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AXIALLY SYMMETRIC FIELDS IN GENERAL RELATIVITY

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ABSTRACT

It is demonstrated that Einstein equations corresponding to stationary axially symmetric vacuum fields allow a special class of solutions admitting a simple interpretation - an observer describes the static Curzon field in the canonical space using a reference system which rotates with constant angular speed. The structure of the solutions remains similar if the interaction of massless scalar fields is considered.

I. INTRODUCTION

In general relativity there are some interesting static solutions of vacuum Einstein field equations, but stationary solutions are quite few. A special class of solutions was first given by Lewis[1] representing the field due to an infinite rotating cylinder in canonical space; one of the interesting features of the solutions is that these are linear combinations of static potentials due to an infinite cylinder, later obtained by Marder[2]. The most interesting exact solutions of the vacuum field equations corresponding to the exterior gravitational field due to a finite rotating body is that of Kerr[3].

The object of the present paper is to demonstrate that axially symmetric vacuum field equations allow a special class of solutions which are linear combinations of the static Curzon[4] potentials. A further generalisation is considered by including the interaction of the long range scalar field following the method of Janis[5].

II. BASIC EQUATIONS

Einstein field equations in empty space are

$$R^{\lambda}{}_{\mu\lambda\nu} = 0 \; ; \qquad (2.1)$$

we consider a stationary axially symmetric line element

$$ds^2 = f dx^{o^2} - e^{2\psi} (dr^2 + dz^2) - \ell d\theta^2 - 2m dx^o d\theta \qquad (2.2)$$

where f, ℓ, m and ψ are functions of (r, z) only. We shall number the coordinates (x^o, r, z, θ) as $(0, 1, 2, 3)$ respectively. Since each component of Ricci tensor vanishes, one can use a Weyl - like canonical coordinate system, that is (van Stockum[6])

$$f\ell + m^2 = r^2 \quad ; \tag{2.3}$$

in this coordinate system the field equations are

$$r\,\Delta\psi\delta_{ij} - \{\,{}^{1}_{ij}\,\} - \frac{1}{4r}(f_i\,\ell_j + f_j\,\ell_i + 2m_i\,m_j) = 0\ , \tag{2.4}$$

$$[(f\ell_i + mm_i)/r]_{,i} = 0 \qquad \text{and} \tag{2.5}$$

$$[(fm_i - mf_i)/r]_{,i} = 0 \quad , \tag{2.6}$$

where $i, j = 1, 2$ and $\Delta \equiv \partial^2/\partial r^2 + \partial^2/\partial z^2$. The repetition of indices denotes summation, f_i means $\partial f/\partial x^i$ etc.

III. SPECIAL SOLUTIONS

From the relation (2.3) one can write

$$f = \frac{r}{r_o}(\cosh\alpha - \sinh\alpha\ \cosh 2u)$$

$$\ell = rr_o(\cosh\alpha + \sinh\alpha\ \cosh 2u) \quad \text{and} \tag{3.1}$$

$$m = r\sinh\alpha\,\sinh 2u$$

where α and u are functions of (r, z) only, and r_o is a constant of dimension length; after some algebraic calculations the field equations can be reexpressed as

$$r\,\Delta\psi\delta_{ij} - \left\{ {1 \atop ij} \right\} - \frac{1}{2r}(\delta_i^1\,\delta_j^1 - r^2\,\alpha_i\,\alpha_j + 4r^2\,\sinh^2\alpha\,u_i\,u_j)=0, \quad (3.2)$$

$$(r\alpha_i)_i + 2r\,\sinh 2\alpha\,u_i\,u_i = 0 \quad \text{and} \quad (3.3)$$

$$(r\,\sinh^2\alpha\,u_i)_i = 0\,. \quad (3.4)$$

We shall consider here now two special cases:

Case 1) $u(r, z)$ = constant.

