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On Supergroups with Odd Clifford Parameters and Non-anticommutative Supersymmetry.

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Abstract

We investigate supergroups with Grassmann parameters replaced by odd Clifford parameters. The connection with non-anticommutative supersymmetry is discussed. A Berezin-like calculus for odd Clifford variables is introduced. Fermionic covariant derivatives for supergroups with odd Clifford variables are derived. Applications to supersymmetric quantum mechanics are made. Deformations of the original supersymmetric theories are encountered when the fermionic covariant derivatives do not obey the graded Leibniz property. The simplest non-trivial example is given by the $N = 2$ SQM with a real $(1, 2, 1)$ multiplet and a cubic potential. The action is real. Depending on the overall sign ("Euclidean" or "Lorentzian") of the deformation, a Bender-Boettcher pseudo-hermitian hamiltonian is encountered when solving the equation of motion of the auxiliary field. A possible connection of our framework with the Drinfeld twist deformation of supersymmetry is pointed out.

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1 Introduction

In this paper we investigate the properties of Lie supergroups whose odd-parameters are Clifford-valued (instead of being Grassmann numbers). The case of the Supersymmetric Quantum Mechanics (s.t. the Lie superalgebra is given by the one-dimensional N-extended supersymmetry algebra) is explicitly discussed. The extension of the approach to odd-Clifford supergroups based on higher-dimensional super-Poincaré superalgebras is immediate.

We produce an extension of the Berezin calculus which takes into account the Clifford property of the odd variables. The supersymmetric fermionic covariant derivatives are derived with standard methods. It is of particular interest the case of the $N = 2$ one-dimensional supersymmetry. The $Cl(2,0)$ and $Cl(1,1)$ ("Euclidean" and respectively "Lorentzian") Clifford generalizations of the ordinary $N = 2$ one-dimensional Grassmann supersymmetry imply fermionic covariant derivatives which do not satisfy the graded Leibniz property. In our framework this fact proves to be crucial to produce genuine deformations of the ordinary $N = 2$ Supersymmetric Quantum Mechanical models. Moreoveor, depending on the type of Clifford deformation and the chosen values of the coupling constants, we naturally induce Bender-Boettcher $\mathcal{PT}$-symmetric pseudo-hermitian hamiltonians [1] from real $N = 2$ supersymmetric actions.

Supergroups with odd Clifford parameters turn to be a very natural framework to describe Non-anticommutative supersymmetric theories. We recall that non-anticommutative supersymmetry has received considerable attention in the last few years. A number of papers [2, 3, 4, 5, 6] have explored the implications of introducing non-anticommutative spinorial coordinates, either as a mathematical possibility or in the string context (see [7, 8] for recent reviews). The work of [6], introducing a non-anticommutative supersymmetry in a 4-dimensional Euclidean superspace, has been particular influential. The construction of nonanticommutative supersymmetric theories in lower dimensions (two, see [9, 10, 11]) or three (see [12]) has later been investigated. In [13] it was pointed out that the one-dimensional framework of the $N = 2$ non-anticommutative supersymmetric quantum mechanics could be important for understanding several mathematical properties of non-anticommutative supersymmetry, as well as exploring physical applications (e.g. to condensed matter physics).

The majority of the recent works on non-anticommutative supersymmetry has been inspired by the non-commutative deformation of the ordinary bosonic theories. Due to this reason, the most employed tool is a deformed Moyal star product applied to fermionic (anticommuting) variables. In [14], for instance, the deformed quantization program of [15] is taken as an inspiration to construct Clifford algebras from Moyal star-products of Grassmann generators.

In this paper we are advocating a somehow complementary viewpoint. We start, from the very beginning, with a Clifford algebra, whose properties are known. The squares of the odd-generators have a mass-dimension $mass^{-1}$; they are therefore naturally associated to a Clifford-deformation mass scale M. By letting $M \to \infty$ we are able to recover, in the limit, the Grassmann case.

It is worth recalling that several different prescriptions have been given in the literature to introduce non-anticommutative deformations of ordinary supersymmetry. In most of the cases, the supersymmetry algebra itself is deformed (see e.g. [6]). On the other hand, as it was already pointed out in [6], the supersymmetry algebra can be restored at the price of introducing fermionic covariant derivatives which do not satisfy the graded Leibniz property. In [6] and several other papers following it, the investigation si restricted to graded Leibniz derivatives which obey the graded Leibniz property. Two main motivations for that are given. The first one concerns chiral (antichiral) superfields; the breaking of Leibniz implies that the product of chiral superfields is no longer chiral. The second motivation concerns the impossibility of integrating by parts. These two motivations can be easily overcome, at least in selected cases. There are interesting theories which do not require the presence of chiral or antichiral superfields. The $N = 2$ supersymmetric quantum mechanics for the *real* $(1, 2, 1)$ superfield (with one auxiliary field), discussed in Section **6**, is an example. Moreover, for this kind of theory, the supersymmetric potentials are manifestly supersymmetric invariant, because the supersymmetry generators, applied to the integrand, produce a total time-derivative applied to the only term surviving the integration over odd-Clifford variables.

We postpone to the Conclusions further discussions of several features of our approach. These features include the connection with pseudohermitian hamiltonians, the connection between the breaking of the graded Leibniz property and the twisted deformation of supersymmetry, the extension of our construction to higher dimensions.

