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by

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CONVOLUTION OPERATORS IN SPACES OF NUCLEARLY ENTIRE FUNCTIONS ON A BANACH SPACE * **

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We shall be concerned with the theorem stated below. Its proof depends on the two propositions indicated afterwards.

Let us start by explaining some of the pertinent notation and terminology. For additional information on the background material, we refer the reader to the bibliography quoted at the end of this article.

We shall be dealing with a complex Banach space E. For each positive integer m = 0, 1, ..., $\mathcal{P}(^m E)$ will denote the Banach space of all continuous m-homogeneous complex-valued polynomials on E. Moreover, $\mathcal{H}(E)$ will represent the vector

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space of all complex-valued functions on E which are holomorphic on the entire E. For each $f \in \mathcal{H}(E)$, we have its Taylor series at the origin

$$f(x) = \sum_{m=0}^{\infty} \frac{1}{m!} \hat{d}^m f(0)(x)$$

for every $x \in E$, and the corresponding differentials of order $m = 0, 1, \ldots$

$$\hat{d}^m f(0) \in \mathcal{P}(^m E) .$$

If E' indicates the dual Banach space to E, we shall have that $\varphi^m \in \mathcal{P}(^m E)$ for every $\varphi \in E'$. We shall denote by $\mathcal{P}_f(^m E)$ the vector subspace of $\mathcal{P}(^m E)$ generated by all φ^m when φ runs over E'. It consists of those elements of $\mathcal{P}(^m E)$ each of which may be represented as a finite sum

$$\varphi_1{}^m + \ldots + \varphi_r{}^m ,$$

where the φ_j belong to E' for each $j = 1, \ldots, r$. An element of $\mathcal{P}(^m E)$ is said to be of finite type in case it lies in $\mathcal{P}_f(^m E)$.

The Banach space $\mathcal{P}_N(^m E)$ of all nuclear m-homogeneous complex -valued polynomials on E is characterized by the following requirements:

(1) $\mathcal{P}_N(^m E)$ is a vector subspace of $\mathcal{P}(^m E)$;

(2) $\mathcal{P}_N(^m E)$ is a Banach space with respect to a norm denoted by $\| \cdot \|_N$ and called the nuclear norm; it is to be distinguished from the current norm on $\mathcal{P}(^m E)$ which is denoted simply by $\| \cdot \|$;

(3) $P_f(^m E)$ is contained and dense in $\mathcal{P}_N(^m E)$ with respect to the nuclear norm;

(4) For each $P \in \mathcal{P}_f(^mE)$, its nuclear norm $\|P\|_N$ is equal to the infimum of the sums

$$\|\varphi_1\|^m + \dots + \|\varphi_r\|^m$$

for all possible representations

$$P = \varphi_1^m + \dots + \varphi_r^m ,$$

where the φ_j belong to E' for each j = 1, ..., r.

A nuclear complex-valued polynomial on E is by definition a complex-valued polynomial on E all of whose homogeneous components are nuclear in the above sense.

A nuclear complex-valued exponential-polynomial on E is defined to be a function on E of the form $P e^{\varphi}$, where P is a nuclear complex-valued polynomial on E and $\varphi \in E'$.

In order to introduce the locally convex space $\mathcal{H}_N(E)$ of all nuclearly entire complex-valued functions on E, let us make the following preliminary considerations.

For every norm α on E which is equivalent to the one originally given on the same vector space, and every subset X of W, we shall say for short that α is X-centered if X is contained in the open ball with respect to α of center at 0 and radius equal to 1.

We shall define $f \in \mathcal{H}(E)$ to be nuclearly entire in case we have

$$\hat{d}^m f(0) \in \mathcal{P}_N(^mE)$$

for each m = 0, 1, ..., and, corresponding to every compact subset K of E, there exists an equivalent norm α on E which is K-centered and is such that

$$\sum_{m=0}^{\infty} \left\| \frac{1}{m!} \hat{d}^m f(0) \right\|_{N\alpha} < +\infty ;$$

here and in the sequel $\| \cdot \|_{N\alpha}$ stands for the nuclear norm of a nuclear homogeneous complex-valued polynomial on E when this vector space is endowed with α rather than its originally given norm. We shall denote by $\mathcal{H}_N(E)$ the vector subspace of $\mathcal{H}(E)$ of all nuclearly entire complex-valued functions on E.

