in h_2^0.

The resulting observation is that, while it is still possible to define a Hopf algebra structure for bilinear second-quantized operators, their "composite nature" is lost in the process. Their Lie algebra is determined by a more fundamental (the "oscillators") level. Their co-structures however, in particular the coproduct, are not. This situation is clearly unsatisfactory. We have seen for instance, for the class of deformations arising from a twist, that an "ideological viewpoint" can be maintained: within a suitable basis the Hopf algebra deformation is carried by the co-structures alone (in particular, the coproduct). A twist deformation of second-quantized operators would be hand-imposed and would spoil the connection with oscillators. Conversely, a twist deformation of the oscillators would not reflect in a twist deformation of second-quantized operators.

This situation can be reconciled by making use of the Wigner's approach to the oscillators algebra. As discussed in Section **6** the troublesome Heisenberg algebras are replaced by superalgebras which do not admit central extension. The ordinary oscillators' energy eigenvalues are recovered as specific highest weight representations. This construction, besides providing the Hopf algebra structure for the ordinary oscillators, allows their extension and deformations. Their extension, already discussed in Wigner's original paper [18], refers to the choice of the highest weight representation. The deformation is induced by such deformations (like the twists, see [24] and [25]) of the original superalgebra which preserve the graded Hopf algebra structure.

All that we have to do at this stage is to re-write the operators X_i (92) and P_i (93) and the number operator N in Weyl-symmetric form:

$$\tilde{X}_i = \frac{1}{2}\int d^D y\; y_i \left(\psi^\dagger(\vec{y})\psi(\vec{y}) + \psi(\vec{y})\psi^\dagger(\vec{y})\right) \tag{124}$$

$$\tilde{P}_i = \frac{1}{2}\int d^D p\; p_i \left(a^\dagger_{\vec{p}} a_{\vec{p}} + a_{\vec{p}} a^\dagger_{\vec{p}}\right) \tag{125}$$

$$\tilde{N} = \frac{1}{2}\int d^D y\; \left(\psi^\dagger(\vec{y})\psi(\vec{y}) + \psi(\vec{y})\psi^\dagger(\vec{y})\right) \tag{126}$$

It can now be easily checked that they satisfy the same algebra as the untilded operators. Note that to facilitate subsequent computations, we have re-written the expression of $\tilde{P}_i$ in momentum space, where it takes the diagonal form. The other two variables $\tilde{X}_i$ and $\tilde{N}$ have similar forms already in coordinate space itself.

If we now declare $\psi(\vec{y})$ and $a_{\vec{p}}$ to be odd, the coproduct of $\tilde{X}_i$ is correctly induced as

$$\begin{aligned}
\Delta(\tilde{X}_i) &= \frac{1}{2}\int d^D y\; y_i(\Delta(\psi^\dagger(\vec{y}))\Delta(\psi(\vec{y})) + \Delta(\psi(\vec{y}))\Delta(\psi^\dagger(\vec{y}))) = \\
&= \frac{1}{2}\int d^D y\; y_i[\psi^\dagger(\vec{y})\psi(\vec{y}) \otimes \mathbf{1} - \psi(\vec{y}) \otimes \psi^\dagger(\vec{y}) + \psi^\dagger(\vec{y}) \otimes \psi(\vec{y}) + \mathbf{1} \otimes \psi^\dagger(\vec{y})\psi(\vec{y}) + \\
&\quad + \psi(\vec{y})\psi^\dagger(\vec{y}) \otimes \mathbf{1} - \psi^\dagger(\vec{y}) \otimes \psi(\vec{y}) + \psi(\vec{y}) \otimes \psi^\dagger(\vec{y}) + \mathbf{1} \otimes \psi(\vec{y})\psi^\dagger(\vec{y})] = \\
&= \tilde{X}_i \otimes \mathbf{1} + \mathbf{1} \otimes \tilde{X}_i,
\end{aligned} \tag{127}$$

the same holding for the coproduct of $\tilde{P}_i$.