The scheme of this paper is as follows. In Section **2** supergroups with odd Clifford parameters are introduced. The supergroups associated to the superalgebras of the one-dimensional N-extended supersymmetry are explicitly discussed. In Section **3** a Berezin-like calculus is presented for odd variables which are no longer anticommuting (Grassmann). In Section **4** we derive the fermionic covariant derivatives for superspaces with odd Clifford variables. The superfield formalism for such superspaces is introduced in Section **5**. The one-dimensional N-extended supersymmetry is investigated and the conditions under which the fermionic covariant derivatives obey the graded Leibniz rule are expressed. The properties of the "Euclidean" and "Lorentzian" Clifford deformations of the $N = 2$ one-dimensional supersymmetry are discussed. In Section **6** the odd-Clifford approach is employed to introduce the Non-Anticommutative Supersymmetric Quantum Mechanics. A detailed analysis of the $N = 2$ SQM in terms of a real $(1, 2, 1)$ superfield with a trilinear superpotential is made. The auxiliary field satisfies (for Euclidean and Lorentzian deformations) an algebraic equation of motion. In the purely bosonic limit the theory is described by a trilinear potential. Depending on the type of deformation and the value of the coupling constant, the effective hamiltonian can be reduced to a Bender-Boettcher $\mathcal{PT}$-symmetric pseudo-hermitian hamiltonian. In the Conclusions we discuss several features of our construction, such as the connection with pseudo-hermitian hamiltonians, the possibility of interpreting the breaking of the graded Leibniz rule of the fermionic covariant derivatives as a non-trivial coproduct within the Drinfeld twist deformation of the supersymmetry, etc. The necessary modifications to accommodate within this framework higher-dimensional supersymmetric theories are mentioned.

2 Supergroups with odd Clifford parameters: the one-dimensional $N = 1, 2$ supersymmetry

Lie superalgebras are $\mathbf{Z}_2$-graded algebras whose generators, split into even and odd sectors, satisfy (anti)-commutation relations (see [16, 17, 18] for a precise definition). Examples of Lie superalgebras include the algebra of the one-dimensional N-extended supersymmetry discussed below, the super-Poincaré algebra, the simple Lie superalgebras ([18]). For ordinary Lie groups the elements connected with the identity are obtained through "exponentiation" of the Lie algebra generators. Similarly, the elements connected to the identity of the Lie supergroups are obtained by exponentiating the Lie superalgebra generators. In the standard construction ([16, 17, 19]), Lie supergroups are locally expressed in terms of bosonic parameters associated to the even generators of the Lie superalgebra, while fermionic, Grassmann-number parameters are associated to the odd generators (being Grassmann, in particular, their square is assumed to vanish).

It is tempting to understand the non-anticommutative formulation of the supersymmetry by relaxing the Grassmann condition for the odd parameters. In the examples here discussed we assume them to satisfy a more general class of algebras. Let's take N odd parameters θ_i ($i = 1, \ldots, N$); we can assume their anticommutators (once conveniently normalized) being expressed through

$$\theta_i\theta_j + \theta_j\theta_i \;=\; 2\eta_{ij}, \tag{2.1}$$

where η_{ij} is a diagonal matrix with p elements $+1$, q elements -1 and r zero elements in the diagonal (therefore $N = p+q+r$). The subsector of r θ_i's with vanishing square is still Grassmann. For $r = 0$, the equation (2.1) reduces to the basic relation of the generators of the $Cl(p, q)$ Clifford algebra.

We can refer to the procedure of replacing Grassmann variables with the (2.1) relation as "Cliffordization" of the supergroup elements. The corresponding supersymmetry will be denoted as "$Cl(p, q, r)$-type". The ordinary supersymmetry is recovered for $p = q = 0$ (therefore, it is of "$Cl(0, 0, N)$-type"). The Cliffordization is here proposed as a framework to understand the features of the "non-anticommutative supersymmetry".

In this paper we are mostly concerned with the example of the one-dimensional N-extended supersymmetry algebra underlying the Supersymmetric Quantum Mechanics. It is explicitly given in terms of a single even generator H (the hamiltonian, in physical applications) and N odd generators Q_i ($i = 1, \ldots, N$), satisfying the (anti)commutation relations

$$\begin{aligned} \{Q_i, Q_j\} &= 2\delta_{ij}H, \\ [H, Q_i] &= 0. \end{aligned} \tag{2.2}$$

Let us discuss first the $N = 1$ case ($Q^2 = H$). The bosonic parameter associated with H is the "time" t, while, for later convenience, we denote as θ_λ the odd-parameter associated with Q. We notice that θ_λ can be expressed as $\theta_\lambda = \lambda\theta$, where λ is a real number and θ is a given odd-parameter of reference.

For both even and odd variables a, b, the conjugation "$*$" is defined (see [20]) satisfying

$$
\begin{aligned}
(ab)^* &= b^* a^*, \\
(a^*)^* &= a.
\end{aligned} \tag{2.3}
$$

Accordingly, an element g of the unitary $N=1$ supergroup is expressed as

$$
g = e^{-iHt} e^{\lambda\theta Q}. \tag{2.4}
$$

We assumed the reality condition

$$
\theta^* = \theta. \tag{2.5}
$$

In mass-dimension, we have

$$
\begin{array}{ll}
[H] = 1, & [t] = -1, \\
[Q] = \frac{1}{2}, & [\theta_\lambda] = -\frac{1}{2}.
\end{array} \tag{2.6}
$$

Since θ^2 must have the correct mass-dimension, it should be expressed in terms of some positive mass scale M. We can distinguish three cases, up to a normalization factor. We can set

$$
\theta^2 = \frac{\epsilon}{M}, \tag{2.7}
$$

with $\epsilon = 0$ (the Grassmann case), $\epsilon = +1$ or $\epsilon = -1$. In terms of θ_λ we have that each θ_λ satisfy, in the three respective cases, the conditions ${\theta_\lambda}^2 = 0$, ${\theta_\lambda}^2 > 0$ or ${\theta_\lambda}^2 < 0$.

As a result we obtain three $N=1$ supergroups associated to the $N=1$ superalgebra (2.2). By setting

$$
X = \sqrt{\frac{\lambda^2 H}{M}} \tag{2.8}
$$

we obtain, for $\epsilon = 0, 1, -1$:
i) $\epsilon = 0$, the *rational* case,

$$
g = e^{-iHt}(1 + \lambda\theta Q), \tag{2.9}
$$

ii) $\epsilon = 1$, the *trigonometric* case,

$$
g = e^{-iHt}(\cos X + I \sin X), \tag{2.10}
$$

where $I = \theta Q\sqrt{\frac{M}{H}}$ and $I^2 = -1$;
iii) $\epsilon = -1$, the *hyperbolic* case,

$$
g = e^{-iHt}(\cosh X + J \sinh X), \tag{2.11}
$$

where $J = \theta Q\sqrt{\frac{M}{H}}$ and $J^2 = 1$.

