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ABSTRACT

On the light of the Einstein-Maxwell-scalar theory it is considered a static, bounded, spherically symmetric distribution of incoherent dust; the constituents of this dust are supposed to be simultaneously sources of gravitational, electrostatic as well as long range scalar fields. Two kinds of scalar fields are considered, the atractive and the repulsive one. Two classes of exact external solutions are encountered, flat at infinity and satisfying the internal boundary conditions: one contains the attractive scalar field, the other the repulsive one. The internal solutions are presented as derivatives of two arbitrary (to some extent) functions of the radial coordinate. All solutions tend straighrforwardly to all known less general solutions. One complete regular example is presented.

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1. INTRODUCTION

Spherically symmetric distributions of electrically charged dust, in static condition, were studied by Bonnor (1960) he verified that the equilibrium could only be mantained when the densities of charge and matter beared a constant ratio, $\sigma = \pm\rho$ in units. Later De and Raychaudhuri (1968) proved that the relation $\sigma = \pm\rho$ was a consequence of the Einstein-Maxwell's equations; this relation was a general requirement of all static incoherent charged dust distributions, provided the system did not show any singularity, no spatial symmetry being necessary. Very recently Wolk et al. (1975) studied a distribution of incoherent dust, the constituents of which were supposed to be the sources of gravitational as well as of repulsive long range scalar field. Using an analysis similar to that of De and Raychaudhuri they found that in static systems free of singularity one should have a proportionallity between the densities of scalar charge and of matter, $s = \pm\rho$ in their units.

We now generalize all these results by considering a static distribution of incoherent dust, charged both in electric as well as in scalar sense. Our scalar field can be either of an attractive kind or of a repulsive kind; like in Teixeira et al. (1975) we are calling attractive (repulsive) a scalar field that produces attraction (repulsion) between scalar charges of the same sign, in static condition, oppositely (similarly) to what happens in electrostatics. For definiteness we have considered a distribution with spherical symmetry, and obtained the external and internal solutions.

2. BASIC EQUATIONS

In the Einstein's equations (Anderson 1967)

$$R^{\mu}_{\nu} = - 8\pi(T^{\mu}_{\nu} - \delta^{\mu}_{\nu}T/2) \tag{1}$$

we take as energy momentum density of our system

$$T^{\mu}_{\nu} = \rho u^{\mu}u_{\nu} + E^{\mu}_{\nu} + K^{\mu}_{\nu} ; \tag{2}$$

here ρ is the mass density of a distribution with velocity u^{μ}, and E^{μ}_{ν} and K^{μ}_{ν} are the energy momentum densities of an electromagnetic and of a long range scalar field.

The tensor E^{μ}_{ν} is given by

$$4\pi E^{\mu}_{\nu} = F^{\mu}_{\alpha}F^{\alpha}_{\nu} - \delta^{\mu}_{\nu}F^{\alpha}_{\beta}F^{\beta}_{\alpha}/4 \quad , \tag{3}$$

where

$$F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} \tag{4}$$

is the electromagnetic field, which satisfies Maxwell's equations

$$F^{\mu\nu}{}_{;\mu} = 4\pi\sigma u^{\nu} \quad , \tag{5}$$

σ being an electric charge density; subscripted and superscripted semicolons mean covariant derivatives.

And the tensor K^{μ}_{ν} is given (Teixeira et al. 1975) by

$$4\pi\gamma K^{\mu}_{\nu} = S^{;\mu}S_{;\nu} - \delta^{\mu}_{\nu}S^{;\alpha}S_{;\alpha}/2 \quad , \tag{6}$$

where $\gamma = +1$ when the scalar field S is attractive, and $\gamma = -1$ for repulsive scalar fields. Both kinds of scalar fields satisfy

$$S^{;\mu}_{;\mu} = - 4\pi\gamma s \quad , \tag{7}$$

where s is the density of the source of S.

