Positivity and Integrability

(Mathematical Physics at the FU-Berlin)

Dedicated to Michael Karowski and Robert Schrader on the occasion of their 65th birthday

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Abstract

Based on past contributions by Robert Schrader to the connection between (higher dimensional) Euclidean and real time QFT and by Michael Karowski to the non-perturbative bootstrap-formfactor approach for factorizing two-dimensional QFTs I review the problem of existence of interacting quantum field theory and present recent ideas and results on rigorous constructions as well as new ideas on an operator-algebraic setting for chiral Euclideanization and temperature duality. The main result of chiral Euclideanization may be compressed into the slogan: *SL(2,Z)-modular group properties of modular forms in thermal chiral QFTs result from the (Tomita-Takesaki) modular theory of operator algebras.*

1 Historical remarks

The title of this essay has been chosen identical to that of a small conference at the FU-Berlin in honor of Michael Karowski and Robert Schrader at the occasion of their sixty-fifth birthday. The history of mathematical physics and quantum field theory at the FU-Berlin, a university which was founded at the beginning of the cold war, is to a good part characterized by "positivity and integrability" [1]. The main point of this paper is to demonstrate that low dimensional QFT, in particular the soluble factorizing models and their chiral short distance limits, relate to the positivity underlying the Euclideanization of low-dimensional theories in a novel and quite subtle way. It turns out that revisiting my colleagues contribution with the hindsight of contemporary progress in advanced QFT is an extraordinary interesting and fruitful task.

Both of my colleagues joined the FU theory group in the first half of the 70s, shortly after I moved there. Robert Schrader arrived after his important contribution [2] to the birth of Euclidean field theory whose proper mathematical formulation he initiated together with Konrad Osterwalder while working at Harvard university under the guidance of Arthur Jaffe; Michael Karowski came from Hamburg where he finished his thesis under Harry Lehmann. Whereas Robert, after his arrival in Berlin, was still in the midst of polishing up the work with he begun with Osterwalder at Harvard [3], Michael was looking for new challenging post-doc problems outside his thesis work. At that time Dashen, Hasslacher and Neveu [4] (DHN) had just published their observations on the conjectured exactness of the quasiclassical particle spectrum of certain 2-dimensional models. There were some theoretical indications [5] and numerical checks [6] pointing to a purely elastic S-matrix in those apparently integrable models which were strongly suggestive of an explanation in the (at that time already discredited) S-matrix bootstrap setting, but now within a more special context of factorizing elastic S-matrices.

It was already clear at that time that such a structural property is only consistent in low spacetime dimensions. In the hands of Michael Karowski together with a group of enthusiastic collaborators and graduate students (B. Berg, H-J. Thun, T.T. Truong, P. Weisz), some of who I had the privilege to attract after my return from the US (University of Illinois, IAS, University of Pittsburg) to Germany, the S-matrix principles based on crossing, analyticity and unitarity behind these (at that time still experimental mathematical) observations were adapted to two-dimensional purely elastic 2-particle scattering. The findings were published in a joint paper [7] which together with a second paper on the subject of formfactors [8] associated with those factorizing bootstrap S-matrices became the analytic basis for systematic model constructions of quantum field theories based on the bootstrap-formfactor program. The aspect of integrability of these models was verified by constructing a complete set of conserved currents. During this time Robert exploited the new Euclidean framework in order to obtain a constructive control of models. For this purpose he had to use a more restrictive sufficient criterion which limited the class of models to those whose short distance behavior is close to that of free fields. Which turned out to be possible in 2-dim. QFT.

These developments took place in a city which was the most eastern outpost of the western world and for this reason played the role of a show-window of free market capitalism and liberty against the planned economy of the eastern block which, together with its geographic isolation, contributed to its at that time well-maintained infrastructure and high quality of life in with, unlike the present situation poverty and social deprivation were virtually unknown. The relative isolation of the city was mollified by a lush funding for visitors.

From those early papers of Michael Karowski and co-workers it became clear that some of the far out speculative conjectures of the Californian (Chew-Stapp...) S-matix bootstrap ideas on uniqueness [1] were without foundation; to the contrary, far from being a unique characterization of a a theory of everything (TOE), the two-dimensional scheme of factorizing S-matrices led to a rich classification of two-dimensional QFT which contained besides the mentioned DHN models many others of physical interest. These new non-perturbative methods attracted a lot of attention; the classification of factorizing S-matrices and the construction of associated integrable models of QFT has remained a fascinating area of QFT ever since. The classification according to factorizing S-matrices which is totally autonomous within QT turns out out to give many more models than the Lagrangian approach which is based on the quantization parallelism to classical field theory.

Through my scientific contacts with Jorge Andre Swieca in Brazil (we both were research associates under Rudolf Haag at the university of Illinois) this line of research took roots at the USP in Sao Paulo and other Brazilian universities and via students of Swieca and Koeberle it led to the formation of a whole group of researchers (E. Abdalla, F. Alcaraz, V. Kurak, E. Marino,..) and also influenced others who nowadays are important members of the Brazilian theoretical physics community.

Besides these two groups which vigorously pursued these new ideas about an S-matrix based construction of low dimensional field theories, there was another independent line with similar aims but stronger emphasis on algebraic structures; this was pursued at the Landau Institute in Moscow (the Zamolodchikov brothers and others [10]) as well as at the Steklov Institute in St. Petersburg (L. Faddeev, F. Smirnov and others [11][12]). There were many scientific exchanges with the people from the Steklov institute.

Unfortunately the FU work (this applies in some lesser degree also to Schrader's contributions) did not generate the interest within the quantum field theoretical establishment in Germany (Lehmann[2],Symanzik, Zimmermann and Haag) which it deserved, and without this support this became an uphill struggle for the FU group. Several of the highly gifted members of the young FU research group had to end their academic careers; when the recognition of their achievements outside of Germany had a positive feedback within it was already too late[3].

After the fall of the Berlin wall the situation with respect to fundamental QFT research in Berlin worsened. The physics department of the Humboldt University was restructured solely by "Wessies"

[1] The S-matrix bootstrap at that time was propagandized as a new theory of everything (apart from gravity). As we all know this was neither the first (it was preceded by Heisenberg's "nonlinear spinor theory") neither the last time. But whereas the old attempts ended in a natural death, the more modern versions are still (not unlike the legendary flying dutchman) circeling over our heads in search of a physical landing place.

[2] There is some irony in the fact that Lehmann at the end of his life became actively interested in low dimensional theories. But at that time his influence on particle physics in Germany was already declining.

[3] One outstanding younger member of the group was H. J. Thun whose joint work with Sidney Coleman [9] on the nature of higher poles in the S-matrix became a standard reference. Despite his impressive start there was no place for his academic carrier in Germany.

(to use the colloquial Berlin slang of those days which has survived up to the present); the home-grown community in QFT had no say and was not asked for advice. Considering the importance of historical continuity in particle theory and the intellectual damage caused by political interference, it is not surprising that the consequences of those negative influences have left their mark and will become even more evident after two of the last FU-QFT innovators go into retirement. Outside attempts to fill fundamental quantum field theoretical research with the passing flow of theoretical fashions did not work.

Notwithstanding these introductory historical remarks, this article is not primarily intended as an essay of past achievements of two of my colleagues, nor about the history of QFT and the ups and downs of mathematical physics in Berlin. I rather intend to demonstrate the relevance of the legacy of these ideas within the actual research. In fact low dimensional QFTs (the favorite research topic of Michael Karowski) require to extend the Euclideanization beyond the Osterwalder-Schrader setting. In particular chiral QFT's permit a new rotational Euclideanization which will be the main new topic of this paper (treated in section 4).