For the purpose of describing the natural locally convex topology on $\mathcal{H}_N(E)$ that we shall use, let us introduce the following concepts.

A seminorm p on the vector space $\mathcal{H}_N(E)$ is said to be nuclearly ported by a compact subset K of E provided that, corresponding to every equivalent norm α on E which is K-centered, there exists some real number $c(\alpha) > 0$ for which the following estimate

$$p(f) \leqslant c(\alpha) \cdot \sum_{m=0}^{\infty} \left\| \frac{1}{m!} \hat{d}^m f(0) \right\|_{N\alpha}$$

holds true for an arbitrary $f \in \mathcal{H}_N(E)$. We notice, as it is standard in similar situations of seminorms ported by compact subsets, that the right-hand side of the above estimate is not necessarily always finite. However, once $f \in \mathcal{H}_N(E)$ and the compact subset K of E are given, there exists some equivalent norm α on E for which the mentioned right-hand side turns to be finite; for such a choice of α the indicated estimate will give us an information on p(f), hence on the seminorm p.

The locally convex topology $\mathcal{J}_{\omega N}$ on $\mathcal{H}_N(E)$ that we shall use is the one defined by all seminorms on $\mathcal{H}_N(E)$ each of which is nuclearly ported by some compact subset of E.

A convolution operator $\mathcal{O}$ in $\mathcal{H}_N(E)$ is defined to be a continuous linear mapping of $\mathcal{H}_N(E)$ into itself commuting with all translations in E, that is $\mathcal{O}\tau_t = \tau_t \mathcal{O}$ for all $t \in E$; here the translation mapping τ_t of $\mathcal{H}_N(E)$ into itself is defined by $(\tau_t f)(x) = f(x-t)$ for all $x \in E$ and an arbitrary $f \in \mathcal{H}_N(E)$, where $t \in E$. Such a convolution operator is actually a constant coefficient linear differential operator of finite or infinite order acting in $\mathcal{H}_N(E)$.

THEOREM: The vector subspace $\mathcal{O}^{-1}(0)$ on which a convolution operator $\mathcal{O}$ in $\mathcal{H}_N(E)$ does vanish is the closure of its vector subspace generated by the nuclear exponential-polynomials that it contains.

As it is standard in this type of an approximation result, the proof of the theorem is reduced via the Hahn-Banach theorem to two propositions: one of them concerns a characterization of Borel transforms of all the elements in the dual space $\mathcal{H}'_N(E)$ to $\mathcal{H}_N(E)$; and the other one refers to a division property between such Borel transforms. We pass now to a description of these propositions.

If $T \in \mathcal{H}'_N(E)$, that is T is a continuous linear form on $\mathcal{H}_N(E)$, its Borel transform $\hat{T}$ is the complex-valued function on

E' defined by

$$\hat{T}(\varphi) = T(e^{\varphi})$$

for every $\varphi \in E'$. The mapping $T \longrightarrow \hat{T}$ is linear and one-to-one. In order to characterize its image set, we shall introduce the following concepts.

Let α be a seminorm on the vector space E. Denote by E_α the completion of the normed space associated to E when this vector space is seminormed by α. We shall say that α is compact in case the natural linear mapping $E \longrightarrow E_\alpha$ is compact. This means that the closed ball in E with respect to the norm originally given in E of center at 0 and radius equal to 1 is totally bounded with respect to α; that is, given any $\varepsilon > 0$, there are $x_1, \ldots, x_r \in E$ such that, for every $x \in E$ for which $\| x \| \leqslant 1$, we can find some $j = 1, \ldots, r$ satisfying $\alpha(x_j - x) < \varepsilon$.