Notice that the ordinary Grassmann case can be recovered from both Clifford cases in the special limit $M \rightarrow \infty$.

The extension of the above procedure for arbitrary N is straightforward. In the following we are mostly interested to the $N = 2$ case. It is characterized by two odd parameters of reference, θ_1, $\theta_2 (\theta_i{}^* = \theta_i)$, satisfying $\theta_i{}^2 = \frac{\epsilon_i}{M}$, where both ϵ_1, ϵ_2 can assume the three values 0, $+1$ and -1. As it will appear in the following, most of the interesting properties of the Cliffordized $N = 2$ supersymmetry are expressed in terms of the product

$$\epsilon = \epsilon_1 \epsilon_2. \tag{2.12}$$

The three cases for ϵ correspond to
i) $\epsilon = 0$, where at least one of the two θ's is Grassmann,
ii) $\epsilon = +1$, the "Euclidean" version of the $N = 2$ Clifford Supersymmetry (realized by either $Cl(2,0)$ or $Cl(0,2)$),
iii) $\epsilon = -1$, the "Lorentzian" version of the $N = 2$ Clifford Supersymmetry (obtained for $Cl(1,1)$).

3 A Berezin-like calculus for odd-Clifford variables

The Berezin calculus sets the rules for the derivation and the integration of odd Grassmann variables [20]. For our purposes we need to introduce a calculus which substitutes the Berezin calculus in the case of odd variables of Clifford type.

For a single Grassmann variable θ the Berezin calculus states that the derivative ∂_θ is normalized s.t. $\partial_\theta \theta = 1$, while giving vanishing results otherwise. The Berezin integration $\int d\theta$ coincides with the Berezin derivation ($\int d\theta = \partial_\theta$). The extension of the Berezin calculus to an arbitrary number of Grassmann variables is straightforward. Notice that, if θ has mass-dimension $[\theta] = -\frac{1}{2}$ as in the previous Section, then ∂_θ has mass-dimension $[\partial_\theta] = \frac{1}{2}$.

We establish now the rules for an analogous calculus in the case of an odd θ s.t. $\theta^2 = \frac{\epsilon}{M} \neq 0$. We introduce an odd derivation ∂_θ for the Clifford θ by assuming that it has the same mass-dimension as the Berezin derivation and coincides with it in the $M \rightarrow \infty$ limit. Therefore

$$\begin{aligned} \partial_\theta \theta &= 1, \\ \partial_\theta 1 &= 0. \end{aligned} \tag{3.13}$$

The application of ∂_θ to the powers θ^n, for integral values $n > 1$, is determined under the assumption that ∂_θ satisfies a graded Leibniz rule. Let f_1, f_2 be two functions of grading $deg(f_1)$, $deg(f_2)$ respectively ($deg(f) = 0$ for a bosonic function f, while $deg(f) = 1$ for a fermionic function); we assume

$$\partial_\theta (f_1 f_2) = (\partial_\theta f_1) f_2 + (-1)^{-deg(f_1)} f_1 (\partial_\theta f_2). \tag{3.14}$$

As a consequence, the application of ∂_θ to even and odd powers of θ is respectively given by

$$\partial_\theta \theta^{2k} = 0,$$

$$\partial_\theta \theta^{2k+1} = \theta^{2k}. \tag{3.15}$$

By requiring the integral of a total derivative to be vanishing we can unambiguously fix

$$\int d\theta\ \theta^{2k} = 0. \tag{3.16}$$

The rule for the integration over odd powers of θ can be set by requiring, as in the Berezin case, that the integration coincides with the derivation ∂_θ. Therefore

$$\int d\theta\ \theta^{2k+1} = \theta^{2k}. \tag{3.17}$$

There is an extra reason motivating (3.17) as the correct prescription for the integration. The even powers of θ are bosonic (even) elements which can be expressed in terms of the mass M, which is expected to play a physical role. We recall that $\theta^{2k} = \frac{\epsilon^k}{M^k}$. The (3.17) prescription allows to perform the substitution $\theta^{2k+1} = \frac{\epsilon^k}{M^k}\theta$ and treat $\frac{\epsilon^k}{M^k}$ as an ordinary bosonic parameter, unaffected by the odd integration.

With the above (3.16) and (3.17) prescriptions, the derivation and integration over an odd Clifford variable are *formally* similar to the Berezin counterparts. All θ-valued fields can be regarded as at most linear in θ, so that the standard Berezin rules for derivation and integration apply. The difference w.r.t. the usual Grassmann case lies in the product of θ-valued fields, since extra contributions arise from the non-vanishing of θ^2. A d-dimensional bosonic θ-valued field Φ can be expressed as

$$\Phi(\theta) = \phi + i\psi\theta. \tag{3.18}$$

In the usual Grassmann case it corresponds to a bosonic component field ϕ of mass dimension $[\phi] = d$, plus its fermionic counterpart ψ of mass-dimension $[\psi] = d + \frac{1}{2}$. In the Clifford case, for $\epsilon = \pm 1$, the bosonic field ϕ is Taylor-expanded in powers of $\frac{1}{M}$:

$$\phi = \sum_{n=0}^{+\infty} \frac{\phi_n}{M^n}. \tag{3.19}$$

Its ϕ_n subcomponents have mass-dimension $[\phi_n] = d + n$. The Grassmann case is recovered by the ϕ_0 subcomponent which survives when taking the $M \to \infty$ limit. The fermionic field ψ is similarly treated.