The contracted Bianchi identities

$$2R^{\mu}_{\nu;\mu} \equiv R_{;\nu} \qquad (8$$

give for our system

$$\rho u_{\mu;\nu}u^{\nu} = \sigma F_{\mu\nu}u^{\nu} + s\, S_{;\mu} \ . \qquad (9)$$

In this work we shall be concerned with spherically symmetric static systems; for such symmetry we use the line element

$$ds^2 = e^{2\eta}(dx^0)^2 - e^{2\alpha}dr^2 - r^2 e^{\alpha-\eta}(d\theta^2+\sin^2\theta\ d\phi^2) \quad , \qquad (10)$$

with η and α functions of r alone; then the components of the Ricci tensor are

$$R^0_0 = -(\eta_{11} + 2\eta_1/r)e^{-2\alpha} \quad , \qquad (11)$$

$$R^1_1 = -(\alpha_{11}-\alpha_1^2/2-\alpha_1\eta_1+3\eta_1^2/2-2\eta_1/r)e^{-2\alpha} \ , \qquad (12)$$

$$R^2_2 = R^3_3 = (\eta_{11}/2-\alpha_{11}/2+\eta_1/r-\alpha_1/r-r^{-2})e^{-2\alpha}+r^{-2}e^{\eta-\alpha} \ , \qquad (13)$$

where a subscript 1 means d/dr. All sources ρ, σ and s are functions of r alone, the same happening to the fields A_μ and S, and to the velocity u^μ ,

$$A_\mu = \delta^0_\mu \Phi(r) \quad , \qquad (14)$$

$$u^\mu = \delta^\mu_0\ e^{-\eta} \quad . \qquad (15)$$

The electromagnetic and scalar energy momentum tensors become

$$8\pi E^\mu_\nu = \Phi_1{}^2\ e^{-2(\eta+\alpha)}\ \text{diag}\ (+,+,-,-) \quad , \qquad (16)$$

$$8\pi K^\mu_\nu = \gamma S_1{}^2\ e^{-2\alpha}\ \text{diag}\ (+,-,+,+) \quad ; \qquad (17)$$

and the electrostatic can scalar field equations are now

$$(r^2 e^{-2\eta}\Phi_1)_1 = 4\pi\sigma r^2 e^{2\alpha-\eta} \quad , \qquad (18)$$

$$(r^2 S_1)_1 = 4\pi\gamma s\, r^2 e^{2\alpha} \quad , \qquad (19)$$

with the Bianchi identity

$$\rho\eta_1 + \sigma e^{-\eta}\Phi_1 + s S_1 = 0 \quad . \qquad (20)$$

3. GENERAL EXTERIOR SOLUTIONS

We put $\rho = \sigma = s = 0$ in the equations of the preceding Section, and obtain

$$\eta_{11} + 2\eta_1/r = e^{-2\eta}\Phi_1{}^2 \quad , \qquad (21)$$

$$\alpha_{11}+3\eta_1{}^2/2-\alpha_1{}^2/2 - \eta_1\alpha_1 - 2\eta_1/r = e^{-2\eta}\Phi_1{}^2 - 2\gamma S_1{}^2 \quad , \qquad (22)$$

$$(\eta_{11}-\alpha_{11})/2 + (\eta_1-\alpha_1)/r + (e^{\eta+\alpha}-1)/r^2 = e^{-2\eta}\Phi_1{}^2 \quad , \qquad (23)$$

$$(r^2 e^{-2\eta}\Phi_1)_1 = 0 \quad . \qquad (24)$$

From the last equation we get immediately

$$\Phi_1 = - q\, r^{-2} e^{2\eta} \quad , \qquad (25)$$

where q is a constant of integration.

The substitution of (25) into (21) gives for η the solution

$$e^{-\eta} = \cosh(d+c/r)+(1+q^2/c^2)^{1/2} \sinh(d+c/r) \quad , \qquad (2\,)$$

where c and d are constants of integration; we impose $\eta = 0$ at infinity, which implies that $d = 0$.