An entire function $f \in \mathcal{H}(E)$ is said to be of compact exponential type in case there exists a compact seminorm α on E such that, for every $\varepsilon > 0$ we can find a real number $c(\varepsilon) \geqslant 0$ for which the following estimate

$$|f(x)| \leqslant c(\varepsilon) . \exp[\alpha(x) + \varepsilon . \| x \|]$$

holds true for every $x \in E$.

In case, however, instead of having an entire function on E, we are dealing with an entire function on E', as it will be precisely our case, we will have the following more stringent notion besides the already acquired one of an entire function of compact exponential type on E'. Letting α denote now a seminorm on E', denote by E'_α the completion of the normed space associated to E' when this vector space is seminormed by α. In case the natural linear mapping $E' \to E'_\alpha$ is continuous, that is α is continuous, we may consider the continuous linear transpose mapping $(E'_\alpha)' \to E''$, which is actually one-to-one in our case, where $(E'_\alpha)'$ stands for the dual Banach space to E'_α and E'' represents the double dual Banach space to E. We shall say that α is E-compact if not only α is compact, that is $E' \to E'_\alpha$ is compact, hence continuous, but in addition the transpose mapping $(E'_\alpha)' \to E''$ which is necessarily compact does map $(E'_\alpha)'$ into the natural image of E in E''; that is, Φ being any α-continuous linear form on E', there is a necessarily unique $x \in E$ such that $\Phi(\varphi) = \varphi(x)$ for every $\varphi \in E'$.

An entire function $F \in \mathcal{H}(E')$ is said to be of E-compact exponential type in case there exists an E-compact seminorm α on E' such that, for every $\varepsilon > 0$ we can find a real number $c(\varepsilon) \geqslant 0$ for which the following estimate

$$|F(\varphi)| \leqslant c(\varepsilon).\exp[\alpha(\varphi) + \varepsilon\|\varphi\|]$$

holds true for every $\varphi \in E'$.

PROPOSITION 1. <u>A complex-valued function</u> F <u>on</u> E' <u>is the Borel transform</u> $\hat{T}$ <u>of some continuous linear form</u> T <u>on</u> $\mathcal{H}_N(E)$ <u>if and only if</u> F <u>is an entire function of</u> E-<u>compact exponential type on</u> E'.

PROPOSITION 2. F_1, F_2 <u>and</u> F_3 <u>being entire complex-valued functions on</u> E' <u>such that</u> $F_1 = F_2F_3$ <u>and</u> F_2 <u>is not identically</u>

zero, then F_3 *will be of* E-*compact exponential type along with* F_1 *and* F_2.

Gupta's previous work was concerned with results analogous to the preceding theorem and propositions for convolution operators in the Frechet space of all nuclearly entire complex-valued functions of bounded type $\mathcal{H}_{Nb}(E)$. The results indicated in the present note were obtained from the bounded nuclear case by a sort of an inductive limit process.

There are other natural possible candidates for the concepts of nuclearly entire functions and of nuclearly entire functions of bounded type. It is not yet known whether they are equivalent to the definitions given here or in Gupta's work, and whether results similar to those we proved for the considered versions of $\mathcal{H}_{Nb}(E)$ and of $\mathcal{H}_N(E)$ can also be established for such alternative concepts.

The same kind of theory should be developped for the spaces $\mathcal{H}(E)$ of all entire complex-valued functions on E, and $\mathcal{H}_b(E)$ of all entire complex-valued functions of bounded type on E, that is for the current holomorphy type.

Once these various cases are settled, there will be hope of establishing a similar theory by means of the concept of a holomorphy type Ⓗ to collect $\mathcal{H}_N(E)$ and $\mathcal{H}(E)$ into $\mathcal{H}_{Ⓗ}(E)$, and to collect $\mathcal{H}_{Nb}(E)$ and $\mathcal{H}_b(E)$ into $\mathcal{H}_{Ⓗb}(E)$.

Finally, through the use of weights on E, it should become possible to collect $\mathcal{H}_{Ⓗ}(E)$ and $\mathcal{H}_{Ⓗb}(E)$ into a single type of a locally convex space of entire complex-valued functions on E leading to a synthesis of these various aspects of the theory.

Detailed proofs of the above mentioned results will appear elsewhere in a joint paper with Gupta.

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