In order to be able to do this, I first have to recall some of the pre-electronic conceptual advances which got lost or failed to get passed in the proper way to the younger generation. A good illustration concerning this matter (of such a loss of knowledge as a result of oversimplifications and distortions caused by globalized fashions) is the fate suffered by the *Euclidean method.* The work of Osterwalder and Schrader and prior contributors as Nelson, Symanzik and Guerra started with a very subtle and powerful problematization of what is behind the so-called Wick rotation. These days I sometimes find myself attending talks in which the speaker did not bother to explain whether he is in the setting of real time QFT or in the Euclidean setting; a related question during such a talk usually raises the speakers eyebrows because to him (in his intellectual state of innocence) it is evident that one can pass from one to the other without being bogged down by the need of justification. The rapid rate in which fundamental knowledge gets lost or substituted by universal phrases is one of the symptoms of a deep crisis in particle physics. I think that this crisis cannot be overcome without the restitution of some of the original depth of these notions.

Indeed the Euclidean theory associated with certain families of real time QFTs is a subject whose subtle and restrictive nature has been lost in many contemporary publications as a result of the "banalization" of the Wick rotation (for some pertinent critical remarks see [13]). The mere presence of analyticity linking real with imaginary (Euclidean) time without checking the validity of the subtle reflection positivity (which is necessary to derive the real time spacelike commutativity as well as the Hilbert space structure) is not of much physical interest in particle physics.

In times of lack of guidance from experiments the most reasonable strategy is to press ahead with the intrinsic logic of the existing framework, using the strong guidance of the past principles and concepts rather than paying too much attention to the formalism. In a previous particle physics crisis, namely that of the ultraviolet divergencies of QFT, it was precisely this attitude and not the many wild speculations in the decade before renormalization theory, which finally led to the amazing progress; in fact renormalization theory was probably the most *conservative affirmation of the underlying causality and spectral principles* of Jordan's "Quantelung der Wellenfelder". What was however radically different was their new mathematical and conceptual implementation.

Although this impressive progress made QFT what it is today, namely the most successful physical theory of all times, it is still suffering from one defect which sets it apart from any other area of theoretical physics. Whereas in other areas the availability of rigorous model illustrations left no room for doubts about the consistency of the underlying physical principles and an axiomatization only served as a kind of esthetical show piece which condenses the mathematical description, the aim of presenting QFT into an axiomatic setting is very different indeed. One reason was that the perturbative constructions (unlike say in mechanics, astronomy and quantum mechanics) did not come with mathematical assertions concerning their convergence and estimates of errors; to the contrary, it is by now well-known that the perturbative series for correlation functions are divergent in all realistic models. Indeed, the best one can hope for in any of the various perturbative settings is asymptotic convergence (which unfortunately is a property which does not reveal anything about the mathematical existence).

This led the birth of an axiomatic framework[4] which was followed by "constructive QFT". Measured

[4]This terminology has often been misunderstood. It has a completely different connotation than say axiomatics of mechanics or thermodynamics, since it results from the realization that it is much harder to do a credible computation for

in terms of the complexity of the problem, the results and the methods by which they were obtained are impressive [26][20]. There was also a certain amount of elegance which clearly came from a very clever use of Euclideanization and/or algebraic properties. Conceptually these methods followed closely the physical ideas underlying renormalized perturbation theory. As in the perturbative approach the main objects to be controlled are correlation functions of pointlike fields, with free fields and their Fock space still playing an important auxiliary role. An important technical step was to establish a measure-theoretical interpretation of the interaction-polynomial to the action relative to the free field measure. As a result of methodical limitations it was virtually impossible to go beyond the very restrictive super-renormalizability requirement and to incorporate real life models (as e.g. the Standard Model). For reasons of a certain imbalance between an unwieldy mathematical formalism and the few (and mostly anyhow expected) results besides the control of existence, the *constructive approach* did not enter standard textbooks but rather remained in the form of reviews and monographs; expert cynics (in particular H. Lehmann and R. Jost) even sometimes referred to it as "destructive QFT". Most practitioners of QFT do not mention this problem or raise their students with the palliative advice that since QFT is anyhow not credible for very short distances, the question of mathematical existence problem is a somewhat academic problem; unfortunately without giving the slightest hint how in physics (i.e. outside of politics) problems can be solved by enlarging them. Recently there has been some renewed interest in a variant of this method which is based on the hope that some progress on the functional analytic control [14] of time-dependent Hamiltonian problems may extend the mathematical range (of a more modern algebraic formulation) of the Bogoliubov S-operator approach beyond what had been already achieved in [15].

For a number of years I have entertained the idea that one is struggling here against a birth-defect of QFT whose effects can be only removed by a some radical conceptional engineering. I am referring here to the that *classical parallelism* called Lagrangian quantization by which Pascual Jordan found the "Quantelung der Wellenfelder"[5]. What is most amazing is the fact that only two years after his discovery he apparently became worried about this kind of quantization [17] not being really intrinsic to quantum physics (shortly after computing the Jordan-Pauli commutator function as a consequence of which he also could have been already aware that pointlike quantum fields as obtained from Lagrangian quantization, besides lacking intrinsicness, are rather singular objects). As the main speaker at an international conference [17] he pleaded to look for a new access to QFT which avoids such "classical crutches" but without proposing a way to implement this idea. Apart from Wigner's solitary representation-theoretical approach to relativistic particles in 1939 (which only in the late 50's became known to a broader public through the work of Wightman and Haag), the first entirely intrinsic setting which avoided field coordinatizations (non-intrinsic generators of spacetime-indexed algebras) was the algebraic approach by Haag [18] (with some mathematical ideas concerning operator algebras taken from Irving Segal), later known as the Haag-Kastler approach. But in some sense the baby was thrown out with the bath water because the conceptual precision and the panoramic vision was not matched by any calculational implementation.

This was at least the situation before the modular theory of operators algebras was incorporated in order to place local quantum physics on a more constructive course. At this point Karowski's contribution (and more generally the meanwhile extensive literature on the classification and construction of factorizing model) enters; these explicit and nontrivial model constructions represent presently a valuable theoretical laboratory for the new constructive ideas based on modular theory; they have led to new results concerning the proof of existence of certain strictly renormalizable models to which we will return later. These new developments suggest that it rather improbable to win the Clay prize for the solution of gauge theory as an isolated subject without revolutionizing the whole of local quantum physics.

The article is organized as follows. The next section reviews some of the ideas which led to the Osterwalder-Schrader results and their role in strengthening the constructive approach to QFT. In the same section I also sketch the more recent framework of modular localization. Unlike the Bargmann-Hall-Wightman analyticity of correlation functions of pointlike fields (operator-valued distributions) which

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a concrete model than it is to understand joint structural properties of a whole class of models (as long as the question of existence is ignored).

[5] His peer Max Born with Heisenberg's support limited Jordan's obsession with quantizing also structures beyond mechanics by banning his calculations to the last section of the joint work. Darrigol [16] reports that when Jordan received Schroedinger's results he already had what was later called a second quantized version. For his radical viewpoint it was apparently sufficient that a structure could be fitted into a classical framework and not whether it was actually part of classical physics.

constitute an important aspect of the Euclidean approach, the analyticity aspects of modular localization do not originate from covariance and locality operators of individual Wightman fields but rather derive from the domain properties of a certain unbounded operator S which characterizes localization aspects of operator subalgebra. This Tomita involution S is the corner stone of the Tomita-Takesaki modular theory of operator algebras. The third section sketches the notion of integrability in QFT which is synonymous with factorization of the S-matrix and relates this property to a special aspect of vacuum polarization. Closely related to those factorizing models are the conformal invariant chiral QFTs whose modular Euclideanization constitutes a nontrivial analog of the O-S Euclideanization. This and the ensuing *temperature duality*, an analog of the Nelson-Symanzik duality, will be the main topic of section 4. In the concluding section I list some related open problems.

2 Positivity and Euclideanization

After the rehabilitation of the divergence-ridden QFT in the form of the postwar renormalized perturbative quantum electrodynamics, and after the subsequent significant conceptual advances in the understanding of the particle-field relation through scattering theory [19][20], the idea gained ground that the importance of QFT for particle physics can be further boosted by understanding more about its model-independent structural properties beyond perturbation theory. The first such general setting was that by Wightman [21]. In harmony with the increasing importance of analytic properties which entered the setting of scattering theory through the particle physics adaptation of the optical Kramers-Kronig dispersion relations, special emphasis was placed on the study of analytically continued correlation functions. These investigations started from the *positive energy momentum* spectrum (expressing the presence of a stable ground state, the vacuum) and the *Lorentz-covariant transformation* properties as well as *locality* (in the form of (anti)commutators vanishing for spacelike distances). This, led to an extension of the original tube domain resulting from the positive energy-momentum spectrum to the Bargmann-Hall-Wightman region of analyticity and its extension by locality. The analytically continued correlation functions turned out to be meromorphic and uni-valued in the resulting "permuted extended tube" region [21].