The extension of the calculus to $N = p + q + r$ odd variables of $Cl(p, q, r)$-type is immediate. In the following we will work with $N = 2$. The two derivatives ∂_{θ_1}, ∂_{θ_2} satisfy $\partial_{\theta_i} 1 = 0$, $\partial_{\theta_i}\theta_j = \delta_{ij}$. The double integration $\int\int d\theta_1 d\theta_2$ is only non-vanishing when applied to the product of θ_1, θ_2 (the even powers of θ_i's are assumed to be replaced in the integrand by the powers in $\frac{1}{M}$, ${\theta_i}^{2k} = \frac{\epsilon_i}{M^k}$, as explained above):

$$\int\int d\theta_1 d\theta_2\ \ \theta_2\theta_1 = 1. \tag{3.20}$$

4 Fermionic covariant derivatives for Clifford-valued superfields

In [21] (see also [22]) the construction of supersymmetric fermionic covariant derivatives for Grassmann variables was discussed. A similar procedure is now adopted to derive supersymmetric fermionic covariant derivatives in the case of odd-Clifford variables. For simplicity let's start discussing the $N = 1$ supersymmetry algebra (2.2). Its supergroup element is given by g, introduced in (2.4). It is convenient, for the moment, to keep explicit the dependence on the λ parameter. The left (right) action of the supersymmetry generator Q on g (Qg and, respectively, gQ) induces the operator Q_L (Q_R), determined in terms of t, θ, λ and their derivatives, s.t.

$$\begin{aligned} Qg &= Q_L g, \\ gQ &= Q_R g, \end{aligned} \tag{4.21}$$

where

$$\begin{aligned} \{Q_L, Q_L\} &= -H, \\ \{Q_L, Q_R\} &= 0, \\ \{Q_R, Q_R\} &= H. \end{aligned} \tag{4.22}$$

Q_L is the covariant fermionic derivative, also denoted as "D", while $Q_R \equiv Q$.

In the Grassmann ($\epsilon = 0$) case, Q_L, Q_R are explicitly given by

$$\begin{aligned} Q_L &= \frac{1}{\lambda}\partial_\theta - i\lambda\theta\frac{\partial}{\partial t}, \\ Q_R &= \frac{1}{\lambda}\partial_\theta + i\lambda\theta\frac{\partial}{\partial t}. \end{aligned} \tag{4.23}$$

The (4.22) algebra is consistently reproduced by setting $\lambda = 1$, allowing in the Grassmann case to deal with an $N = 1$ superspace depending only on the time parameter t and the Grassmann variable θ. The hamiltonian H is expressed as $H = i\frac{\partial}{\partial t}$. We have

$$\begin{aligned} D &= \partial_\theta - i\theta\frac{\partial}{\partial t}, \\ Q &= \partial_\theta + i\theta\frac{\partial}{\partial t}. \end{aligned} \tag{4.24}$$

In the odd-Clifford case (for $\epsilon = \pm 1$), a solution to the (4.21) equations is provided by

$$\begin{aligned} Q_L &= \partial_\theta\partial_\lambda - i\theta\partial_t\int d\lambda + i\frac{\epsilon}{M}\partial_\theta\partial_t\int d\lambda, \\ Q_R &= \partial_\theta\partial_\lambda + i\theta\partial_t\int d\lambda - i\frac{\epsilon}{M}\partial_\theta\partial_t\int d\lambda. \end{aligned} \tag{4.25}$$

When Q_L, Q_R are constrained to be applied to superfields whose dependence on λ is given by $\exp(\lambda)$, then both the λ-derivation ∂_λ and the λ-integration $\int d\lambda$ act as identity. The (4.22) algebra is reproduced by $D \equiv Q_L$, $Q \equiv Q_R$ (with dropped λ-dependence), given by

$$\begin{aligned} D &= \partial_\theta - i\theta\partial_t + i\frac{\epsilon}{M}\partial_\theta\partial_t, \\ Q &= \partial_\theta + i\theta\partial_t - i\frac{\epsilon}{M}\partial_\theta\partial_t. \end{aligned} \tag{4.26}$$

They are, respectively, the fermionic covariant derivative and the supersymmetry generator expressed in terms of an $N = 1$ superspace parametrized by the time t and an odd Clifford variable θ s.t. $\theta^2 = \frac{\epsilon}{M}$. The extra terms (proportional to $\frac{1}{M}$) appearing on the r.h.s. of the odd Clifford case w.r.t. the Grassmann case have a purpose. They compensate for the non-vanishing θ^2 to provide the correct supersymmetry transformations for the component fields of an $N = 1$ superfield. Set a bosonic superfield $\Phi = \phi + i\psi\theta$. Its supersymmetry transformation $\delta_\varepsilon\Phi = \varepsilon Q\Phi$ gives, for its component fields in both Grassmann and odd-Clifford cases,

$$\begin{aligned} \delta_\varepsilon\phi &= -i\varepsilon\psi, \\ \delta_\varepsilon\psi &= \varepsilon\partial_t\phi. \end{aligned} \tag{4.27}$$

The generalization of the above construction to the $N = 2$ case with two odd Clifford variables θ_j (${\theta_j}^2 = \frac{\epsilon_j}{M}$, $j = 1, 2$) is immediate. The two fermionic covariant derivatives D_j and the two supersymmetry generators Q_j are respectively given by

$$\begin{aligned} D_j &= \partial_{\theta_j} - i\theta_j\partial_t + i\frac{\epsilon_j}{M}\partial_{\theta_j}\partial_t, \\ Q_j &= \partial_{\theta_j} + i\theta_j\partial_t - i\frac{\epsilon_j}{M}\partial_{\theta_j}\partial_t. \end{aligned} \tag{4.28}$$

They satisfy the algebra

$$\begin{aligned} \{D_i, D_j\} &= -\delta_{ij}H, \\ \{D_i, Q_j\} &= 0, \\ \{Q_i, Q_j\} &= \delta_{ij}H. \end{aligned} \tag{4.29}$$

5 The $1D$ $N = 1$ and $N = 2$ superfields in the odd Clifford formalism

Let's denote with A_k a set of $N = 1$ superfields expanded in the odd Clifford variable θ. The grading $deg(A_k)$ specifies the bosonic ($deg(A_k) = 0$) or fermionic ($deg(A_k) = 1$) character of A_k.