The subtraction of (21) from (23) yields the solution

$$e^{\eta+\alpha} = (f/r)^2 \sinh^{-2} (g+f/r) \quad , \tag{27}$$

whith f and g constants of integration; the imposition of $\eta+\alpha = 0$ at infinity demands $g = 0$.

Finally (22) gives for $S_1{}^2$ the solution

$$S_1{}^2 = \gamma(f^2 - c^2)/r^4 \quad ; \tag{28}$$

this solution is compatible with (19) which in our exterior region is expressed by $(r^2 S_1)_1 = 0$.

So the exterior solution of our problem, which tends to flatness at infinity is (subscript e for exterior)

$$g_{00} = \exp(2\eta_e) = \left[\cosh c/r+(1+q^2/c^2)^{1/2} \sinh c/r\right]^{-2} \quad , \tag{29}$$

$$g_{rr} = - \exp (2\alpha_e) = - (f/r)^4 \sinh^{-4}(f/r) \exp(-2\eta_e) \quad , \tag{30}$$

$$g_{\theta\theta} = - r^2 \exp (\alpha_e-\eta_e) \quad , \tag{31}$$

$$\Phi_1 = - q\, r^{-2} \exp (2\eta_e) \quad , \tag{32}$$

$$S_1{}^2 = \gamma(f^2-c^2)/r^4 \quad , \tag{33}$$

with q, c and f constants to be associated somehow to the gravitational, electric and scalar charges. While the constant q must be real in order to have the usual physical meaning in (32), the constants c and f can be real or immaginary, independently. This spherically symmetric exterior solution (29-33) is consistent with mo 'e general results already obtained by Teixeira et al. (1975).

The familiar Reissner-Nordström exterior solution

$$g_{00} = 1-2m/r'+q^2/r'^2 \quad , \quad g_{r'r'} = -\,g_{00}^{-1} \quad , \quad g_{\theta\theta} = -r'^2 \quad , \quad \Phi = q/r$$

is obtained by putting c = f and next performing the radial coordinate transformation $r' = m + f \coth f/r$ with $m^2 = q^2+f^2$; and Yilmaz's (1958) one-parameter repulsive scalar field solution

$$ds^2 = e^{-2c/r}(dx^0)^2 - e^{2c/r}(dr^2+r^2d\theta^2+r^2 \sin^2\theta \, d\phi^2),$$

$$S_1{}^2 = c^2/r^4$$

is obtained by putting q=f=0.

4. INTERIOR SOLUTIONS

We consider now a static, spherically symmetric distribution of incoherent dust, charged both electrostatically and scalarly (long range, either attractive $\gamma = +1$ or repulsive $\gamma = -1$); we assume that all distributions $\rho(r)$, $\sigma(r)$ and $s(r)$ are regular. Our set of equations is now

$$\eta_{11}+2\eta_1/r = 4\pi\rho\, e^{2\alpha} + e^{-2\eta}\Phi_1{}^2 \quad , \qquad (34)$$

$$\alpha_{11}+3\eta_1{}^2/2 - \alpha_1{}^2/2-\eta_1\alpha_1-2\eta_1/r = -\,4\pi\rho\, e^{2\alpha}+e^{-2\eta}\Phi_1{}^2-2\gamma S_1{}^2 \quad , \qquad (35)$$

$$(\eta_{11}-\alpha_{11})/2 + (\eta_1-\alpha_1)/r + (e^{\eta+\alpha}-1)/r^2 = 4\pi\rho e^{2\alpha}+ e^{-2\eta}\Phi_1{}^2 \quad , \qquad (36)$$

$$(r^2\, e^{-2\eta}\Phi_1)_1 = -\,4\pi\sigma r^2\, e^{2\alpha-\eta} \quad , \qquad (37)$$

$$(r^2 S_1)_1 = 4\pi\rho s\, r^2\, e^{2\alpha} \quad , \qquad (3\;)$$

with the contracted Bianchi identity

$$\rho\eta_1 + \sigma e^{-\eta}\,\Phi_1 + s\,S_1 = 0 \quad .$$

The subtraction of (34) from (36) gives again the solution (27); however in our interior system we can only have regularity at the origin when both constants f and g vanish in such a way that

$$\eta_i + \alpha_i = 0 \qquad , \tag{40}$$

where the subscript i means internal.