Already before these mathematical results the Euclidean region, which resulted by letting the time component be pure imaginary, attracted the attention of Schwinger since in his formalism it led to considerable computational simplification. The next step was taken by Symanzik [22] who observed from his functional integral manipulations that the analytic continuation to imaginary times (Wick rotation) for bosonic theories highlighted a positivity property which was well-known from the continuous setting of statistical mechanics. Nelson [23] succeeded to remove the somewhat formal aspects and achieved a perfect placement into a mathematically rigorous setting of an autonomous stochastic Euclidean field theory (EFT) in which the most important conceptual structure was the Markov property. Guerra [24] applied this new setting to control the vacuum energy coming from certain polynomial interactions in bosonic 2-dimensional QFT and emphasized its usefulness in establishing the existence of the thermodynamic limit. This Euclidean field formalism was limited to fields with a canonical short distance behavior; but even in this limited setting *composite fields* with worse short distance behavior permit no natural incorporation into this probabilistic setting. The limitation was intimately related to the property of Nelson-Symanzik positivity, which basically is the kind of positivity which Schwinger functions should obey if they were to describe a (continuous) thermal classical field mechanics.

The breakthrough for the understanding of Euclideanization in QFT beyond this restriction was achieved in the work of Osterwalder and Schrader [2]. They had to sacrifice the stochastic interpretation of EFT which was then substituted by a certain *reflection positivity condition* as well a (physically less amenable) growth condition on n-point Schwinger functions for $n \rightarrow \infty$. If one is less ambitious and only asks for a sufficient condition on Schwinger functions, one obtains a formulation which turns out to be quite useful for the constructive control for certain superrenormalizable low-dimensional QFTs.

One reason why the constructive control of higher dimensional QFTs presents a serious obstacle is that the reflection positivity does not harmonize well with the idea of (Euclidean invariant) ultraviolet cutoffs. For this reason one encounters difficulties with ultraviolet cutoffs in a functional Euclidean integral setting; in general one does not even know whether such a cutoff is consistent with the quantum theory setting i.e. whether one can pass to an operator setting before having removed the Euclidean cutoff.; not to mention all the other requirements as e.g. cluster properties, asymptotic scattering limits etc. which are indispensable in order to uphold the physical interpretation of a theory. Of course cutoff versions are strictly auxiliary constructs and as such may violate such properties, but then the problem of securing the recovery of the local quantum physical properties in the cutoff-removal becomes a hairy problem. On the other hand it is possible to formulate the process of O-S Euclideanization by starting from the more algebraic setting of AQFT which avoids the use of (necessarily singular) point-like field coordinatizations; in this case one has problems to specify concrete interactions. A formulation of the algebraic approach in the lattice setting for which the concepts and their mathematical implementation of the O-S Euclideanization allow a very simple presentation of high pedagogical value can be found in [25].

One property within the Nelson-Symanzik setting which turned out to be extremely useful in controlling the removal of infrared regularizations (thermodynamic limit) is the Euclidean spacetime duality. This *Nelson-Symanzik duality* is suggested by the formal use of the Feynman-Kac Euclidean functional integral representation. Let us consider thermal correlation functions at inverse temperature β for a 2-dimensional enclosed in a periodic box (rather interval). The KMS condition for the correlation functions at imaginary times (and equal real components) reduces to a β-periodicity property. Since the Euclidean functional representation treats space and time on equal footing, the duality under a change of x and t_E accompanied by an exchange of the box- with the thermal- periodicity is obvious from the F-K representation. A mathematical physics derivation without requiring the validity of F-K can be found in a recent paper [27]. In the last section we will use this property as an analogy of the chiral *temperature duality*. A model-independent systematic adaptation of the O-S Euclideanization to the imaginary time thermal setting can be found in a recent review paper [37].

In the remainder of this section I will recall the modular localization setting for the convenience of the reader. This is a preparatory step for the content of the last section. The salient properties of the modular aspects of QFT can be summarized as follows [28].

- Modular localization is an adaptation of the modular Tomita-Takesaki theory to QFT in the setting of operator algebras (AQFT). The modular properties are not associated to individual local covariant fields but rather to the operator algebra $\mathcal{A}(\mathcal{O})$ which is generated by smeared fields if one limits the test function supports to a fixed spacetime region $\mathcal{O}$. Modular theory [29] is based on the idea that one learns a lot about the properties of operator algebras $\mathcal{A}$ by studying the unbounded antilinear and (as it turns out) closed operator S defined as

$$S A\Omega = A^*\Omega,\ A \in \mathcal{A} \tag{1}$$

where Ω is a cyclic (i.e. $\mathcal{A}\Omega$ is dense in H) and separating (i.e. there is no nontrivial $A \in \mathcal{A}$ which annihilates Ω). Its importance results from the fact that operator algebras (including the localized generators of symmetries) encode not only scattering theory but also the complete content of particle physics [20]. Interesting properties arise from the polar decomposition of S which is traditionally written as

$$S = J\Delta^{\frac{1}{2}} \tag{2}$$

The resulting unbounded positive operator Δ (the "polar" part) generates via its one-parametric unitary group Δ^{it} a modular automorphism group σ_t of $\mathcal{A}$ and the "angular" part J, the so-called Tomita conjugation, is an antiunitary involution which maps the operator algebra into its commutant $\mathcal{A}'$

$$\sigma_t(\mathcal{A}) \equiv \Delta^{it}\mathcal{A}\Delta^{-it} \subset \mathcal{A},\ J\mathcal{A}J = \mathcal{A}' \tag{3}$$

where I used a condensed notation with $\mathcal{A}$ as a short hand instead of its individual operators $A \in \mathcal{A}$. The crucial property of the Tomita S which is behind all this algebraic richness is the fact that S is "transparent" in the sense that $domS = ranS = dom\Delta^{\frac{1}{2}}$, $S^2 = 1$ on $domS$. I am not aware of the existence of such unusual unbounded involutive operators outside modular theory (one does not even find them in the mathematical physics literature e.g. in Reed-Simon). This theory begins to unfold its magic power within QFT[6] once one realizes (as was first done by Bisognano and Wichmann [30])

[6] Actually the constructive power of the modular approach only began to unfold after a seminal paper by Borchers [32] which led to a flurry of additional remarks [33][34] and finally gave rise to the theory of modular inclusions and modular intersections [35][36].

that not only any pair $(\mathcal{A}(\mathcal{O}), \Omega = vacuum)$ with a nontrivial spacelike disjoint $\mathcal{O}'$ is "standard" in the sense of modular theory, but even more: for the standard pair $(\mathcal{A}(\mathcal{W}),\Omega)$ with $\mathcal{W}$ a wedge region, the modular group acts as the unique W-preserving Lorentz-boost and the Tomita reflection is (up to a rotation which depends on the choice of W) equal to the physically extremely significant TCP symmetry. Whereas the unitary turns out to be $\Delta^{it}_{\mathcal{W}}$ is part of "kinematics" i.e. uniquely determined once the representation theory of the Poincaré group (the spectrum of particles) is known, the J_W contains profound dynamical information. If we assume that we are in the standard LSZ setting of scattering theory[7] then the J of an interacting theory is connected to its interaction-free asymptotic counterpart J_0 through the scattering matrix

$$J = S_{scat} J_0 \tag{4}$$

i.e. whereas in the interaction-free case the modular data for the wedge algebra are constructed in terms of the relevant representation of the Poincaré group, the presence of interactions enriches the modular theory of wedge algebras through the appearance of S-matrix in J. For models for which the bootstrap construction of their S-matrix can be separated from the construction of their fields (these are the factorizing models of the next section) the knowledge of the modular wedge data can be used for their explicit construction. The guiding ides is that knowing the modular data for the wedge algebra uniquely fixes the modular operators for all the other causally complete region. Although there is no geometro-physical interpretation a la Bisognano-Wichmann for the modular objects of smaller causally closed spacetime regions (spacelike cones, double cones), there is no problem in constructing them through the process of algebraic intersections in terms of wedge algebras (such $\mathcal{O}$ are necessarily causally closed)

$$\mathcal{A}(\mathcal{O}) = \bigcap_{\mathcal{W}\supset\mathcal{O}} \mathcal{A}(\mathcal{W}) \tag{5}$$