In the odd Clifford case the ordinary superfield multiplication must be replaced by the (anti)symmetrized $*$-multiplication defined as follows

$$A_1 * A_2 = \frac{1}{2}(A_1A_2 + (-1)^{deg(A_1)deg(A_2)}A_2A_1). \tag{5.30}$$

There are several reasons motivating (5.30) as the correct prescription. We notice at first that the $*$-multiplication preserves the reality condition. If A_1, A_2 are real, then $A_1 * A_2$ is real for $deg(A_1)deg(A_2) = 0$, imaginary for $deg(A_1)deg(A_2) = 1$.

The $*$-multiplication induces the $*$-(anti)commutation relations defined through

$$[A_1, A_2\}_* \;=\; A_1 * A_2 - (-1)^{deg(A_1)deg(A_2)} A_2 * A_1.$$

$[A_1, A_2\}_*$ is always vanishing, guaranteeing that the superfields $*$-(anti)commute.

The $N = 1$ covariant derivative D (4.26) satisfies a graded Leibniz property w.r.t. the $*$-multiplication. Indeed

$$D(A_1 * A_2) \;=\; (DA_1) * A_2 + (-1)^{deg(A_1)} A_1 * (DA_2). \tag{5.31}$$

The $N = 2$ supersymmetry requires the introduction of two, θ_1, θ_2, odd variables. It admits two irreducible representations [23], the real (also denoted as $(1, 2, 1)$, with one auxiliary field) and the $(2, 2)$ chiral representation.

A real bosonic $N = 2$ superfield Φ is given by

$$\Phi \;=\; \phi + i\psi_1\theta_1 + i\psi_2\theta_2 + if\theta_1\theta_2, \tag{5.32}$$

with real bosonic, ϕ and f, component fields of mass-dimension d and $d+1$, respectively (f is the auxiliary field). The real component fermionic fields ψ_1 and ψ_2 have mass-dimension $d + \frac{1}{2}$.

The chiral $(2, 2)$ representation is realized in terms of constrained *complex* superfields. Let Υ denote a complex superfield. In terms of the $N = 2$ covariant derivatives D, $\overline{D}$, given by

$$\begin{aligned} D &= D_1 - iD_2, \\ \overline{D} &= D_1 + iD_2, \end{aligned} \tag{5.33}$$

where D_1,D_2 have been introduced in (4.28), the chirality condition for Υ reads as

$$\overline{D}\Upsilon \;=\; 0 \tag{5.34}$$

(the antichirality condition is obtained by replacing $\overline{D}$ with D).

For $\epsilon_1 = \epsilon_2 = \rho = \pm 1$ (namely, the Euclidean $N = 2$ Clifford supersymmetry described at the end of Section **2**) the bosonic chiral superfield Υ is expressed in terms of its complex component fields φ, ξ as

$$\Upsilon \;=\; \varphi + i\frac{\rho}{M}\dot{\varphi} + \xi\theta - \frac{i}{2}\dot{\varphi}\theta\overline{\theta} \tag{5.35}$$

(here $\theta = \theta_1 + i\theta_2$, $\overline{\theta} = \theta_1 - i\theta_2$, while the dot denotes, as usual, the time-derivative).

The (anti)symmetrized $*$-multiplication is introduced for $N = 2$ superfields as in $N = 1$. However, unlike the $N = 1$ case, the $N = 2$ covariant derivatives D_1, D_2 do not satisfy a graded Leibniz property for $\epsilon = \epsilon_1\epsilon_2 \neq 0$ (see (2.12)). In order to preserve a graded Leibniz property, for $\epsilon \neq 0$, the $*$-multiplication must be modified with an extra term proportional to $\frac{1}{M^2}$. Given two $N=2$ superfields A, B, the $\hat{*}$-multiplication, defined as

$$A\hat{*}B = A*B+\frac{\epsilon}{M^2}\partial_{\theta_1}\partial_{\theta_2}A\cdot\partial_{\theta_1}\partial_{\theta_2}B \tag{5.36}$$

is such that it preserves the graded Leibniz property for D_i, $i=1,2$,

$$D_i(A\hat{*}B) = (D_iA)\hat{*}B+(-1)^{-degA}A\hat{*}D_iB. \tag{5.37}$$

Alternatively, the breaking of the graded Leibniz rule for the $*$-multiplication can be expressed as a non-vanishing $\Delta_i(A,B)$, where

$$\Delta_i(A,B) = D_i(A*B)-(D_iA*B+(-1)^{-deg(A)}A*D_iB). \tag{5.38}$$

For bosonic superfields A, B s.t. $A=\phi_A+i\psi_{1A}\theta_1+i\psi_{2A}\theta_2+if_A\theta_1\theta_2$ and $B=\phi_B+i\psi_{1B}\theta_1+i\psi_{2B}\theta_2+if_B\theta_1\theta_2$, $\Delta_1(A,B)$ is, e.g., given by

$$\Delta_1(A,B) = -i\frac{\epsilon}{M^2}(f_B\dot{\psi}_{2A}+f_A\dot{\psi}_{2B}+(f_A\dot{f}_B+\dot{f}_Af_B)\theta_1). \tag{5.39}$$

6 The Non-anticommutative Supersymmetric Quantum Mechanics in the Clifford approach

The constant supersymmetric kinetic term of the $N=1$ superfield (expanded in the odd Clifford variable θ) $\Phi=\phi+i\psi\theta$ ($\theta^2=\frac{\epsilon}{M}$) of mass-dimension $d=0$ is given by the $N=1$ action

$$S_{N=1} = \frac{i}{2m}\int dt\int d\theta(\dot{\Phi}*D\Phi). \tag{6.40}$$

The $N=1$ derivative D is given in (4.26), while $\int d\theta$ is the odd Clifford integration specified by (3.16) and (3.17). In component fields the kinetic action reads as

$$S_{N=1} = \frac{1}{2m}\int dt(\dot{\phi}^2-i\dot{\psi}\psi) \tag{6.41}$$

and coincides with the $N=1$ constant kinetic action in the Grassmann case.

