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SOME PROBLEMS IN EXTENDING AND LIFTING CONTINUOUS LINEAR TRANSFORMATIONS *

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Our starting point is the classical Hahn-Banach theorem, which became a simple and important tool in Functional Analysis and its applications. In one of its different forms, it asserts that, if φ is a continuous linear form defined on a vector subspace S of a real or complex normed vector space E, then φ can be extended to a continuous linear form ϕ defined on all of E, with the same norm, that is $\|\phi\| = \|\varphi\|$. This result can also be stated as a separation theorem for convex sets, in various equivalent ways which will not be considered here. There arises, naturally, the question as to whether the Hahn-Banach theorem generalizes to continuous linear transformations, instead of continuous linear forms. It has been known for many years that this is not the case, in general. A Banach space E, indeed, need not have a continuous projection into one of its closed vector subspace S, meaning that the identity map of S may fail to have a continuous linear extension to a map from E into S, as shown by Banach and Mazur, Fichtenholtz and Kantorovich, Murray and other authors. We are, then, led

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to the following question. Given two Banach spaces E and F and a continuous linear transformation φ defined on a vector subspace S of E, which we may assume to be closed, having its values in F, when does there exist a continuous linear extension Φ of φ to E, with values in F and the same norm, that is $\|\Phi\| = \|\varphi\|$? In such a case, we shall say shortly that (E, F, S, φ) has the extension property. Due to the lack of an interesting answer to this problem, some variations of it have been considered by different authors, who investigated this extension problem by not holding E, F, S and φ all as given data. Analogous situations have occured in the study of extension problems pertinent to Algebra and Topology which lead to the notion of injective objects in a given category. The dual notion of projective objects arises in the dual situation of lifting problems.

Given a Banach space F, the problem of determining necessary and sufficient conditions in order that (E, F, S, φ) should have the extension property for any E, S and φ, that is, for any Banach space E, any closed vector subspace S of E and any continuous linear transformation φ of S into F, there should exist a continuous linear extension Φ of φ to E with values in F and the same norm $\|\Phi\| = \|\varphi\|$, was first solved in the real case by the author, Goodner and Kelley, and in the complex case by Hasumi. We shall then say shortly that F the extension property of type ∞. The classical Hahn-Banach theorem means, of course, that F has the extension property of type ∞ if its dimension is 1. A real or complex Banach space F has the extension property of type ∞ if and only if it has the projection property consisting in that, for any Banach space E containing F as a Banach subspace, there is a projection of norm 1 of E into F, provided of course $F \neq 0$. For real Banach spaces, it has been known for a long time, as an easy and natural generalization of the method for proving the classical real Hahn-Banach theorem, that if F is a complete vector lattice with a metric order unit e, that is an element $e \in F$, $e > 0$, such that $-\lambda e \leqslant x \leqslant \lambda e$ is equivalent to $\lambda \geqslant \| x \|$ for any $x \in F$, then F has the extension property of type ∞. Such a metric order unit is unique. Moreover, the metric order unit e is an extreme of the unit closed ball $U = \left\{ x; x \in F, \| x \| \leqslant 1 \right\}$ of F, meaning that it is impossible to write $e = \lambda a + (1 - \lambda)b$, with $a, b \in U$, $a \neq e$, $b \neq e$, $0 < \lambda < 1$. It turns out that this apparently special situation covers the general case of the extension property of type ∞ for real Banach spaces, in the following sense. If a real Banach space F has the extension property of type ∞ and e is an extreme point of U, then there is one and only one may of making F into a complete vector lattice so that e becomes a metric order unit. Actually the order relation on F thus associated to e is described explicitly by defining $x \geqslant 0$ when we can write $x = \lambda (e + u)$ with $\lambda \geqslant 0$ real and $u \in U$. The question of the existence of at least one extreme point in the unit closed ball of a real Banach space having the extension property of type ∞ was first settled affirmatively by Kelley in an indirect way and later by Aronszajn and Panitchpakdi by a direct proof. Anyhow this seems to be one exceptional case of existence of at least one, and actually of sufficiently many extreme points in a closed convex set, which does not follow obviously from the Krein-Milman theorem in the sense that one fails to known how to endow the vector space at hand with a separated locally convex topology under which the aforesaid set becomes compact. Since the extreme point e of U is not unique and so the order relation well defined by each e is not unique, the preceding characterization of those real Banach spaces having the extension property of type ∞ involves a certain arbitrariness. A more direct characterization is as follows. A collection $\mathscr{C}$ of non empty sets is said to have the binary intersection property when, given any subcollection of $\mathscr{C}$ such that any two members of it intersect, it follows that all members of this subcollection have a non empty intersection. Then, a necessary and sufficient condition for a real Banach space F to have the extension property of type ∞ is that the collection of all closed balls, with arbitrary centers and radii, of F should have the binary intersection property. From this point of view, the validity of the classical real Hahn-Banach theorem amounts to the fact that the collection of all non empty compact intervals in the real line R has the binary intersection property. A Banach space F with the extension property of type ∞ has a simple functional representation. In fact, if K denotes a compact space and $\mathscr{C}(K, R)$ is the Banach space of all real continuous functions on K, then $\mathscr{C}(K, R)$ is a complete vector lattice if and only K is a stonean space or an extremally disconnected space, meaning that the closure of every open set is again open. Since the unit function e on K is a metric order unit for $\mathcal{C}(K, R)$, we conclude that $\mathcal{C}(K, R)$ has the extension property of type ∞ if K is stonean. Conversely, if $\mathcal{C}(K, R)$ has the extension property of type ∞, then K is stonean, as it follows immediately from the binary intersection property characterization. The classical real Hahn-Banach theorem corresponds to the case in which K is reduced to a single point. More generally, a real Banach space F has the extension property of type ∞ if and only if F is metrically isomorphic to the Banach space $\mathcal{C}(K, R)$ corresponding to a suitable stonean compact space K, which is uniquely determined by F up to homeomorphisms. Different proofs of this functional representation theorem exist in the litterature. Let, indeed F be a real Banach space with the extension property of type ∞. Firstly, if e is an extreme point of the unit closed ball U of F, then by making F into a complete vector lattice with e as a metric order unit, F becomes an (M)-space in the sense of Kakutani, which therefore is orderly and metrically isomorphic to a Banach space $\mathcal{C}(K, R)$, where the compact space K has to be stonean since F and hence $\mathcal{C}(K, R)$ are complete vector lattices. Secondly, a direct proof of such a metric isomorphism of F to $\mathcal{C}(K, R)$ on a stonean compact space K was provided by Kelley, without a priori using existence of an extreme point on the unit closed ball U of F and thus proving such an existence as a by-product of the functional representation. As shown by Hasumi in an interesting way, similar results hold in the complex case. A complex Banach space F has the extension property of type ∞ if and only if F is metrically isomorphic to the