The impressive constructive power of this theory already shows up in its application to the Wigner representation theory of positive energy representations of the Poincaré group. The results obtained by combining Wigner's theory with modular localization go beyond the well-known results of Weinberg on three counts:

- The spatial version of the modular localization method associates string-like localized fields with Wigner's enigmatic family of massless infinite spin representations whereas previous attempts [31] already showed that these representations are incompatible with point-like localization.
- For massless finite helicity representations (photons, gravitons,...) only the "field strength" whose scale dimension increases with the helicity are pointlike whereas the "potentials" with would-be dimension one turn out to be string-like[8] i.e. pointlike potentials are incompatible with the Wigner representation theory.
- The structural analysis carried out by Buchholz and Fredenhagen on massive theories with a mass gap suggests strongly that the setting of interactions may be significantly enlarged by permitting interactions to possess a string-like localization structure. If one wants to implement this idea in a perturbative setting one needs massive string-localized free fields. The application of modular localization leads to *scalar string-like localized fields* for arbitrary spins.

There is another important representation theoretical result from modular localization for d=1+2 dimensional QFT which according to my best knowledge cannot be derived by any other method. It is well-known that the (abelian in this case) spin in this case can have anomalous values which activates the representation theory of the universal covering $\tilde{P}(3)$ first studied by Bargmann. Combined with

[7] The LSZ asymptotic convergence of Heisenberg operators towards free (incoming or outgoing) particle operators is guarantied by spacelike locality and the assumption of gaps which separate the one-particle massive state from the continuum [20].

[8] In the strict Heisenberg-Wigner spirit of observables one rejects unphysical ghosts (which formally make potentials covariant and pointlike) even though they are only computational catalyzers in order to obtain observables at the end of the computation.

the modular localization theory one is able to determine the localized subspaces and a "preemptive" one-particle version of a plektonic spin-statistics theorem. In this case the transition from the Wigner representation to the QFT can however not be done in a functorial way since there is an inherent vacuum polarization related with nontrivial braid-group statistics [38].

It is interesting to compare the setting of modular localization with the O-S Euclideanization. The former also leads to analyticity properties and to euclidean aspects but in this case they are not coming from Fourier-transformed support properties and their covariant extension but rather encode domain properties of unbounded operators. The connection with analyticity properties and "Euclideanization" is through the KMS-property of $\mathcal{A}(\mathcal{O})$ expectation values in the state implemented by the vector Ω. The defining equation for S shows that all vectors of the form $A\Omega$ are in $dom\Delta^{\frac{1}{2}}$, which means that these vectors $\Delta^{iz}A\Omega$ are analytic in the strip $-Imz < \frac{1}{2}$.

The most attractive and surprising property of this formalism is the encoding of geometry of localization in domain properties (and a fortiori in analyticity) in the sense $S_{\mathcal{O}_1} \subset S_{\mathcal{O}_2}$ if $\mathcal{O}_1 \subset \mathcal{O}_2$ and $S_{\mathcal{O}_1} \subset S^*_{\mathcal{O}_2}$ if $\mathcal{O}_1 >< \mathcal{O}_2$ [40]. Such an intimate relation between domain (and range) properties of unbounded and geometric localization properties is unique in particle physics and is not met anywhere else in physics (this probably explains why it is not treated in books on mathematical-physics methods as Reed-Simon)

In the present stage of development of the modular formalism does not permit a general classification and constructive control in the presence of interactions. As the Euclidean formalism it is limited to certain low dimensional QFT but for quite different reasons. What is interesting is that the families of low dimensional models covered by the two settings are quite different. Whereas for the Euclidean approach the limitation is the traditional one coming from short distance properties, the present limitation of the modular approach has nothing to do with short distance properties of pointlike fields but rather is tied to the existence of generators of wedge algebras $\mathcal{A}(\mathcal{W})$ with simple physical properties, so called tempered vacuum-polarization-free generators (PFG). Some details will be explained in the next section. It turns out that this requirement is equivalent to the S-matrix being factorizing.

The test of existence of a model (which has been defined in terms of its generators for the wedge-localized algebra) in the modular approach is not related to its good short distance behavior, but rather consists in the nontriviality of algebraic intersections.

In the last section I will present a Euclideanization based on modular localization which shows its analogy to the O-S setting.

3 Tempered PFG, integrable QFT, factorizing models

Modular localization offers a surprising new way to obtain new insight into field theoretic integrability and the classification of factorizing models. It also contains the somewhat hidden link between the Osterwalder-Schrader positivity and its analytic setting and the extremely nontrivial peculiarities of positivity and analyticity in low-dimensional models as conformal chiral theories and factorizing massive models.

As before we assume the existence of isolated one-particle mass-shells which is sufficient for the validity of scattering theory. The starting point is the following definition which then leads to two theorems.

Definition 1 *A vacuum-**p**olarization-**f**ree-**g**enerator (PFG) of a localized algebra $\mathcal{A}(\mathcal{O})$ is a (generally unbounded) operator $G^{\#}$ affiliate to this algebra which applied to the vacuum creates a one-particle state without vacuum polarization admixture*

$$G\Omega = one - particle\ vector \tag{6}$$

$$G^*\Omega = one - particle\ vector \tag{7}$$

It is clear that a (suitably smeared) free field is a PFG for any compactly localized operator algebra $\mathcal{A}(\mathcal{O})$ generated by a free field, but it takes some amount of thinking to see that the inverse also holds[9] i.e. the existence of a PFG for any compact causally complete subwedge region $\mathcal{O}$ implies that G is a

[9] This is the content (apart from the unnecessary requirement of covariance of the generators) of the Jost-Schroer theorem [21].

smeared pointlike covariant free field (and hence that also the superselection-sector generated by G are those of a free field).

On the other hand the (in/out) particle creation/annihilation operators are affiliated to the global algebra. This places the wedge region into the very interesting position of a borderline case; the application of modular theory shows that PFGs in interacting theories do exist in that case i.e. in more intuitive physical terms: *the wedge localization is the best compromise between particles and fields in interacting QFTs*. A closer examination reveals that if one demands that PFGs are "tempered" in the sense that they have domains which are stable under spacetime translations (i.e. domain properties similar to a Wightman theory), the S-matrix is necessarily purely elastic [39]. This in turn reduces the possibilities (excluding "free" models with braid group statistics in d=1+2) to d=1+1 dimensional interacting theories; and in that case one indeed has the rich class of factorizing S-matrices as illustrative examples.