Let us discuss now the $N=2$ Supersymmetric Quantum Mechanics for the real $(1,2,1)$ superfield $\Phi=\phi+i\psi_1\theta_1+i\psi_2\theta_2+if\theta_1\theta_2$ introduced in (5.32). The two odd Clifford variables θ_i satisfy ${\theta_i}^2=\frac{\epsilon_i}{M}$. The parameter $\epsilon=\epsilon_1\epsilon_2$ has been introduced in (2.12). The $\int d\theta_1 d\theta_2$ integration is defined in (3.20). The covariant derivatives D_1, D_2 are given in (4.28). Concerning the superfields multiplication two options are equally admissible for $\epsilon\neq 0$. Either superfields are multiplied w.r.t. the (anti)symmetrized $*$-multiplication (formula (5.31) applied to $N=2$ superfields), or w.r.t. the modified $\hat{*}$-multiplication (5.36) which guarantees the graded Leibniz rule for D_i's. In both cases the action, whose integrand is written in terms of superfields and covariant derivatives, is manifestly $N=2$ supersymmetric.

As it happens for deformed Moyal products, due to the property of the $\int d\theta_1 d\theta_2$ integral, bilinear combinations with $*$-multiplication or $\hat{*}$-multiplication produce the same results as in the ordinary Grassmann case (${\theta_1}^2 = {\theta_2}^2 = 0$). The effects of the odd Clifford variables can only be detected for k-linear products with $k \geq 3$ (trilinear terms and beyond).

The $N = 2$ free kinetic action of the real superfield Φ can be written as

$$S_{N=2,kin.} = \frac{1}{2m} \int dt \int d\theta_1 d\theta_2 (D_1\Phi * D_2\Phi). \tag{6.42}$$

It reads, in component fields,

$$S_{N=2,kin.} = \frac{1}{2m} \int dt (\dot{\phi}^2 + f^2 - i\dot{\psi}_1\psi_1 - i\dot{\psi}_2\psi_2). \tag{6.43}$$

The general $N = 2$ action is

$$S_{N=2} = S_{N=2,kin.} + S_{N=2,pot.}, \tag{6.44}$$

where $S_{N=2,pot.}$ is the potential term.

For the $N = 2$ harmonic oscillator the potential term is quadratic in Φ,

$$S_{N=2,pot.} = i\frac{\omega}{2} \int dt \int d\theta_1 d\theta_2 (\Phi * \Phi), \tag{6.45}$$

with ω an adimensional constant.

It is required at least a trilinear potential to spot the difference between the Grassmann and the odd Clifford realization of the $N = 2$ supersymmetry. The most general trilinear potential can be written either as

$$S_{N=2,pot.} = i \int dt \int d\theta_1 d\theta_2 (c_1 \Phi * \Phi * \Phi + c_2 \Phi * \Phi + c_3 \Phi), \tag{6.46}$$

or

$$\hat{S}_{N=2,pot.} = i \int dt \int d\theta_1 d\theta_2 (c_1 \Phi \hat{*} \Phi \hat{*} \Phi + c_2 \Phi \hat{*} \Phi + c_3 \Phi). \tag{6.47}$$

The two potentials coincide for $\epsilon = 0$.

In (6.46) and (6.47) the coefficients c_i's are real and the i normalizing factor is introduced to ensure the reality of the $N = 2$ potential.

Without loss of generality the c_2 constant can be set equal to zero ($c_2 = 0$) through a shift $\Phi \mapsto \Phi' = \Phi + c$, for a suitable value c. The constant c_1 can be normalized s.t.

$$c_1 = \frac{1}{6}, \tag{6.48}$$

leaving the trilinear potential depending on a single real parameter $\alpha = c_3$.

In the Grassmann case ($\epsilon_1 = \epsilon_2 = 0$), the full $N = 2$ action with the trilinear potential is explicitly given by

$$S_{N=2,Gr.} = \int dt (K - V), \tag{6.49}$$

where K is the kinetic term

$$K = \frac{1}{2m}(\dot{\phi}^2 - i\dot{\psi}_1\psi_1 - i\dot{\psi}_2\psi_2), \tag{6.50}$$

and V is the potential

$$V = -(\frac{f^2}{2m} - i\phi\psi_1\psi_2 + \frac{1}{2}\phi^2 f + \alpha f). \tag{6.51}$$

This result is reproduced in the $\epsilon = 0$ odd Clifford case (6.46) and, no matter which is the value of ϵ, in the (6.47) case. For the trilinear potential we are guaranteed that, ensuring the graded Leibniz property for the covariant fermionic derivatives, the resulting odd Clifford action coincides in components with the ordinary component fields action for Grassmann supersymmetry. This equivalence is preserved for more general potentials and more general theories. On the other hand, genuine odd Clifford deformations of the ordinary Grassmann supersymmetry are recovered when the graded Leibniz property of the fermionic covariant derivatives is broken. In the trilinear example above it corresponds to the choice of the (6.46) $N = 2$ potential for either $\epsilon = -1$ (the "Lorentzian" $N = 2$ odd Clifford supersymmetry) or $\epsilon = 1$ (the "Euclidean" $N = 2$ odd Clifford supersymmetry).

It is worth stressing the result of our analysis, that the Non-Anticommutative Supersymmetry, within the odd Clifford approach, can be understood arising from the breaking of the graded Leibniz property. In the following subsection we discuss in detail the deformed trilinear potentials obtained for $\epsilon = \pm 1$ (the (6.46) prescription is understood) and spot the differences w.r.t. the $\epsilon = 0$ case.

Before starting this analysis let us point out, as a side remark, that the fermionic component fields entering the superfields are assumed to be Grassmann. In principle this assumption can be further relaxed, the fermionic fields can be taken as odd Clifford fields with a non-vanishing square. However, at least for the classes of actions here discussed, no gain is made since the overall effect is reproduced by a shift in the coupling constants entering the potential term.

