Stochastic Equations for Spin-2 Fields in the Quasi-Maxwellian Formulation

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ABSTRACT

In the present note we suggest a stochastization procedure for the spin-2 field using the Quasi-Maxwellian formulation. Such scheme is completely analogous of the stochastic quantization of the electromagnetic field. The present approach is free of the gauge difficulties that appear in the conventional geometric Einstein variables.

Key-words: Quantization; Gravity; Stochastic equations.

1 Introduction

Quantum systems described by the Schrödinger equation allow a stochastic interpretation. This remarkable result, due to Nelson [1, 2], provided the grounds for several stochastic approaches in quantum field theory. Among these, of particular interest is the stochastic formulation for the electromagnetic field proposed by Guerra and Loffredo[3], who obtained stochastic equations for the classical fields $E(x,t)$ and $B(x,t)$ in the case of an Euclidean space. The important lesson to be learned from this advance is that one can now consider the development of alternative methods for quantizing classical fields. In the present note we will suggest a stochastization procedure for the gravitational field (expressed in the so-called Quasi-Maxwellian formulation), which is closely related to such stochastic "quantization" of the electromagnetic field. For the sake of simplicity, we will consider the free field case, and work within the weak field approximation. In the present analysis no further care is given to the Riemannian nature of space-time geometry, because in this approximation one may as well consider an Euclidean 3-space when making use of the 3+1 separation.

2 Quasi-Maxwellian Equations in $3+1$ Dimensions

For the purposes of the present paper it is convenient to employ the so-called Quasi-Maxwellian (QM) representation of Einstein's Theory of General Relativity (GR). The main reason for this choice is the striking structural similarity between the QM formulation and Maxwell's electrodynamics. We will explore here precisely this similarity and take the stochastic treatment of electrodynamics as a paradigm in order to achieve the formal stochastization of the gravitational field.

In the QM approach, the basic quantities that describe the gravitational field are the electric and magnetic parts of the Weyl conformal tensor. They are defined in a standard way by:

$$E_{\mu\nu} = W_{\alpha\mu\beta\nu} V^{\alpha} V^{\beta} \tag{1}$$

and

$$B_{\mu\nu} = W^{*}_{\alpha\mu\beta\nu} V^{\alpha} V^{\beta} \tag{2}$$

with respect to a field of arbitrary observers endowed with (unit) 4-velocities V^{α}. This observer defines an inertial system. The rest space associated to an inertial system defined by a given observer constitutes a 3-dimensional hypersurface $\mathcal{H}$. As usual, the projection operation on hypersurface $\mathcal{H}$ is accomplished by means of the quantity $h_{\mu\nu}$, defined as:

$$h_{\alpha\beta} = g_{\alpha\beta} - V_{\alpha} V_{\beta}$$

Indeed, from this definition it follows that

$$h_{\alpha\beta} V^{\beta} = 0.$$

We will assume here that the QM formulation is the fundamental framework for the description of gravitation and will also limit ourselves to the weak field approximation. In this case the QM equations take the form:

$$E^{ij}{}_{,j} = 0 \tag{3}$$

$$B^{ij}{}_{,j} = 0 \tag{4}$$

$$\dot{E}^{ij} = -\frac{1}{2} B^k_{l,m}\; h^{(i}_k\; \epsilon^{j)ml} \tag{5}$$

$$\dot{B}^{ij} = \frac{1}{2} E^k_{l,m}\; h^{(i}_k\; \epsilon^{j)ml} \tag{6}$$

Following the standard procedure suggested by Guerra and Loffredo, let us define a complete real orthonormal basis of 3-dimensional tensor spherical harmonics $F^{(n)}_{ij}(x)$ $(n = 0, 1, 2, ...)$. By construction, the $F^{(n)}_{ij}(x)$ are eigenfunctions of the Laplacian operator $\nabla^2 \equiv g^{ij}\, \nabla_i \nabla_j$ and obey a tranversality condition (3, 4). They therefore satisfy the following relations:

$$\int F_{ij\,(n)}(x) F^{ij}_{(n')}(x) = \delta_{nn'} \tag{7}$$

$$\sum_n F^{(n)}_{ij}(x) F^{(n)}_{lk}(y) = \delta^T_{il}(x-y)\delta^T_{jk}(x-y) \tag{8}$$

$$F^{ij}_{(n)}(x)_{,j} = 0 \tag{9}$$

$$\nabla^2\; F^{(n)}_{ij}(x) = k^2 F^{(n)}_{ij}(x) \tag{10}$$

where

$$\delta^T_{ij}(x-y) = (2\pi)^{-3} \int exp[ip.(x-y)[\delta_{ij} - |p|^{-2} p_i p_j] d^3p.$$

Introducing the canonical co-ordinates $(p_n(t),\, q_n(t))$ and expanding the electric and magnetic parts $E_{ij}(x,t)$ and $B_{ij}(x,t)$ of the Weyl tensor in terms of the basis $F_{ij}(x)$ we obtain:

$$E_{ij}(x,t) = \sum_n F^{(n)}_{ij}(x) p_n(t) \tag{11}$$

$$B^{ij}(x,t) = \frac{k^2}{2} \sum_n F^k_{l,m}\; h^{(i}_k \epsilon^{j)ml} q_n(t) \tag{12}$$

Substituting the above expansion in the QM equations (5, 6) we find that

$$\dot{q}_n(t) = p_n(t) \tag{13}$$

and

$$\ddot{q}_n(t) + \frac{k^2}{4} q_n(t) = 0 \tag{14}$$

As one would expect these equations are nothing but the equations of motion of n independent harmonic oscilators with co-ordinates $q_n(t)$.