Theorem 2 *[39] Tempered PFGs are only consistent with purely elastic S-matrices, and (excluding statistics beyond Bosons/Fermions), elasticity and non-triviality are only compatible in d=1+1.*

The (assumed) crossing property for formfactors excludes connected elastic 3-particle contributions [10] so that the factorizing models actually are the only ones whose wedge algebras are generated by PFGs. This approach culminates in the recognition that the generators of the Zamolodchikov-Faddeev algebras are actually the Fourier transforms of the tempered wedge-localized PFGs; in this way the computational powerful but hitherto (in the LSZ scattering setting) conceptually somewhat elusive Z-F operator algebra acquires a physical spacetime interpretation. Since there are some still unresolved fine points concerning wedge-localization in the presence of bound states (associated with certain S-matrix poles in the physical rapidity strip), I will for simplicity assume that there are none. For models with a continuous coupling strength (e.g. the Sine-Gordon model) this is achieved by limiting the numerical value of the coupling parameter. Let us further assume that the particle is spinless in the sense of the Lorentz-spin. Then the following theorem holds

Theorem 3 *Let $Z^{\#}(\theta)$ be scalar Z-F operators i.e.*

$$\begin{aligned} Z(\theta)Z^*(\theta') &= S_2(\theta-\theta')Z^*(\theta')Z(\theta) + \delta(\theta-\theta') \\ Z(\theta)Z(\theta') &= S_2(\theta'-\theta)Z(\theta')Z(\theta) \\ \phi(x) &= \frac{1}{\sqrt{2\pi}}\int (e^{ip(\theta)x(\chi)}Z(\theta) + h.c.)d\theta \end{aligned} \tag{8}$$

then the non-local fields generate a wedge-localized algebra $\mathcal{A}(\mathcal{W})$ and the coefficient functions S_2 are the two-particle scattering matrix contributions of a purely elastic factorizing scattering matrix S_{scat}

As already stated in the previous section, the algebras for compact spacetime regions and their pointlike field generators are constructed by forming intersections of wedge algebras. The relevant calculations are very simple in the case of interaction-free fields associated with the various families of positive energy Wigner representations [41][28]. For the case at hand they are slightly more involved, reflecting the fact that although the PFG generators are still on-shell, the creation/annihilation components have a more complicated algebraic structure than the standard ones. There are two strategies to be followed depending on what one wants to achieve.

If the aim is to establish the existence of the model in the algebraic setting, then one must find a structural argument which secures the nontriviality of intersections of wedge algebras associated to causally complete spacetime regions. For the case at hand (for which the double-cone localized algebras can be obtained by the intersection of two opposite wedges) the property of *modular nuclearity* which only refers to a wedge-localized algebra is sufficient to show nontriviality . There are some recent interesting results by Lechner [42].

The underlying physical idea is that the nontriviality is already encoded into the structure of the wedge algebra generators. In particular in d=1+1, a property called *modular nuclearity* of the wedge algebra (referring to the cardinality of phase space degree of freedoms [43]) secures the nontriviality of double

[10] Private communication by Michael Karowski.

cone intersections which is tantamount to the existence of the model in the framework of local quantum physics. Since the proof uses the S-matrix in an essential way it is not surprising that certain properties, which were extremely hard to obtain in the old approach based on Euclideanization (as e.g. the condition of asymptotic completeness), are a quite easy side result of the nontrivial existence arguments.

If on the other hand the aim is to do explicit calculations of observables beyond the S-matrix, then the determination of the formfactor spaces is the right direction to follow. In that case one makes a Glaser-Lehmann-Symanzik-like Ansatz, but instead of expanding the desired localized Heisenberg operator in terms of incoming creation/annihilation operators, ones uses the Z-F operators instead

$$A = \sum \frac{1}{n!} \int_C ... \int_C a_n(\theta_1, ...\theta_n) : Z(\theta_1)...Z(\theta_n) : d\theta_1...d\theta_n \tag{9}$$

Whereas in the GLZ case the coefficient functions are expressible in terms of mass-shell projections of retarded functions, the coefficient functions in (9) are connected multiparticle formfactors

$$\begin{aligned} \langle \Omega \left| A \right| p_n, ..p_1 \rangle^{in} &= a_n(\theta_1, ...\theta_n), \quad \theta_n > \theta_{n-1} > ... > \theta_1 \\ {}^{out}\langle p_1, ..p_l \left| A \right| p_n, ..p_{l+1} \rangle^{in}_{conn} &= a_n(\theta_1 + i\pi, ...\theta_l + i\pi, \theta_{l+1}, ..\theta_n) \end{aligned} \tag{10}$$

which are boundary values of analytic functions in the rapidity variables. If we are interested in operators localized in a double cone $A \in \mathcal{A}(\mathcal{D})$ we should look for the relative commutant $\mathcal{A}(\mathcal{D}) = \mathcal{A}(\mathcal{W}) \cap \mathcal{A}(\mathcal{W}_a)'$ with $\mathcal{D} = \mathcal{W} \cap \mathcal{W}'_a$ and $\mathcal{W}_a$ being the wedge obtained by spatial shifting $\mathcal{W}$ to the right by a. In terms of the above Ansatz this means that the looked for $A's$ should commute with the generators of $\mathcal{A}(\mathcal{W}_a)$ i.e.

$$[A, U(a)\phi(f)U(a)^*] = 0 \tag{11}$$

Since the shifted generators are linear in the Z-F operators and the latter have rather simple bilinear commutation relations, it is possible to solve the recursive relation (the kinematical pole relation) iteratively and characterize the resulting spaces of connected formfactors. Although such recursive formfactor calculations are not a substitute for the existence proof (since bilinear forms (formfactors) need not result from operators), the combination with the previous existence argument would show that the bilinear forms are really particle matrix elements of genuine operators. In the pointlike limit $a \to 0$ the equation characterizes the space of formfactors of pointlike fields. In this case one obtains a basis of this space by invoking the covariance properties of the Lorentz spin. After splitting off a common rather complicated factor shared by all connected formfactors, the remaining freedom is encoded in momentum (rapidity) space polynomial structure which is similar but more complicated than the analogous structure for formfactors of Wick polynomials.

For all QFT which are not factorizing (i.e. in particular for higher dimensional theories) there are no tempered PFGs which generate wedge algebras. So one could either try to find out whether the rather unwieldy non-tempered PFGs (whose existence follows from modular theory [39]) may be used for this purpose, or one could give up the PFG restriction and work with generators in the form of powerseries in the incoming field whose numerical coefficient (together with the S-matrix in terms of in-fields) function must then be iteratively determined; this case would amount to a new kind of modular-based perturbation theory. A scenario for such a construction may look as follows. Starting in zeroth order with generators which are linear in the incoming creation/annihilation operators, one defines first order generators of the commutant (localized in the opposite wedge) by using the perturbative first order S-matrix in $J = J_0 S_{scat}^{(1)}$. This leads to a first order correction for the coefficient functions of first order generators of the opposite wedge. The hope would be that the imbalance in the commuting property with the original generator would then require a second order correction of S as well as a correction in the coefficient function and that this, similar to the iterative Epstein-Glaser approach for pointlike fields could serve as a perturbative analog of the on-shell bootstrap-formfactor program which bypasses correlation functions of singular fields (and the distribution theory of the Epstein-Glaser approach) and leads to a fresh start for a construction program which is also capable to handle the unsolved problem of existence.

In the context of the bootstrap formfactor program for factorizing models one observes an unexpected (and may be even undeserved) simplicity in the analytic dependence of the formfactors on the coupling strength. There is always a region around zero in which the coupling dependence is analytic. According to general structural arguments there is however no reason that the yet unknown correlation functions will inherit this property. This raises the general question: do on-shell observables have better analytic properties in the coupling than off-shell operators?

It is very interesting to compare the constructive control one has on the basis of the Osterwalder-Schrader setting with that for models constructed in modular bootstrap formfactor program. In the first case the restriction comes from short distance properties; in the almost 40 year lasting history of attempts of field theoretic constructions it has not been possible to go beyond superrenormalizable models (mainly $P\phi_2$). On the other hand all the known factorizing models have strictly renormalizable interactions (e.g. the sine-Gordon model interaction is nonpolynomial) and there seems to be no overlap between the requirement of super-renormalizability and factorizability. The weakness of one construction method is the strength of the other. If one could break the limitation set by factorizability as indicated above, then the constructive approach would change in favor of the modular wedge generator approach without the super-renormalizability restriction. One would also expect a clarification about the existence of QFT in more than 4 spacetime dimensions for which the standard renormalizability requirement would negate the existence of any interacting solution.