6.1 The $N = 2$ trilinear potential for $\epsilon = 0, -1, 1$

The trilinear potential, for $\epsilon = \pm 1$, induces an action whose kinetic term K coincides with eq. (6.50), while the potential V is given by

$$V = -\left(\frac{1}{2m}f^2 + \frac{1}{2}\phi^2 f - i\phi\psi_1\psi_2 + \frac{1}{6}\frac{\epsilon}{M^2}f^3 + \alpha f\right). \tag{6.52}$$

In the $\epsilon = 0$ Grassmann supersymmetry, for generic potentials, the equation of motion of the auxiliary field f is a linear equation. For $\epsilon \neq 0$, the equation of motion for f is an algebraic equation (a second order equation for the above example of the trilinear potential), with several branches of solutions. The prescription to correctly pick up a branch is discussed in the following.

For simplicity, and without loss of generality, it is convenient to identify the mass-scale M with the mass-scale m entering (6.42). We can therefore set $M = m$. The main features of the potential can be understood by taking its purely bosonic sector, consistently setting all fermionic fields to zero ($\psi_1 = \psi_2 = 0$). In the $\epsilon = 0$ Grassmann case, solving the equation of motion for f and inserting back into V, we obtain

$$\frac{V}{m} = \frac{1}{8}(\phi^2 + 2\alpha)^2. \tag{6.53}$$

The corresponding theory admits two invariances: supersymmetry and $\mathbf{Z}_2$-invariance $\phi \mapsto -\phi$. We can distinguish three cases according to the value of α. We have
i) $\alpha > 0$: the $\mathbf{Z}_2$-invariance is exact, while the supersymmetry is spontaneously broken,
ii) $\alpha = 0$: both the $\mathbf{Z}_2$-invariance and the supersymmetry are exact and, finally,
iii) $\alpha < 0$: the supersymmetry is exact, while the $\mathbf{Z}_2$-invariance is spontaneously broken (the "mexican hat"-shape potential).

This analysis can be repeated for $\epsilon = \pm 1$. We obtain the following equation of motion for the auxiliary field f:

$$f_\pm = m(-\epsilon \pm \sqrt{1 - \epsilon(2\alpha + \phi^2)}). \tag{6.54}$$

By specializing to $\epsilon = -1$ (the "Lorentzian" case) and setting

$$x = \sqrt{1 + 2\alpha + \phi^2}, \tag{6.55}$$

we obtain two branches for the potential V:

$$\frac{V_\pm}{m} = \pm\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{6}. \tag{6.56}$$

For $\alpha \geq -\frac{1}{2}$, x is always real. The branches have to be chosen s.t. V is bounded below. Therefore

$$\frac{V}{m} = \frac{1}{3}|x^3| - \frac{1}{2}x^2 + \frac{1}{6}. \tag{6.57}$$

Three cases have to be distinguished according to the value $\alpha \geq -\frac{1}{2}$. We have
i) $\alpha > 0$: the $\mathbf{Z}_2$-invariance $\phi \mapsto -\phi$ is exact, while the supersymmetry is spontaneously broken,
ii) $\alpha = 0$: both the $\mathbf{Z}_2$-invariance and the supersymmetry are exact and,
iii) $-\frac{1}{2} \leq \alpha < 0$: the supersymmetry is exact, while the $\mathbf{Z}_2$ invariance is spontaneously broken (this case corresponds to a deformed version of the "mexican hat" potential).

In the three cases above, x belongs to the real axis. On the other hand x is constrained to satisfy $|x| \geq \sqrt{1 + 2\alpha}$ (the whole real axis is recovered for the special value $\alpha = -\frac{1}{2}$).

In the Lorentzian $\epsilon = -1$ case, for $\alpha \geq -\frac{1}{2}$, we obtained real potentials which are deformations of the "Grassmann" potential (6.53). On the other hand, the reality condition (for the classical theory, the hermiticity condition is understood for its quantum version) for the $N = 2$ odd Clifford action written in terms of the $N = 2$ superfield requires α to be an unconstrained real parameter. In particular, the values $\alpha < -\frac{1}{2}$ are allowed. In the Lorentzian case, such values correspond to a potential expressed in terms of x, where now x takes value on the whole real axis and on the part of the imaginary axis constrained to $|x| \leq \sqrt{-2\alpha - 1}$.

In the Euclidean $\epsilon = 1$ case, the two branches of the potential are still furnished by equation (6.56). On the other hand, the x variable is now expressed in terms of the real field ϕ as

$$x = \sqrt{1 - 2\alpha - \phi^2}. \tag{6.58}$$

In the Euclidean odd Clifford supersymmetry the x variable always takes some of its values on (part of) the imaginary axis. We can indeed distinguish three separate cases according to the value of the α parameter. We have
i) for $\alpha > \frac{1}{2}$, x takes values on the part of the imaginary axis satisfying the constraint $|x| \geq \sqrt{2\alpha - 1}$;
ii) for $\alpha = \frac{1}{2}$, x takes value on the whole imaginary axis;
iii) for $\alpha < \frac{1}{2}$, x takes value on the whole imaginary axis *and* the part of the real axis satisfying the constraint $|x| \leq \sqrt{1 - 2\alpha}$.

6.2 On $N = 2$ NAC supersymmetry and $\mathcal{PT}$-hamiltonians

Let us specialize now our discussion to the Euclidean-deformed $\alpha = \frac{1}{2}$ case. For this special choice of α, we have $x = i\phi$, s.t. the purely bosonic effective action S for ϕ is given by

$$S = \int dt \left(\frac{1}{2}\dot{\phi}^2 + \frac{i}{3}\phi^3 - \frac{1}{2}\phi^2 - \frac{1}{6} \right) \tag{6.59}$$

(we set $m = 1$ for simplicity).