We next add (34) and (35), consider (40) and get the quadratic first order relation

$$\eta_1{}^2 - e^{-2\eta}\Phi_1{}^2 + \gamma S_1{}^2 = 0 \quad ; \tag{41}$$

from the system (39) and (41) we get

$$\Phi_1 = -\,Q\,e^{\eta}\,\eta_1 \quad , \tag{42}$$

$$S_1{}^2 = \gamma(Q^2 - 1)\eta_1{}^2 \ , \tag{43}$$

where the function Q(r) is a combination of ρ, σ and s given by

$$(Q^2 - 1)s^2 - \gamma(\rho - Q\sigma)^2 = 0 \quad . \tag{44}$$

The interior solution of our system can then be specified by the two functions $\eta(r)$ and Q(r); having chosen these two functions we get Φ_1 and S_1 from (42) and (43), respectively ; next we obtain

$$4\pi\rho = (\eta_{11}+2\eta_1/r)e^{2\eta} - \Phi_1{}^2 \quad , \tag{45}$$

$$4\pi\sigma = -\,r^{-2}\,e^{3\eta}\,(r^2\,e^{-2\eta}\Phi_1)_1 \quad , \tag{46}$$

$$4\pi s = \gamma\, r^{-2}\, e^{2\eta}\, (r^2 S_1)_1 \quad . \tag{47}$$

We see from (43) that $Q^2(r)$ must be everywhere greater (less) than unity, for attractive (repulsive) scalar fields. And we see that the choice of $\eta(r)$ and $Q(r)$ are not completely arbitrary, since the density of mass $\rho(r)$ in (45) must be positive . Some small additional restrictions on η and Q will arise from the continuity conditions on the boundary of the sphere. One acceptable choice of $\eta(r)$ and $Q(r)$ will be given at the end of Section 5.

Bonnor's (1960) well known electrostatic solutions correspond to the particular case $Q^2=1$; his radial coordinate r' is related to our r by $r' = r \exp(-\eta)$. Another particular case is that of vanishing density of electric charge, a situation which can be represented by $Q = 0$; then (43) demands that $\gamma = -1$. Indeed, in absence of electrostatic repulsion a repulsive scalar field is required for balancing the gravitational attraction; this particular case was studied by Yilmaz (1958).

5. COMPLETE SOLUTIONS; ONE EXAMPLE

In matching the regular interior solutions with the asymptotically flat exterior solutions we impose the continuity of the fields g_{00}, g_{rr}, Φ, S, and of the radial derivatives dg_{00}/dr, $d\Phi/dr$, dS/dr on the boundary r_0 of the sphere.

The continuity of g_{00} and g_{rr} implies, from (29), (30) and (40), that $f = 0$, or equivalently $\alpha_e = -\eta_e$. Then from (33) we see that for attractive scalar fields ($\gamma = +1$) the parameter c must be immaginary, $c = i\, b$ (b real), and from (29) one notes that we must have $b^2 < q^2$ for $\gamma = +1$. And we see that for repulsive scalar fields ($\gamma = -1$) the parameter c is real. So if we define a positive constant m according to

$$m^2 = q^2 - \gamma b^2 \tag{48}$$

we have two different expressions for the exterior g_{00} metric coefficient:

$$g_{00} = \exp(2\eta_e) = \left[\cos b/r + (m/b)\sin b/r\right]^{-2} \quad (\gamma = +1) \quad , \tag{49}$$

$$g_{00} = \exp(2\eta_e) = \left[\cosh b/r + (m/b)\sinh b/r\right]^{-2} \quad (\gamma = -1), \tag{50}$$