In this case (3.4) is identically satisfied and (333) reduces to

$$(r\,\alpha_i)_i = 0 \quad ; \quad (3.5)$$

one class of solutions of this equation is

$$\alpha = \log r/r_o + 2a/\rho \quad (3.6)$$

where $\rho = (r^2 + z^2)^{1/2}$ and r_o and a are constants. And one can easily verify that

$$\psi = \frac{a}{\rho}(1 - ar^2/2\,\rho^3) \quad (3.7)$$

is a solution of (3.2). If we define the constants

$$\omega = \tanh u \quad \text{and} \quad \gamma = (1 - \omega^2)^{-1/2} \quad (3.8)$$

we get from (3.1).

$$f = \gamma^2(e^{-2a/\rho} - \omega^2\,r^2\,e^{2a/\rho}/r_o^2\,)$$

$$\ell = \gamma^2(r^2\,e^{2a/\rho} - \omega^2\,r_o^2\,e^{-2a/\rho}) \quad \text{and} \quad (3.9)$$

$$m = \gamma^2 \omega r \, (r e^{2a/\rho} / r_o - r_o e^{-2a/\rho} / r) \; .$$

The effect of rotation is represented by the terms with ω. Since with $\omega = 0$ equations (3.7) and (3.9) represent the Curzon metric elements, the constant a can be interpreted as the mass of the Curzon particle.

If one makes the purely local transformation of coordinate differentials (Lewis)

$$dx^o = dx^{o'} \cosh u - r_o \, d\theta' \sinh u \quad \text{and}$$
$$r_o d\theta = r_o d\theta' \cosh u - dx^{o'} \sinh u \quad , \qquad (3.10)$$

the metric (2.2) reduces to the quadratic canonical form

$$ds^2 = e^{-2a/\rho} \, dx^{o'2} - e^{2a(1-ar^2/2\rho^3)/\rho} (dr^2+dz^2) - r^2 e^{2a/\rho} \, d\theta'^2 ; \qquad (3.11)$$

so when u = constant equations (3.10) and (3.11) suggest a very simple interpretation of the solution:an observer in the canonical space (r,z,θ) describes the static fields in the canonical space (r,z,θ') using a reference system which rotates with constant angular speed whose measure is given by $|\omega| < 1$. In this particular case, the static field corresponds to Curzon field.

Case 2) $u = u[\alpha(r,z)]$

In this case one obtains from (3.3) and (3.4)

$$d^2 u/d\alpha^2 + 2 \coth\alpha \; du/d\alpha - 2 \sinh 2\alpha (du/d\alpha)^3 = 0 \; ; \qquad (3.12)$$

the general solution of this equation is

$$u = u_o + \frac{1}{2} \log\left[\coth\alpha \mp (\cosh^2\alpha + K^2)^{1/2}\right] \qquad (3.13)$$

with u_o and K constants of integration.

We now define the function $\phi(u)$ by

$$d\phi/du = \pm 2K \sinh^2 \alpha \quad , \tag{3.14}$$

then we have from (3.13)

$$\phi = \log \left[K \cosh \alpha + (1 + K^2 \sinh^2 \alpha)^{1/2} \right] \tag{3.15}$$

apart from an additive constant of integration.

Equations (3.15) and (3.13) reduce (3.1) to

$$f = \frac{r/r_o}{1-\omega^2} (\beta_1^2 e^{-\phi} - \omega^2 \beta_2^2 e^{\phi}) \quad ,$$

$$\ell = \frac{rr_o}{1-\omega^2} (\beta_1^{-2} e^{\phi} - \omega^2 \beta_2^{-2} e^{-\phi}) \quad \text{and}$$

$$m = \frac{r\omega}{1-\omega^2} (\beta_2 e^{\phi}/\beta_1 - \beta_1 e^{-\phi}/\beta_2) \quad , \tag{3.16}$$

where the constants ω , β_1 and β_2 are related to u_o and K by

$$\beta_1^2 = (1-K)^2 [(1+K)e^{u_o} - e^{-u_o}][(1-K)e^{u_o} + e^{-u_o}]^{-1} \quad , \tag{3.17}$$

$$\beta_2^2 = (1-K)^{-2}[(1-K)e^{u_o} - e^{-u_o}][(1+K)e^{u_o} + e^{-u_o}]^{-1} \quad \text{and} \tag{3.18}$$

$$\omega^2 = [(1-K^2) e^{2u_o} - e^{-2u_o} - 2K] [(1-K^2) e^{2u_o} - e^{-2u_o} + 2K]^{-1} \; ; \tag{3.19}$$

with (3.16), equation (2.4) reads now

$$r\,\Delta\psi\delta_{ij} - (\delta_i^1\,\delta_j^1 - \delta_i^2\,\delta_j^2)\psi_1 - (\delta_i^1\,\delta_j^2 + \delta_i^2\,\delta_j^1)\psi_2 + \frac{1+\omega^2}{2r(1-\omega^2)}(\delta_i^1\,\delta_j^1 - r^2\phi_i\,\phi_j) = 0. \tag{3.20}$$