Factorizing models are very closely related to chiral conformal theories which "live" (in the sense of modular localization) on a (compactified) lightray. On the one hand there is the general relation of a QFT to its scale invariant short distance limit. Many different massive theories have the same critical limit i.e. belong to one *short distance universality class*. If one only looks at factorizing models than Zamolodchikov has presented conditions under which one can invert this relation in a formal setting of a perturbed conformal theory [44]. On the other hand there is a conceptually quite different relation between d=1+1 massive theories to their chiral *holographic projection [45]*. In that relation the algebraic substrate and the Hilbert space in which it is represented remains unchanged and only the spacetime indexing of the algebras is radically changed in a way that cannot be encoded in a simple geometric relation between the chiral fields on the lightray; with other words holographic projection unlike the scaling limit only rearranges the spacetime ordering device of the matter substrate but not the matter substrate itself. But different from the AdS-CFT holography, the *lightfront holography is also a class property* i.e. without enlargement of the Hilbert space there are many ambient theories which are holographic inverses of a QFT on a null-surface. Only if one had the luck to find generators of the holographic projection which are covariant under the ambient Poincaré group, as it is the case with the Z-F generators in factorizing models, the holographic inverse is uniquely fixed. The rather complicated connection between pointlike generators of the ambient algebra and those of its holographic projection prevent an understanding of this relation by a straightforward inspection.

4 A modular analog of O-S and of the Nelson-Symanzik duality

Those cases in which the Schwinger functions associated with the O-S Euclideanization and its thermal generalization (leading to the Nelson-Symanzik duality) admit a stochastic interpretation possess a very intriguing low dimensional analogy in the setting of the modular localization: *modular Euclideanization* and *temperature duality*. This will be explained in the following.

The issue of understanding Euclideanization in chiral theories became particularly pressing after it was realized that Verlinde's observation about chiral partition functions [11] is best understood by incorporating it into a wider quantum field theoretic setting involving angular parametrized thermal n-point correlation functions of observable fields Φ_i in the superselection sector $\mu \equiv [\rho_\mu]$ associated with the localized DHR endomorphism ρ_μ [20]

$$\begin{aligned}\langle\Phi(\tau_1,...\tau_n);\beta\rangle_\mu &:\equiv tr\,|_{H_\mu} e^{-\beta(L_0-\frac{c}{24})}\prod_{k=1}^{n}\left(e^{i2\pi\tau_k L_0}\Phi_k(0)e^{-i2\pi\tau_k L_0}\right).\\ \langle\Phi(\tau_1,..\tau_n);\beta\rangle_{\mu_t} &= \langle\Phi(\tau_n+i\beta,\varphi_1,..\varphi_{n-1});\beta\rangle_{\mu_t}\end{aligned}$$

where the first line defines the angular thermal correlations (with unit periodicity) in terms of a L_0- Gibbs trace at inverse temperature β on observable fields in the representation π_{ρ_α}. Gibbs states are special

[11] Verlinde discovered a deep connection between fusion rules and those modular properties which were first observed by Kac and Peterson in the context of characters of loop groups.

unnormalized KMS states i.e. states whose correlations fulfill the analytic KMS property in the second line. Being defined in terms of traces, they are independent of the concrete choice of the endomorphism ρ_μ and only depend on the equivalence class $\mu = [\rho_\mu]$ (i.e. on the superselected charge and not on its spatial distribution). Their zero point function which is the Gibbs trace of the identity operator, defines the L_0- partition functions. In contrast to the previously used ground states, such thermal correlations are *independent on the particular localization of charges* $loc\rho_\alpha$. This is the result of the unitary invariance of the trace and consequently they only depend on the equivalence class i.e. on the sector $[\rho_\alpha] \equiv \alpha$, which makes them *valuable objects to study the sector structure* (classes of inequivalent representations of the observable algebra). These correlation functions[12] fulfill the following amazing thermal duality relation

$$\left\langle \Phi(i\tau_1,\ldots,i\tau_n);\beta\right\rangle_\mu = \left(\frac{2\pi}{\beta i}\right)^a \sum_\nu S_{\mu\nu} \left\langle \Phi\left(\frac{2\pi}{\beta}\tau_1,\ldots,\frac{2\pi}{\beta}\tau_n\right);\frac{4\pi^2}{\beta}\right\rangle_\nu \qquad (12)$$
$$a = \sum_i dim\Phi_i$$

where the right hand side is a sum over thermal expectation at the inverse temperature $\frac{4\pi^2}{\beta_t}$ (the thermal symmetry point being $\beta = 2\pi$) at real angles which are related to the imaginary values on the left hand side by a scaling factor $\frac{2\pi}{\beta_t}$. The multiplicative scaling factor in front which depends on the scaling dimensions of the fields Φ_i is just the one which one would formally write if the transformation $i\tau \to \frac{2\pi}{\beta_t}\tau$ were an ordinary conformal transformation law (ignoring the fact that such a complex scale transformation is not part of the unitarily implemented conformal symmetry).

The way in which this identity comes about is very interesting but before presenting a proof the reader should notice of the analogy with the thermal version of the previously (second section) presented Nelson-Symanzik duality for massive two-dimensional theories. Since chiral theories are localized on the (compactified) lightray, this analogy consists of an interchange of the real angle with its imaginary scaled version; the stretching factor $\frac{2\pi}{\beta_t}$ together with the inverse temperature on the right hand side corresponds to the interchange of the two periodicities[13] in the N-S duality except that the duality now interchanges the real with the imaginary angular-parametrized lightray component which (apart from the symmetry point $\beta = 2\pi$) requires the presence of this scaling factor. The appearance of the linear combination of all (finite in rational models) superselection sectors weighted with the Verlinde matrix S has no obvious counterpart in the N-S setting. In simple models as e.g. the multi-component abelian current model [45] the proof of the temperature duality relation can be reduced to properties of the Jacobi Θ-functions and the Dedekind eta-function. The Kac-Peterson-Verlinde character relations

$$\chi_\alpha(\tau) = \sum_\beta S_{\alpha\beta}\chi_\beta(-\frac{1}{\tau}),\ \chi_\alpha(\tau) \equiv tr_{H_{\rho_\alpha}} e^{-2\pi\beta_t\left(L_0^{\rho_\alpha} - \frac{c}{24}\right)}\mathbf{1} \qquad (13)$$

is a special case of the (12) for $\Phi = 1$ (the thermal expectation of the identity operator or partition function). The relevant Verlinde matrix S is the one which diagonalizes the Z_N lattice fusion rules and together with a certain diagonal matrix T generates a unitary representation of the modular group $SL(2,\mathbb{Z})$ whose generators are T: $\tau \to \tau + 1$ and S: $\tau \to -\frac{1}{\tau}$. Mathematically these objects are $SL(2,\mathbb{Z})$ modular forms in the sense of Hecke and many of their modular identities were discovered by the Indian mathematical wizard Ramanujan. Similar modular identities have been proven for partition functions of minimal chiral models by Poisson-resummation technique [46]. The relevant question in the present context is whether QFT is capable to give a structural prove for all partition functions associated with chiral models with a finite number of superselection rules. Under certain technical assumption within the setting of vertex operators[14] Huang recently presented a model-independent proof [47] of the character relation (13) with Verlinde's definition of S. Huang's proof does not really reveal the deep local quantum

[12]The conformal invariance actually allows a generalization to complex Gibbs parameters τ with $Im\tau = \beta$ which is however not neede in the context of the present discussion.

[13]In local theories the KMS condition at equal (real) times turns into imaginary time periodicity.

[14]The Vertex framework is based on pointlike covariant objects, but unlike Wightman's formulation it is not operator-algebraic i.e. the star operation is not inexorably linked to the topology of the algebra as in C^*algebras of quantum mechanical origin. Although it permits a generalization beyond two dimensions [48], the determination of classifications and representations of higher-dimensional vertex-algebras remains an open problem.

physical principles which the analogy to the N-S duality suggests. The only family of thermal n-point functions which have been explicitly computed are those of maximally extended multicomponent abelian current algebras [45], in that case the proof of (12) can be derived from a known property of the Jacobi Θ-functions. A profound structural understanding of (12) in analogy to the Nelson-Symanzik duality can be given in the setting of *modular Euclideanization.* In the following I will sketch such a derivation.