3 Stochastic Quantization

Let us now proceed to establish the stochastic quantization of the gravitational field. In order to follow the same procedure as in the electrodynamical case we need first to present some useful definitions.

We start by promoting each of the co-ordinates $q_n(t)$ to a random Markov process, for which the following stochastic differential equations are postulated:

$$dq_n(t) = v_n^+(q,t)dt + dw_n(t) \tag{15}$$

where $v_n^+ = D^+[q_n(t)]$. The quantity (D^+) is the so-called forward time derivative and $dw_n(t)$ are the differentials of Wiener processes for each independent $q(t)$. According to the definition of a Wiener process, the mean and co-variance of dw_n are given in a standard way by:

$$E[dw_n(t)] = 0$$

and

$$E[dw_n(t)dw_{n'}(t)] = \nu\ \delta_{nn'}dt$$

The operator $E[]$ represents the expectation value and ν is the so-called diffusion parameter (which can be associated to Planck's constant $\hbar$).

If we define the acceleration $\ddot{q}_n$ in Nelson's way [1], i.e.,

$$\ddot{q}_n = \frac{1}{2}(D_+D_- + D_-D_+)[q_n] \tag{16}$$

and use (15), the equation (14) becomes

$$D_+[v_n^-] + D_-[v_n^+] + \frac{k^2}{2}q_n(t) = 0 \tag{17}$$

where, correspondly, $v_n^- = D^-[q_n(t)]$. Operator D^- is defined as the backward time derivative. These are the stochastic equations for n independent harmonic oscilators, as it should.

We are now prepared to apply the method of stochastic quantization of eletromagnetic fields outlined above to the Quasi-Maxwellian linear form of Einstein's theory of gravity. In this vein, the application of the method yields the promotion of the magnetic part B_{ij} to a Markov process, while the eletric part splits into a pair of functions $E^{\pm}_{ij}(B_{ij};x,t)$. Substituting the equation (15) in the expansions of both E_{ij} and B_{ij} (eqs. 11 and 12) we obtain:

$$dB_{ij}(x,t) = \frac{1}{2}E^+_{kl,m}(B_{ij};x,t)\ h^k_{(i}\ \epsilon_{j)}{}^{ml} + dW_{ij}(x,t) \tag{18}$$

$$D_\pm B_{ij}(x,t) = \frac{1}{2}E^\pm_{kl,m}(B_{ij};x,t)\ h^k_{(i}\ \epsilon_{j)}{}^{ml} \tag{19}$$

$$D_+E^{ij}_- + D_-E^{ij}_+ = -B^k_{l,m}\ h^{(i}_k\epsilon^{j)ml} \tag{20}$$

These results give us the stochastic counterpart of the QM equations, in which the new, stochastized fields are given by:

$$B_{ij}(x,t) = \sum_n F^n_{ij}(x) q_n(t)$$

$$dW_{ij}(x,t) = \sum_n F^n_{ij}\, dw_n(t)$$

$$E^{\pm}_{ij}(B_{ij}; x, t) = k^2 \sum_n F^n_{lk,m} h^k_{(i} \epsilon^{lm}_{j)} v^{\pm}_n(q,t)$$

4 Conclusion

In this paper we presented a new approach aimed at the stochastic quantization of the weak gravitational field by means of the Quasi-Maxwellian formulation. As one should expect, the result thus obtained is strongly reminiscent of Guerra & Loffredo's derivation for the eletromagnetic field case. The reason of this analogy is that the canonical quantization of spin-2 fields is completely similar to the quantization of spin-1 fields in this formulation; indeed, such quantization has been carried out both in terms of Fierz's variables and in the linear case [5]. However, in canonical quantization one deals with two spin-2 fields and consequently some additional condition is required to eliminate one of them, the root of this difficulty being the need of defining a potential in order to accomplish the quantization procedure. Here, we have the very definite advantage of quantizing the fields directly, through the equations of motion; hence, the two field problem is absent. Of course, there remains the question of determining whether, in the present formalism, all the information available with the application of the canonic quantization method can also be obtained.

Other advantage of the stochastic approach, according to Davidson[4], is that the use of the stochastic formulation to quantize the gravitational field may reinforce the geometrical interpretation of quantum gravity. Unfortunately, this aspect of the problem couldn't be analysed here, due to the option of applying the Quasi-Maxwell formulation. On the other hand, in the present approach one is completely free of the gauge difficulties that appear in connection with conventional geometric Einstein variables.

References

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