in both cases $\gamma = \pm 1$ we have the exterior quantities

$$g_{rr} = r^{-2} g_{\theta\theta} = -\exp(-2\eta_e) \quad , \tag{51}$$

$$\Phi_{1e} = -q\, r^{-2} \exp(2\eta_e) \quad , \tag{52}$$

$$S_{1e} = \gamma\, b/r^2 \quad . \tag{53}$$

The continuity of g_{00} and dg_{00}/dr implies that the internal $\eta_i(r)$ used in (42) - (47) must satisfy

$$\eta_i(r_0) = \eta_e(r_0) \quad , \quad \eta_{1i}(r_0) = \eta_{1e}(r_0) \quad , \tag{54}$$

where $\eta_e(r)$ is that given in (49) or (50). Finally the continuity of $d\Phi/dr$ and of dS/dr on the boundary r_0 give both the same restriction, say from (42) and (52)

$$Q(r_0) = q\left[r_0^2\, \eta_{1e}(r_0)\right]^{-1} \exp\left[\eta_e(r_0)\right] \quad . \tag{55}$$

Since there is a number of restrictions on the possible functions $\eta_i(r)$ and $Q(r)$, we close this Section by giving the complete solution of one particular system: let the spherical distribution have $m = 5\varepsilon$ and $q = 4\varepsilon$, with $\varepsilon \ll 1$; we are considering units such that $G = c = 1$ and the radius $r_0 = 1$. Then from (48) we get $\gamma = -1$ (repulsive scalar field), and $b = \pm 3\varepsilon$. From (50) one obtains to the first order in ε

$$\eta_e(r) = -5\,\varepsilon/r \quad ,$$

and from (32) and (33)

$$\Phi_{1e} = -4\varepsilon/r^2 \quad , \quad S_{1e} = \mp\, 3\varepsilon/r^2 \; .$$

Let us choose the functions

$$\eta_i(r) = \lambda r^2 + \mu \quad , \quad Q(r) = \nu(r-1/2) \quad ,$$

with λ, μ, ν three constants such that the three boundary equations (54), (55) be satisfied; in our linear approximation we get from these equations

$$\eta_i(r) = 5(r^2-3)\varepsilon/2 \quad , \quad Q(r) = 4(2r-1)/5 \quad .$$

Then from (4.9) to (4.14) we obtain after trivial calculations

$$\Phi_{1i} = 4\varepsilon r(1-2r) \quad , \quad S_{1i} = \pm\varepsilon r(9+64r - 64r^2)^{1/2} \; ,$$

$$4\pi\rho = 15\varepsilon \quad , \quad 4\pi\sigma = 4\varepsilon(8r - 3) \quad ,$$

$$4\pi s = \pm 256\varepsilon(r-0.98)(r+0.108)(9+64r+64r^2)^{-1/2} .$$

One sees that in this example the mass density ρ is constant (to first order of approximation in ε), the density σ of electric charge changes sign nearly at $r=3/8$, the electric field $-\Phi_1$ changes the orientation at $r=1/2$, and the density s of scalar charge is everywhere finite and changes sign near the boundary, at $r = 0.98$.

6. DISCUSSIONS

One sees from the asymptotic expressions of (49) -(53) that the parameters m, q and b represent in the weak field approximation the mass, the electric charge and the scalar charge of the sphere, respectively. While in the absence of scalar charges one observes that $q^2 = m^2$ (with a relation between the corresponding densities, $\sigma^2 = \rho^2$), and in the absence of electric charge one has $b^2 = m^2$ (with the relation $s^2 = \rho^2$), in our general case one has $m^2 - q^2 + \gamma b^2 = 0$, but the relation (44) between the corresponding densities involves a somehow arbitrary function $Q(r)$; in the particular example of the end of Section 5 we had $\rho^2 - \sigma^2 - s^2 = 0$ only at the origin.