The fact that the above relation (12) involves analytic continuation to imaginary rotational lightray coordinates suggests that one should first look for a formulation in which the rotational Euclideanization has a well-defined operator-algebraic meaning. On the level of operators a positive imaginary rotation is related to the Moebius transformation $\tilde{\Delta}^{i\tau}$ with the two fixed points $(-1,1)$ via the formula

$$e^{-2\pi\tau L_0} = \Delta^{\frac{1}{4}}\tilde{\Delta}^{i\tau}\Delta^{-\frac{1}{4}} = \tilde{\Delta}_c^{i\tau} \tag{14}$$

where $\Delta^{i\tau}$ and $\tilde{\Delta}^{i\tau}$ represents the 0ne-parametric $SL(2,R)$ Moebius subgroups with fixpoints $(0,\infty)$ respectively $(-1,1)$ and $\tilde{\Delta}_c^{i\tau}$ the corresponding subgroup of the $SU(1,1)$ compact presentation of the Moebius group with $z = (e^{-i\frac{\pi}{2}}, e^{i\frac{\pi}{2}}) = (-i,i)$ being fixed (the subscript c denotes the compact picture description). Note that $Ad\Delta^{\frac{1}{4}}$ acts the same way on $\tilde{\Delta}^{i\tau}$ as the Cayley transformation AdT_c, where the T_c is the matrix which represents the fractional acting Cayley transformation

$$T_c = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \tag{15}$$

Ignoring for the moment domain problems for $\Delta^{\frac{1}{4}}$, the relation (14) gives an operator representation for the analytically continued rotation with positive imaginary part $(t>0)$ in terms of a Moebius transformation with real rapidity parameter. If we were to use this relation in the vacuum representation for products of pointlike covariant fields Φ where the spectrum of L_0 is nonnegative, we would with obtain with $\Phi(\tau) = e^{2\pi i\tau L_0}\Phi(0)e^{-2\pi i\tau L_0}$

$$\begin{aligned} \langle\Omega\left|\Phi_1(i\tau_1)..\Phi_n(i\tau_n)\right|\Omega\rangle^{ang} &= \langle\Omega\left|\Phi_1(\tau_1)_c..\Phi_n(\tau_n)_c\right|\Omega\rangle^{rap} \\ &= \omega_{2\pi}(\Phi_1(\tau_1)_c..\Phi_n(\tau_n)_c)^{rap} \end{aligned} \tag{16}$$

The left hand side contains the analytically continued rotational Wightman functions in the vacuum state. As a result of positivity of L_0 in the vacuum representation this continuation is possible as long as the imaginary parts remain ordered i.e. $\infty > t_1 > ... > t_n > 0$. On the right hand side the fields are at their physical boundary values parametrized with the *rapidities* of the compact $\tilde{\Delta}_c^{it}$ Moebius subgroup of $SU(1,1)$. Note that this rapidity interpretation implies a restriction since the rapidities associated with $x = th\frac{\tau}{2}$ cover only the interval $(-1,1)$. The notation in the second line indicates that this is a KMS state at modular temperature $\beta_{mod} = 1$ $(\beta_{Hawking} = 2\pi\beta_{mod} = 2\pi)$, in agreement with the well-known fact that the restriction of the global vacuum state to the interval (-1,1) becomes a state Ω_1 at fixed Hawking-Unruh temperature $\beta_{Hawking} = 2\pi$. Note that only the right hand side is a Wightman *distribution in terms of a standard* $i\varepsilon$ *boundary prescription*, whereas the left hand side is an *analytic function (i.e. without boundary prescription* which converts the analytic function into a Wightman distribution*).* This significant conceptual (but numerical harmless) difference is responsible for the fact that in the process of angular Euclideanization of chiral models *the analytic KMS condition*[15] *passes to a periodicity property of Wightman boundary values and vice versa.*

As explained before, the analogy with the generalized Nelson-Symanzik setting for temperature duality suggests to start from a *rotational thermal representation* in the chiral setting. For simplicity let us first assume that our chiral theory is a model which possess besides the vacuum sector no other positive energy representations. Examples for such a situation are lattice extension of multicomponent Weyl algebras with selfdual lattices (e.g. the Leech lattice....). In group theoretic terms these selfdual lattices correspond to special finite groups whereas in terms of QFT these are the cases in which the only sector is the vacuum-sector (or in terms of AQFT these are cases in which Haag duality holds even for multiply connected localization regions [49]) In this case $S = 1$ in the above matrix relation (12). With the Gibbs temperature fixed at the symmetry point $\beta = 2\pi$, and in accordance with what was said about the

[15] Contrary to popular believes KMS is not equivalent to periodicity in time but it leads to such a situation if the the involved operators commute inside the correlation function (e.g. spacelike separated observables).

interchange of KMS with periodicity in the process of angular Euclideanization, we expect the selfdual relation

$$
\begin{aligned}
&\langle\Omega_1|\Phi(i\tau_1)...\Phi(i\tau_n)|\Omega_1\rangle^{rot} = (i)^{ndim\Phi}\left\langle\Omega_1^E\right|\Phi^E(\tau_1)...\Phi^E(\tau_n)\left|\Omega_1^E\right\rangle^{rot} \\
&\langle\Omega_1|\Phi(\tau_1)...\Phi(\tau_n)|\Omega_1\rangle^{rot} \equiv tr(e^{-\pi L_0}\Phi(\tau_1)...\Phi(\tau_n)e^{-\pi L_0}) \\
&\Phi^E(\tau_1)^\dagger \equiv \tilde{J}\Phi^E(\tau_1)\tilde{J} = \Phi^E(-\tau_1)^*,\ \left[\tilde{J}, L_0\right] = 0
\end{aligned}
\tag{17}
$$

where the analyticity according to a general theorem about thermal states [50][51] limits the $\tau's$ to the unit interval and requires the ordering $1 > \tau_1 > ... > \tau_n > 0$. The thermal Gibbs states were conveniently written in the Hilbert space inner product notation with the help of the Hilbert-Schmidt operators $\Omega_1 \equiv e^{-\pi L_0}$, in which case the modular conjugation is the action of the Hermitian adjoint operators from the right on Ω_1 [20]. Since KMS and the periodicity match crosswise on both sides of (17), the only property to be checked is the positivity of the right hand side (which attributes to the right hand side the status of an autonomous theory). In analogy to the Osterwalder-Schrader Euclideanization *this positivity must refer to another star operation* i.e. that the analytic continued correlations on the imaginary axis must now be Wightman distributions (with a ε-boundary prescription) with the new star-conjugation given by the Ad action of the modular conjugation $\tilde{J}$ which interchanges the right with the left halfcircle and because of $L_0 = H + \tilde{J}H\tilde{J}$ commutes with L_0. They are distributions which fulfill a Wightman positivity with respect to this new star-operation[16]. The label E on $\Phi(\tau_1)$ denotes the Euclideanization for this we need the star conjugation associated with $\tilde{J}$. The modular group of $\Phi^E(\tau_1) = \Phi(i\tau)$ is $e^{-2\pi\tau L_0}$ and the modular conjugation is the Ad action of $\tilde{J}$ which changes the sign of τ as in the third line (17). Whereas the modular conjugation in the original theory maps a vector $A\Omega_1$ into $\Omega_1 A^*$ with the star being the Hermitian conjugate, the Euclidean modular conjugation acts as $A^E\Omega_1^E \rightarrow \Omega_1^E\tilde{J}A^E\tilde{J} \equiv \Omega_1^E\left(A^E\right)^\dagger$. This property is at the root of the curious self-conjugacy (17).