This action induces a Bender-Boettcher [1, 24, 25] $\mathcal{PT}$-symmetric hamiltonian H (with $p = \dot{\phi}$)

$$H = \frac{1}{2}p^2 - \frac{i}{3}\phi^3 + \frac{1}{2}\phi^2 + \frac{1}{6}, \tag{6.60}$$

invariant under the coupled transformations (see [24])

$$\begin{aligned} \mathcal{P} &: \quad \phi \mapsto -\phi, \quad p \mapsto -p, \\ \mathcal{T} &: \quad \phi \mapsto \phi, \quad\quad p \mapsto -p, \quad i \mapsto -i. \end{aligned} \tag{6.61}$$

It is worth stressing the fact that our original odd-Clifford $N = 2$ supersymmetric action for ϕ, ψ_1, ψ_2, f (no matter which Clifford deformation and which real value of the α parameter are taken) satifies the reality condition. It's only after solving the equation of motion for the auxiliary field f that the imaginary unit i appears (for the above-discussed cases) in the reduced action. What we succeeded here is to directly link a $\mathcal{PT}$-symmetric hamiltonian with a Non-anticommutative $N = 2$ supersymmetric quantum mechanical system.

For $\alpha \neq \frac{1}{2}$, in terms of the x variable, we get an action with a non-constant kinetic term. The explicit investigation of the properties of these actions will be left for the future.

We conclude this Section mentioning that the trilinear superpotential has been explicitly discussed here for its simplicity. The $N = 2$ odd-Clifford framework for the real $(1, 2, 1)$ multiplet allows the construction of actions for a general class of superpotentials, whose properties can be analyzed within the scheme here outlined.

7 Conclusions

Some of the issues of the Non-anticommutative supersymmetry based on supergroups with odd-Clifford variables deserve further comments.

To our knowledge, the first paper mentioning a connection between Non-anticommutative supersymmetry and $\mathcal{PT}$-symmetric (pseudohermitian) hamiltonians is [26]. In that work, a model introduced in [13] was investigated in detail. The pseudohermitian property of the hamiltonian (called "cryptoreality" in [26]) was discussed in terms of the [27] conjugation transformation $\widetilde{H} = e^R H e^{-R}$ relating the pseudohermitian hamiltonian H to its self-adjoint $\widetilde{H}$ counterpart. In [26] it was further pointed out that such "cryptoreal" hamiltonians, with real spectrum and a unitary evolution operator, could define a consistent supersymmetric Non-anticommutative theory in a Minkowski space-time. In the [27] approach, the key issue to the reality of the spectrum of the pseudohermitian hamiltonians is the existence of the conjugation transformation, rather than the presence of a $\mathcal{PT}$-symmetry. On the other hand, as we have seen, our $N = 2$ odd-Clifford framework to Non-anticommutative supersymmetry provides in a very natural way pseudohermitian hamiltonians. Essentially, the complexity of the hamiltonian is "artificially induced" by solving the equation of motion of the auxiliary field. The action, written in terms of the whole set of $N = 2$ component fields, is real. It is therefore quite natural to pose the question whether the pseudohermitian property of a generic hamiltonian could be recovered from the existence of an underlying extended non-anticommutative supersymmetry. Due to the growing importance of the investigations on pseudohermitian hamiltonians, this issue deserves a careful investigation.

Concerning the violation of the graded Leibniz property for covariant fermionic derivatives based on odd-Clifford supersymmetries, some works, discussing related results, should be signaled [28, 29, 30]. In these works the non-anticommutative supersymmetry is formulated as a Drinfeld twist deformation of the Hopf algebra structure of (a given) supersymmetry algebra. The twist deformation implies, in particular, a deformed coproduct Δ for the fermionic generators Q_i of the superalgebra, s.t. $\Delta(Q_i) = Q_i \otimes \mathbf{1} + \mathbf{1} \otimes Q_i + (\ldots)$, where $(\ldots)$ denotes the extra terms arising from the deformation. In [31] it was discussed a physical interpretation of the coproduct in the construction of tensored multiparticle states. Let g be a bosonic Lie algebra generator associated to, let's say, a hamiltonian H, the undeformed coproduct $\Delta_0(g) = g \otimes \mathbf{1} + \mathbf{1} \otimes g$ is interpreted, e.g., as the addition of energy for a two-particle state ($E_{1+2} = E_1 + E_2$). The results of [28, 29, 30] admit the physical interpretation that the supersymmetry transformation δ_ϵ acting on the product of two (let's say bosonic) superfields Φ_1, Φ_2 is such that, in the deformed case, $\delta_\epsilon(\Phi_1\Phi_2) \neq (\delta_\epsilon\Phi_1)\Phi_2 + \Phi_1(\delta_\epsilon\Phi_2)$. This corresponds to our finding, see equation (5.39), with the bosonic fields multiplied by the $*$-product (the fermionic generators Q_j are recovered from the covariant derivatives D_j after replacing the imaginary unit $i \mapsto -i$, see (4.28)).

In this work we have explicitly discussed supergroups based on odd Clifford variables associated to the one-dimensional N-extended superalgebra. The generalization of the present construction to superPoincaré algebras in higher dimensions is straightforward. The odd generators are spinors and carry spinorial indices. To preserve the Lorentz covariance, in the real case, the odd cordinates θ_α must satisfy anticommutation relations such as $\{\theta_\alpha, \theta_\beta\} = C_{\alpha\beta}$, where $C_{\alpha\beta}$ is a constant charge conjugation matrix, symmetric in the $\alpha \leftrightarrow \beta$ exchange. This construction is only possible for spacetimes, see [32], admitting a symmetric charge conjugation matrix. Alternatively, for complex, Dirac, odd coordinates, non-vanishing constant anticommutation relations can be imposed in the presence of a hermitian A matrix discussed in [32]. The Weyl projection, when needed, can also be accommodated in this framework. The extension of the Berezin calculus to these cases follows from the rules of the odd-Clifford calculus here discussed. The fermionic covariant derivatives can be similarly obtained.

The present results can find several possible applications. Quite naturally, the $N = 2$ odd-Clifford one-dimensional supersymmetry can be discussed in the context of the deformation of the non-relativistic $N = 2$ supersymmetric integrable systems in $1 + 1$-dimensions (such as the $N = 2$ KdVs and analogous $N = 2$ KP-reduced hierarchies). Furthermore, the analysis of two-dimensional superconformal models is particularly interesting in order to understand the role played by the mass-scale M entering the Clifford relations for odd variables. It is also quite natural to extend the present investigation to four or higher-dimensional supersymmetric field theories.

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