Also in connection with the relation $m^2 = q^2 - \gamma b^2$ one should remark that for attractive scalar fields ($\gamma = +1$) the parameter m can only be interpreted as a mass parameter when $b^2 < q^2$; a classical picture to see the origin of this result has already been tried by Teixeira et al. (1975).

In case of vanishing total scalar source ($b \to 0$) the two external metric coefficients (49) and (50) approach each other, giving as a limiting case.

$$ds^2 = (1+m/r)^{-2}(dx^0)^2 - (1+m/r)^2 \left[dr^2 + r^2(d\theta^2 + \sin^2\theta \; d\phi^2)\right] \quad ;$$

this result was already obtained by Papapetrou (1947), his radial coordinate r' being related to our r by $r' = r+m$.

The generation of the external spherically symmetric solution (50) - (53) according to recent prescriptions (Teixeira et al. 1975) presents an interesting peculiarity. One has to start from the Schwarschild line element

$$ds^2 = (1-2f/r')(dx^0)^2-(1-2f/r')^{-1}dr'^2-r'^2d\Omega^2,$$

next perform the coordinate transformation $1-2f/r' = \exp(-2f/r)$ and get

$$ds^2 = e^{-2f/r}(dx^0)^2 - e^{2f/r} \left[(f/r)^4\sinh^{-4}(f/r)dr^2+f^2\sinh^{-2}(f/r)d\Omega^2\right];$$

from this static spherically symmetric vacuum solution, the prescriptions lead to the generalized electrovac solutions (29) - (33). However, we have seen that the continuity conditions on the boundary of the sphere have imposed that $f=0$. Since f is the original Schwarzschild mass, we see that the original vacuum solution was indeed a flat solution. So we have had to use the mass f only to the extent required to the imposition of our particular spatial symmetry, the spherical one.

It is known (De and Raychaudhuri, 1968) that two distributions of incoherent dust charged only electrically, in static equilibrium, are insensitive to each other, in the sense that their mutual gravitational attraction balances the electrical repulsion. This is generally not the case when the dust has an additional scalar charge. Indeed, consider two spheres ($j=1,2$) with densities of mass $\rho_j(r)$, of electric charge $\sigma_j(r)$, and of scalar charge $s_j(r)$, and corresponding parameters m_j, q_j, b_j; the densities are such that each sphere would have an internal static equilibrium in the absence of the other. When these two spheres are put in the presence of each other, they will feel not only a mutual force proportional to $(m_1m_2 - q_1q_2 + \gamma b_1b_2)R^{-2}$: also the various fields originated by each sphere will destroy the original internal equilibrium of the other one, since in general $m_1\rho_2 - q_1\sigma_2 + \gamma b_1s_2 \neq 0$; This is a peculiarity of structures based purely on long range interactions; distributions containing short range fields (Teixeira et al. 1975a) are of a more difficult mathematical treatment, but allow the systems to have their internal structures more blinded against influences coming from neighbouring similar systems.

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REFERENCES

- Anderson J.L. 1967, Principles of Relativity Physics (New York: Academic Press); we use the notations and conventions of this book.

- Bonnor W.B. 1960, Z.F.Phys. 160,59-65.

- De U.K. and Raychaudhuri A.K., 1968, Proc. Roy. Soc. 303A,47-51.

- Papapetrou A.,1947, Proc. Roy. Irish Acad. 51A,191-204.

- Teixeira A.F. da F., Wolk I. and Som M.M., 1975, J. Phys. A(Static Electrovacs with Scalar Fields, to appear).

- Teixeira A.F. da F., Wolk I. and Som M.M., 1975a Phys. Rev. D (Tentative Relativistic Model for Neutral Yukawa Systems , to appear).

- Wolk, I., Teixeira A.F. da F. and Som M.M. , 1975, Lett. Nuovo Cim. 12, 319-20.

- Yilmaz, H.,1958, Phys. Rev. 111, 1417-25.