The antipode is also properly induced as

$$\begin{aligned}
S(\tilde{X}_i) &= \frac{1}{2}\int d^D y\; y_i(S(\psi(\vec{y})\psi^\dagger(\vec{y}))S(\psi^\dagger(\vec{y})\psi(\vec{y})) = \\
&= \frac{1}{2}\int d^D y\; y_i[(-1)^{|\psi(\vec{y})||\psi^\dagger(\vec{y})|}S(\psi^\dagger(\vec{y}))S(\psi(\vec{y})) + (-1)^{|\psi^\dagger(\vec{y})||\psi(\vec{y})|}S(\psi(\vec{y}))S(\psi^\dagger(\vec{y}))] = \\
&= \frac{1}{2}\int d^D y\; y_i \left(-\psi^\dagger(\vec{y})\psi(\vec{y}) - \psi(\vec{y})\psi^\dagger(\vec{y})\right) = \\
&= -\tilde{X}_i,
\end{aligned} \tag{128}$$

as well as

$$S(\tilde{P}_i) = -\tilde{P}_i \tag{129}$$

Co-unit poses no problem since $\varepsilon(\psi(\vec{y})) = \varepsilon(a_{\vec{p}}) = 0$, leading to $\varepsilon(\tilde{X}_i) = \varepsilon(\tilde{P}_i) = 0$.

Expressions (124-126) yield the expected (absence of) deformation for $\tilde{P}_i$:

$$\begin{aligned}
\tilde{P}_i^{\mathcal{F}} &= \frac{1}{2}\int d^D p\; p_i \left(a^{\dagger\mathcal{F}}_{\vec{p}} a^{\mathcal{F}}_{\vec{p}} + a^{\mathcal{F}}_{\vec{p}} a^{\dagger\mathcal{F}}_{\vec{p}}\right) = \\
&= \frac{1}{2}\int d^D p\; p_i \left(a^\dagger_{\vec{p}} e^{\frac{i}{2\hbar^2}\theta_{ij}p_iP_j} e^{\frac{-i}{2\hbar^2}\theta_{ij}p_iP_j} a_{\vec{p}} + a_{\vec{p}} e^{\frac{-i}{2\hbar^2}\theta_{ij}p_iP_j} e^{\frac{i}{2\hbar^2}\theta_{ij}p_iP_j} a^\dagger_{\vec{p}}\right) = \\
&= \tilde{P}_i,
\end{aligned} \tag{130}$$

as well as for $\tilde{X}_i$, as can be seen in momentum space:

$$\tilde{X}_i^{\mathcal{F}} = \frac{i\hbar}{4}\int d^D p \left(a_{\vec{p}}^{\dagger\mathcal{F}} \overleftrightarrow{\partial}_{p_i} a_{\vec{p}}^{\mathcal{F}} + a_{\vec{p}}^{\mathcal{F}} \overleftrightarrow{\partial}_{p_i} a_{\vec{p}}^{\dagger\mathcal{F}}\right) = \tilde{X}_i - \frac{1}{2\hbar^2}\theta_{ij}p_j\hbar\tilde{N}, \tag{131}$$

i.e., it reduces to the previously obtained deformation (37) at the one particle ($\mathcal{N} = 1$) limit.

8 Hopf algebra and quantum statistics

We discuss now the relation between Wigner's oscillators (Wigner's approach and extension of the standard oscillator algebra), Hopf algebras and quantum statistics. Related issues have been discussed in [26] and [27]. However, some comments are necessary. It is sufficient to discuss the single bosonic oscillator which, in Wigner's approach, is related to a given highest weight representation of $osp(1|2)$. The vacuum state $|0>$ is assumed to be bosonic. Due to the fermionic nature of a^+, which belongs to the odd sector of the $osp(1|2)$ superalgebra, applying integer powers ${a^+}^k$ to the vacuum produces a tower of states which are, alternately, bosonic and fermionic. If $\lambda = \frac{1}{2}$ ($H'|0>= \lambda|0>$) one can introduce the fermion-number operator N_F, which in the $osp(1|2)$ Wigner realization of the oscillator algebra, can be expressed as

$$N_F \;=\; \frac{1}{2}(1 + e^{2\pi i H'}). \tag{132}$$

The bosonic sector is recovered through the superselection rule $N_F = 0$ (the corresponding projector is $\mathbf{1} - N_F$). The fermionic sector has eigenvalue $N_f = 1$. The energy eigenvalues E_n^{bos} of the bosonic states are given by $E_n^{bos} = \lambda + n$, for $n = 0, 1, 2, \ldots$, while the energy eigenvalues E_n^{fer} of the fermionic eigenstates are $E_n^{fer} = \lambda + \frac{1}{2}n$ for $n = 1, 2, 3, \ldots$.