Banach space $\mathscr{C}(K, C)$ of all complex continuous functions on a suitable stonean compact space K, which is uniquely determined by F up to homeomorphisms. Stonean spaces bear a natural relationship to complete boolean algebras. The open-closed subsets of a topological space K form a boolean algebra a(K). Conversely, as Stone showed, every boolean algebra A is isomorphic to the boolean algebra a(K) of a suitable totally disconnected compact space K, which is uniquely determined by A up to homeomorphisms. Moreover, A is a complete boolean algebra if and only if the representation space K is a stonean space. Hence, there are as many real or complex Banach spaces with the extension property of type ∞, up to metric isomorphisms, as there are stonean compact spaces, up to homeomorphisms, or complete boolean algebras, up to isomorphisms. The easy case in which F is finite dimensional deserves one word. If $n = \dim F < \infty$, then F has the extension property of type ∞ if and only if F has a basis $e_1, \ldots, e_n$, whose elements are unique appart from their order and signs, such that $\|x\| = \max\{|x_1|, \ldots, |x_n|\}$ if $x = \sum x_1 e_1 \in F$, that is if and only if the balls in F are parallelepipeds.

Given a Banach space S, there arises the problem of determining when, for every Banach space E containing S as a Banach subspace, any Banach space F and any continuous linear transformation φ of S into F, there should exist a continuous linear extension Φ of φ to E with values in F and the same norm $\|\Phi\| = \|\varphi\|$. The answer is that S should have the extension property of type ∞, so that this problem leads to the same category of Banach spaces as the previous one.