From (3.4) and (3.14) we get

$$(r\,\phi_i)_i = 0\ ; \tag{3.21}$$

a special class of solutions of this equation is

$$e^{\phi} = \frac{r}{r_o}\,e^{2a/\rho} \tag{3.22}$$

and then (3.20) give

$$\psi = \frac{1+\omega^2}{1-\omega^2}\,\frac{a}{\rho}\left(1-\frac{ar^2}{2\rho^3}\right) \tag{3.23}$$

apart from an additive constant. Solutions (3.7) to (3.9) and (3.16) to (3.23) are similar in structure.

One can transform the metric (2.2) with elements (3.16) and (3.23) into the fundamental quadratic form

$$ds^2 = \frac{r}{r_o}\,e^{-\phi}\,dx^{o\,\prime 2} - e^{2\psi}(dr^2 + dz^2) - r_o\,r\,e^{\phi}\,d\theta^{\prime 2} \tag{3.24}$$

with

$$x^o = \gamma\beta_1^{-1}\, x^{o'} - \omega\gamma\beta_2^{-1}\, r_o\, \theta' \qquad \text{and}$$

$$r_o\theta = \gamma\beta_1\, r_o\, \theta' - \omega\gamma\beta_2\, x^{o'} \quad , \tag{3.25}$$

where as before $\gamma = (1 - \omega^2)^{-1/2}$.

IV. GRAVITATIONAL FIELD COUPLED WITH MASSLESS SCALAR FIELD

The interaction of massless scalar field with gravitational field can be easily obtained following the method prescribed by [redacted] Janis et al. for the static case. In presence of massless scalar field V Einstein equations take the form

$$R^{\mu}_{\nu} = - \kappa\, V^{\mu}\, V_{\nu} \tag{4.1}$$

where $\kappa = 8\,\pi G/c^4$. Since the only surviving components of $V_{\mu}(r,z)$ are V_i where $i = 1,2$, we have in addition to equations (3.3) and (3.4)

$$r\Delta\psi\, \delta_{ij} - \{ {}^{1}_{ij} \} - \frac{1}{2r}\,(\delta^1_i\, \delta^1_j - r^2\, \alpha_i\, \alpha_j + 4\, r^2\, \sinh^2\alpha\;\; u_i\, u_j) = - \kappa(-g)^{-1/2}\, V^i V_j . \tag{4.2}$$

Now we define

$$\varphi = 2a\; A/\rho \tag{4.3}$$

where $A^2 = 1 + \kappa B^2/2$ and B is a constant such that

$$V = - Ba/\rho \quad ; \tag{4.4}$$

since $(r\,\varphi_i)_i = 0$, φ is also a solution of (3.5) when u = const, so that we have now

$$\alpha = \varphi + \log r/r_o \tag{4.5}$$

and

$$\psi = \frac{a}{\rho}\left(A - \frac{ar^2}{2\rho^3}\right) \; ; \tag{4.6}$$

the remaining metric elements then take the form

$$f = \gamma^2 (e^{-2aA/\rho} - \omega^2 r^2 \, e^{2aA/\rho}/r_o^2) \quad ,$$

$$\ell = \gamma^2 (r^2 \, e^{2aA/\rho} - \omega^2 r_o^2 \, e^{-2aA/\rho}) \quad \text{and}$$

$$m = \gamma^2 \omega r (r e^{2aA/\rho}/r_o - r_o \, e^{-2aA/\rho}/r) \; . \tag{4.7}$$

For $\omega = 0$ the expressions (4.7) reduce to the metric elements obtained by Gautreau[7] for axially symmetric static Curzon field coupled with massless scalar field.

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