The standard oscillator Hilbert space is therefore recovered from the $osp(1|2)$ $\lambda = \frac{1}{2}$ highest weight representation by taking the bosonic sector. This is tantamount to start, from the very beginning, by looking at a specific highest weight representation of $su(1|1)$ (the $osp(1|2)$ bosonic subalgebra), whose generators (suitably normalized) are $H, E^{\pm}$.

On the other hand, limiting the construction to the $su(1|1)$ bosonic algebra would prevent rexpressing the hamiltonian in terms of a bilinear combination of oscillators (the anticommutator (up to a normalizing factor) of $a^{\pm}$).

The theory derived from $osp(1|2)$ is more general because it produces both bosonic and fermionic eigenstates of the hamiltonian. Taking the bosonic projection is an extra requirement which needs not to be necessarily implemented.

Contrary to what stated in [27], the $osp(1|2)$ spectrum is *not* supersymmetric. The reason is due to the fact that there is no degeneracy of the positive energy eigenvalues between fermionic and bosonic states. In order to fulfil a true supersymmetry, the presence of at least one bosonic oscillator and one fermionic oscillator satisfying the Heisenberg (respectively bosonic and fermionic) algebra is required. In the Wigner interpretation, this coupled system of oscillators has to be replaced by its associated superalgebra. The results of [23], recalled in Section **6**, show that one such superalgebra is $osp(2|2)$. For such superalgebra it is indeed possible to introduce a hamiltonian operator (bilinear w.r.t. the

odd generators of $osp(2|2)$) s.t. its spectrum produces the supersymmetric degeneracy with a one-to-one correspondence of bosonic and fermionic eigenstates for every positive eigenvalue of the energy. The detailed construction will be produced elsewhere. Other superalgebras are related with other systems of oscillators; [22], e.g., relates a 3-dimensional oscillator to the $osp(3|2)$ superalgebra.

The interpretation of the coproduct follows now the one given in [27]. Let us focus on the $su(1,1)$ algebra (the generalization to other algebras and superalgebras is straightforward) expressed by the $H, E^{\pm}$ generators satisfying

$$\begin{aligned} [H, E^{\pm}] &= \pm 2E^{\pm}, \\ [E^{+}, E^{-}] &= H. \end{aligned} \tag{133}$$

In the Wigner's interpretation, the hamiltonian for the harmonic oscillator can be expressed as $\mathbf{H} = \frac{\omega}{2}H$. If one starts with a highest-weight vector $|0\rangle$ s.t. $E^{-}|0\rangle = 0$ and $H|0\rangle = \mu|0\rangle$, therefore, for $su(1|1)$ the hamiltonian $\mathbf{H}$ admits eigenvalues $\omega(\frac{\mu}{2} + m)$ when applied to its $E^{+m}|0\rangle$ eigenvector. Setting $\widetilde{|m\rangle} = E^{+m}|0\rangle$, with straightforward computations, one can introduce the normalized state

$$|m\rangle = \frac{1}{\sqrt{(-1)^m m! \prod_{j=0}^{m-1}(\mu + j)}}\widetilde{|m\rangle}, \tag{134}$$

where

$$\langle m|m\rangle = 1. \tag{135}$$

The integer m can be regarded both as an energy level or as an m-particle state. Let us call $\mathcal{H}$ the Hilbert space associated to the highest weight representation of $\mathbf{H}$. The coproduct $\Delta^n(E^{+m})$ induces the map $E^{+m}|0\rangle \in \mathcal{H} \mapsto \mathcal{H} \otimes \ldots \otimes \mathcal{H}$ ($\equiv \mathcal{H}^{\otimes n+1}$). The j-th Hilbert space in the tensor product can be referred to as the j-th slot. The hamiltonian acting on the tensor product is $\Delta^n(\mathbf{H})$. The vacuum state in $\mathcal{H}^{\otimes n+1}$ is given by the tensor product $|0\rangle \otimes \ldots \otimes |0\rangle \equiv |\mathbf{0}\rangle$. The probability that an m-particle state can be realized with m_j particles at the j-th slot (s.t. $\sum_j m_j = m$) depends on the highest weight μ.