Given a Banach space E, the problem of determining necessary and sufficent condition in order that, for any Banach space F, any closed vector subspace S of E and any continuous linear transformation φ of S into F, there should exist a continuous linear extension ϕ of φ to E with values in F and the same norm $\|\phi\| = \|\varphi\|$, was first solved in the real case by Kakutani. It was also treated in the real case by Phillips and solved in the complex case by Bohnenblust. We shall say shortly that E has the extension property of type 2. Then it turns out that a necessary and sufficent condition that E should have the extension property of type 2 is that E be a Hilbert space in case E has dimension at least equal to 3, since E always has the extension property of type 2 when its dimension is at most equal to 2. This is actually a problem in dimension 3, in the sense that the general case dim $E \geqslant 3$ reduces to the special case dim $E = 3$. A related problem is as follows. Let us denote by $\mathcal{L}(S, F)$ and $\mathcal{L}(E, F)$ the Banach spaces of all continuous linear transformations of S and E into F, respectively. Then we have the natural restriction map of $\mathcal{L}(E, F)$ into $\mathcal{L}(S, F)$. The problem is then that of, given a Banach space E, to determine necessary and sufficient conditions in order that, for any Banach space F and any closed vector subspace S of E, there should exist a metric isomorphism of $\mathcal{L}(S, F)$ into $\mathcal{L}(E, S)$ such that the composition $\mathcal{L}(S, F) \rightarrow \mathcal{L}(E, F) \rightarrow \mathcal{L}(S, F)$ is the identity, in other words, $\mathcal{L}(S, F)$ gets identified to a Banach subspace of $\mathcal{L}(E, F)$ in such a way that the restriction map becomes a projection into that subspace. This is a problem of simultaneous extensions type, but it also can be looked as a lifting problem. As Kakutani showed, it is necessary and sufficient that E be a Hilbert space, under the same restriction on the dimension and for the same reason.

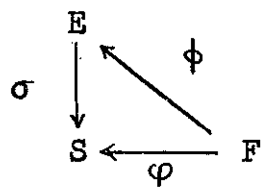
The extension problem has as its dual the lifting problem, firstly studied by Grothendieck in a systematic way in the case of linear continuous transformations in Banach spaces. In an equivalent but slightly more general form, the previously considered extension problem can be rephrased by considering three Banach spaces E, F and S, a metric isomorphism σ of S into E (which was the identity in the previous case), a continuous linear transformation φ of S into F and then asking for a continuous linear transformation ϕ of E into F such that $\phi\,\sigma = \varphi$ and $\|\phi\| = \|\varphi\|$. More generally, we may say that

(ext)
$$\begin{array}{ccc} E & & \\ \sigma\uparrow & \searrow^{\phi} & \\ S & \xrightarrow[\varphi]{} & F \end{array}$$

(E, F, S, σ) has the extension property when the natural mapping of $\mathcal{L}$(E, F) into $\mathcal{L}$(S, F) given by $\phi \longrightarrow \phi\,\sigma = \varphi$ is a metric homomorphism of $\mathcal{L}$(E, F) into $\mathcal{L}$(S, F), meaning that, if η is the closed vector subspace of $\mathcal{L}$(E, F) where this homomorphism vanishes, then the natural mapping of $\mathcal{L}(E, F)/\eta$ into $\mathcal{L}$(S, F) is a metric isomorphism between these two Banach spaces, which amounts to saying that given any continuous linear transformation

φ of S into F and $\varepsilon > 0$, there exists a continuous linear transformation ϕ_ε of E into F such that $\phi_\varepsilon \sigma = \varphi$ and $\|\phi_\varepsilon\| < \|\varphi\| + \varepsilon$. The previously used strict definition of the extension property corresponds to having a strict metric homomorphism, that is to having $\|\phi\| = \|\varphi\|$ in place of $\|\phi_\varepsilon\| < \|\varphi\| + \varepsilon$. Since the concept of a metric isomorphism σ of S into E has as its dual the concept of a metric homomorphism σ of E into S, we are led to the following situation. Let E, F and S be three Banach spaces and σ be a metric homomorphism of E into S. Then we say that (E, F, S, σ) has the lifting property when the natural mapping of $\mathcal{L}(F, E)$ into $\mathcal{L}(F, S)$ given by $\phi \longrightarrow \sigma\phi = \varphi$ is a metric homomorphism of $\mathcal{L}(F, E)$ into $\mathcal{L}(F, S)$. This means that, given any continuous linear transformation φ of F into S and $\varepsilon > 0$, there exists a continuous linear transformation ϕ_ε of F into E