There are two changes to be taken into consideration if one passes to a more general situation. The extension to the case where one starts with a β Gibbs state, which corresponds in the Hilbert-Schmidt setting to $\Omega_\beta = e^{-\pi\beta L_0}$, needs a simple rescaling $\tau \rightarrow \frac{1}{\beta}\tau$ on the Euclidean side in order to maintain the crosswise correspondence between KMS and periodicity. Since the Euclidean KMS property has to match the unit periodicity on the left hand side, the Euclidean temperature must also be $\frac{1}{\beta}$ i.e. the more general temperature duality reads

$$
\langle\Omega_\beta|\Phi(i\tau_1)...\Phi(i\tau_n)|\Omega_\beta\rangle^{rot} = \left(\frac{1}{\beta i}\right)^{ndim\Phi}\left\langle\Omega_{\frac{1}{\beta}}^E\right|\Phi^E(\frac{1}{\beta}\tau_1)...\Phi^E(\frac{1}{\beta}\tau_n)\left|\Omega_{\frac{1}{\beta}}^E\right\rangle^{rot} \tag{18}
$$

The positivity argument through change of the star-operation remains unaffected. This relation between expectation values of pointlike covariant fields should not be interpreted as an identity between operator algebras. As already hinted at the end of section 2 one only can expect a sharing of the analytic core of two different algebras whose different star-operations lead to different closure. In particular the above relation does not represent a symmetry in the usual sense of Wigner.

The second generalization consists in passing to generic chiral models with more superselection sectors. As usual the chiral system of interests is assumed to be rational i.e. there is a finite number of sectors. In that case the mere matching between KMS and periodicity does not suffice to fix the relation between the two sides (because all sectors are periodic as well as KMS and one does not know which sectors to match). Here a closer examination (at the operator level, taking the Connes cocycle properties versus charge transportation around the circle into account) reveals that the *statistics character* matrix S [52] enters as in (12) as a consequence of the well-known connection between the *invariant* content (in agreement with the sector $[\rho]$ dependence of rotational Gibbs states) of the *circular charge transport* and the statistics character matrix [53][54]. For those known rational models for which S has been computed, this matrix S in the sense of charge transport around the circle (identical to the statistics character matrix S introduced by Rehren) turns out to be identical to the Verlinde matrix S which diagonalizes the fusion

[16]Note the close analogy to the Nelson-Symanzik setting where the positivity and its associated star-operation of the measure theoretical Euclidean (stat. mech.) description is reprocessed via the reflection positivity into the positivity and associated star operation of the real time QFT. In both cases there exists a "star-less" subalgebra which is shared and which via completion with the different stars leads to the two star algebras related by Euclideanization [25].

rules [55] and which together with a diagonal phase matrix τ generates a unitary representation of the modular group $SL(2,Z)^{17}$. Confronting the previous zero temperature imaginary angular correlation (16) with the asymptotic limit of the finite temperature identity, one obtains the well-known Kac-Wakimoto relations as an identity between the temperature zero limit and the double limit of infinite temperature (the chaos state) and short distances on the Euclidean side.

As mentioned before the temperature duality relation for the partition function (13) was first derived for loop group chiral theories by Kac and Peterson and arguments for its general validity for rational chiral theories were given by Verlinde. The resulting identities are mathematically fascinating because they shed a new light on relations between so-called *modular forms in the sense of Hecke*; the surprising aspect being that the arise here from the physical principles of QFT specialized to chiral models. What is even more surprising is that they are special objects whose generalization are n-point functions of an autonomous thermal chiral QFT fulfilling the temperature duality relations (12). And to raise the surprise to a climax: they are derived from chiral QFT via the modular operator algebraic setting even though the two uses of the word modular have no apparent connection. *This is one of the rare cases in which the accidental use of a word for two apparently different mathematical concepts is a posteriori justified by the use of localization and stability (energy positivity, KMS) principles underlying QFT.* It is an interesting mathematical question whether any SL(2,Z) modular form occurs as a partition function of a thermal chiral model. Some conjectures of an analogous kind, e.g. the question whether every compact group is encoded in form of the D-R group dual in the localized observable net of a suitable QFT, received a positive rigorous answer.

The thermal duality only has a simple form in terms of expectations of pointlike fields where it almost takes the appearance of a symmetry relation. In terms of operator algebras it looses this simple presentation in terms of an identity between (analytically continued) correlation functions; as for the Nelson-Symanzik duality its operator algebraic basis is the existence of a *dense analytic algebra of operators* in a Hilbert space which can be equipped with two star-operations which admits two different extensions to an operator C^*-algebra, each one with its own positivity. We hope to present more details in a separate paper.

Modular operator theory is also expected to play an important role in bridging the still existing gap between the Cardy [57] Euclidean boundary setting and that in the recent real time operator algebra formulation by Longo and Rehren [58]

5 Open problems, concluding remarks

The comparison of the constructive results obtained in the O-S setting and in the old bootstrap-formfactor approach built on the Smirnov construction recipes with the more recent constructions based on modular localization theory gives rise to a wealth of unsolved basic problems of QFT. Here are some of them.

- The O-S Euclideanization and the modular setting are related in a deep and still largely unknown way. In order to learn something about this connection one may start with the Wigner representation setting. Recently Guerra has spelled out what the O-S Euclideanization means in the simplest context of the spinless one-particle Wigner space [59]. On the other hand most problems concerning the modular localization setting have been explicitly answered for all positive energy representations [28]. It would be very interesting to translate these results into the O-S setting.

- The old "bootstrap dream" which had its glorious come-back in the bootstrap-formfactor program for 2-dim. factorizing models remained unfulfilled beyond factorizing; the new modular setting not only explains why a general pure S-matrix approach is not feasible but also indicates that if one views the S-matrix construction as part of a wider framework which aims at generators of wedge localized algebras, this dream still may find its realization in a new construction of QFT which bypasses correlation functions of (necessarily singular) correlation functions of pointlike generators.

[17] Whereas relativistic causality already leads to an extension of the standard KMS β-strip analyticity domain to a β-tube domain [56], conformal invariance even permits a complex extension of the temperature parameter to τ with $Im\tau > 0$. For this reason the chiral theory in a thermal Gibbs state can be associated with a torus in the sense of a Riemann surface, but note that in *no physical sense* of localization this theory lives *on* a torus.

With new hindsight and a new conceptual setting one should revisit the properly re-formulated old problems.

- The d=1+2 massive Wigner representation can have anomalous (not semiinteger) spin which leads to plektonic (braid group) statistics. The simplest abelian family of representations is that of Z_N-anyons. Such representations activate representations of the Poincaré group in which the Lorentz part is represented through the Bargmann covering $\widetilde{SO}(2,1)$. The modular theory of these string-like representations has been worked out in [60] and it is known that d=1+2 anomalous spin representations are the only Wigner representations whose associated QFT has vacuum-polarization which prevent the standard on-shell free field realization [38]. It would be very interesting to understand how an O-S like Euclidean formulation would look like.

- Up to now models of QFT have been "baptized" and studied in the setting of Lagrangian quantization (either canonical of functional integral). More recently the bootstrap-formfactor setting led to models which do not possess a Lagrangian description (e.g. the Z_N model in [61] whose natural description is in terms of Z_N braid group statistics). There are indications that interactions in terms of string-localized fields (which apparently do not permit a Euler-Lagrange characterization) extends the possibilities for formulating interactions. The ghostfree potential for the pointlike physical fields of zero mass finite helicity representations (e.g. the vectorpotential associated with the electromagnetic field) are necessarily string-localized [28]. Also in this case it should be possible to use these objects outside the Lagrangian framework directly in the implementation of interactions. Such a description would be particularly interesting for higher helicities as in the case of the graviton. Closely related is the question whether interacting QFT exists in $d > 4$ in which case the Lagrangian approach does not permit any renormalizable solution. Last not least an intrinsic description of QFT "without the classical crutches" of Lagrangian quantization is an old dream of Pascual Jordan, the protagonist of "Quantelung der Wellenfelder". The continuation of the ongoing attempts may still lead to a fulfillment of this dream.

- The relation between heat bath thermal behavior and the purely quantum thermal manifestations of vacuum polarization (Hawking, Unruh, Bekenstein) have received a lot of recent attention from a modular point of view [62][63]; many properties have not yet been adequately understood. Quantum field theory, which in the frame of mind of some string theorist has become a historical footnote of their theory, had already been declared dead on two previous occasions; first in the pre-renormalization ultraviolet crisis of the 30s and then again by the protagonists of the S-matrix bootstrap in the 60s. But each time a strengthened rejuvenated QFT re-appeared. It is clear that an area which still produces many fundamental new questions is very far from its closure.

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