We explicitly discuss the $n = 1$, $m = 2$ example. The (unnormalized) state $\Delta(E^{+2})|\mathbf{0}\rangle$ is given by

$$\Delta(E^{+2})|\mathbf{0}\rangle = \sqrt{2\mu(\mu+1)}(|2\rangle \otimes |0\rangle + |0\rangle \otimes |2\rangle) - 2\mu|1\rangle \otimes |1\rangle. \tag{136}$$

The normalized state is

$$|\mathbf{2}\rangle = \frac{1}{\sqrt{4\mu(1+2\mu)}}\Delta(E^{+2})|\mathbf{0}\rangle. \tag{137}$$

The probability to recover, e.g., one particle in the fist slot and one particle in the second slot is given by $P_{11} = |(\langle 1|\otimes\langle 1|)|\mathbf{2}\rangle|^2$ (and similarly for the other cases $P_{20} = |(\langle 2|\otimes\langle 0|)|\mathbf{2}\rangle|^2$ and $P_{02} = |(\langle 0| \otimes \langle 2|)|\mathbf{2}\rangle|^2$). These probabilities are explicitly given by $P_{11} = \frac{\mu}{2\mu+1}$, $P_{20} = P_{02} = \frac{\mu+1}{4\mu+2}$. The (Bose-Einstein) equipartition, $P_{11} = P_{20} = P_{02} = \frac{1}{3}$ is recovered for the highest weight $\mu = 1$.

This analysis can be repeated for other algebras and superalgebras, recovering as well the Fermi-Dirac statistics. An important final comment is the following: not only deformations of the algebra (the deformed coproduct) can change the usual framework, but also the choice of the vacuum energy (specified by the highest weight μ). Statistics can therefore be deformed with two different prescriptions.

9 Conclusions

We have shown that it is indeed possible to define a Hopf algebra structure on the universal enveloping algebra $\mathcal{U}(h)$ of the Heisenberg algebra h and deform it to $\mathcal{U}^{\mathcal{F}}(h)$ by applying the abelian twist *à la* Drinfeld, which satisfies the co-cycle condition trivially, provided that the role of the central extension is identified properly. To be more precise, we require the central extension to be regarded as a generic element of the Lie algebra, enjoying the same co-algebra structure as the other generators x_i and p_j of the Heisenberg algebra h and not merely as a multiple of the identity (which by definition is included in $\mathcal{U}(h)$) of the corresponding Lie group. We have shown that the deformed generators $x_i^{\mathcal{F}}$ and $p_j^{\mathcal{F}}$, the primitive elements spanning the linear sub-space of $\mathcal{U}^{\mathcal{F}}(h)$ under the deformed commutator $[,]_{\mathcal{F}}$, have an isomorphic structure to those of the undeformed version. In particular, the commuting $[x_i, x_j] = 0$ subalgebra maps again into the corresponding commuting version under complete deformation: $[x_i^{\mathcal{F}}, x_j^{\mathcal{F}}]_{\mathcal{F}} = 0$. The non-commutativity is obtained only in the hybrid case, i.e. when the ordinary commutators of the deformed variables are computed $[x_i^{\mathcal{F}}, x_j^{\mathcal{F}}] = i\theta_{ij}$. We have also shown how this implies that the deformed $x_i^{\mathcal{F}}$'s no longer have a vectorial transformation property under $SO(D)$ rotation (for $D > 2$) to conform to constant and non-transforming θ_{ij} tensor. The parallel of this analysis involving bosonic variables is also valid for fermionic variables.
The position and momentum generators of the Heisenberg algebra can be regarded as composite objects expressed in terms of integrated bilinears of the Schrödinger fields/oscillators. The oscillator algebra can be upgraded to its universal enveloping algebra, with its own Hopf algebra structure, and can be deformed by a twist element. However, this Hopf algebra structure does not induce the appropriate Hopf algebra structure for the Heisenberg algebra defined by the composite position and momentum generators. The failure is at the level of co-algebra; indeed, due to certain ambiguous roles played by the central extension, the correct co-product is not induced. A solution to this problem and a full Hopf algebra mapping can be obtained by getting rid, altogether, of the central charge and by making use of the Weyl ordering. Indeed, using the Wigner's prescription, we can regard the bosonic Schrödinger oscillators as odd generators of an appropriate super-algebra, such as $osp(1|2n)$. By correctly taking into account the odd nature of the oscillators in the Wigner's prescription, it is indeed possible to induce a Hopf algebra structure on $\mathcal{U}(h)$ and deform it to get $\mathcal{U}^{\mathcal{F}}(h)$.

We have also discussed the implication of this construction for quantum statistics, showing that both the deformed co-product and the choice of the vacuum energy corresponding to the highest weight representation can give rise to deformations for ordinary Bose-Einstein and Fermi-Dirac statistics.

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