(lift)



such that $\sigma\phi_\varepsilon = \varphi$ and $\|\phi_\varepsilon\| < \|\varphi\| + \varepsilon$. Easy examples show us that we cannot expect in general to have a strict metric homomorphism in this ample definition, that is to replace $\|\phi_\varepsilon\| < \|\varphi\| + \varepsilon$ by $\|\phi\| = \|\varphi\|$. The situation described by the diagram (ext) leads to the diagram (lift) if we pass to the dual spaces and transposed maps. Conversely, the diagram (lift) gets transformed into the diagram (ext) by passage to the dual spaces and transposed maps. However, of course, the two situations are not freely equivalent because not every Banach space is a dual space and not every continuous linear transformation between Banach spaces is a transposed transformation. In the finite dimensional case, the (ext) and (lift) diagrams are truly equivalent through duality and transposition and such a remark shows its usefulness even when we treat the general case. The problem of, given a real Banach space F, to determine necessary and sufficient conditions in order that (E, F, S, σ) should have the lifting property for any E, S and σ, in the ample sense, not in the strict sense, was first considered by Kothe and Grothendieck, who proved that F has such a lifting property if and only if F is metrically isomorphic to the Banach space $\ell^1(I)$ of all summable real functions on a suitable set I, whose cardinal number is uniquely determined by F. Then we say that F has the lifting property of type 1. A Banach space F has the lifting property of type 1 if and only if it has the metric homomorphism property consisting in that, for any Banach space E having a metric homomorphism σ into F and any $\varepsilon > 0$, there is a continuous linear transformation ϕ_ε of F into E such that $\sigma \phi_\varepsilon$ is the identity mapping of F and $\|\phi_\varepsilon\| < 1 + \varepsilon$. Some interesting variations of the lifting problem were encountered by Grothendieck in the theory of metric and topological tensor products of Banach and topological vector spaces. Given a real Banach space F, then the following conditions on F are equivalent (1) for any real Banach space X and any closed vector subspace Y of X, the natural mapping $F \hat{\otimes} Y \longrightarrow F \hat{\otimes} X$ is a metric isomorphism of the first space into the second one; (2) the dual Banach space F' has the extension property of type ∞; (3) for any real Banach space X and any closed vector subspace Y of X, every continuous linear transformation φ of F into $S = X'/Y^{\perp}$ can be lifted to a continuous linear transformation ϕ of F into $E = X'$ with the same norm $\|\phi\| = \|\varphi\|$; (4) for any real Banach space X and any closed vector subspace Y of X, every continuous linear transformation φ of F into $S = X''/Y^{\perp\perp}$ can be lifted to a continuous linear transformation ϕ of F into the bidual X'' with the same norm $\|\phi\| = \|\varphi\|$; (5) F is metrically isomorphic to a space $L^1(\mu)$ of all real integrable functions with respect to a suitable positive measure μ on a locally compact space, which is not unique. In the statements of the preceding conditions, it is sufficient to assume that X and Y are finite dimensional and then (3) and (4) become identical.

Given a Banach space S, there arises the problem of determining when, for every Banach space E having a metric homomorphism σ into S, any Banach space F, any continuous linear transformation φ of F into S and $\varepsilon > 0$, there should exist a continuous linear transformation ϕ_ε of F into E such that $\sigma\phi_\varepsilon = \varphi$ and $\|\phi_\varepsilon\| < \|\varphi\| + \varepsilon$. The answer is that S should have the lifting property of type 1, so that we fall back in the same category of Banach spaces as in a previous situation.

We finish the present exposition by listing some as yet unsettled problems:

(1) The classical result concerning a compact convex set in a separated locally convex topological vector space being

the closed convex hull of its set of extreme points is the Krein-Milman theorem. On the other hand, the Kelley-Aronszajn-Panitchpakdi work shows us that, if F is a separated locally convex topological vector space and K is a closed convex bounded subset of F such that the collection of sets in the family $\{\lambda K + a\}$ for $\lambda \in R$, $\lambda > 0$ and $a \in F$ has the binary intersection property, then K is the closure of the convex hull of the set of its extreme points. Since compactness and the binary intersection property have some features in common, there should exist a nice result containing both the Krein-Milman theorem and the Kelley-Aronszajn-Panitchpakdi work.

(2) If E and F are locally convex topological vector spaces and φ is a continuous linear transformation defined on a vector subspace S of E with values in F, there arises the problem of determining when φ has a continuous linear extension ϕ to E with values in F. We thus have a topological extension problem parallel to the metric extension problem considered above. The question of studying the topological extension property and a similarly defined topological lifting property has not been completely clarified. In particular, if F is a Banach space which has the topological extension property for any Banach space E, any closed vector subspace S of E and any continuous linear transformation of S into F, can F be renormed so as to have the metric extension property of type ∞? A cartesian product of real or complex lines and, more generally, of Banach spaces with the metric extension property of type ∞, is a simple example of a topological

vector space F having the topological extension property for any E, S and φ. What about the converse ? Dual questions for the lifting problem.

(3) If F is a real Banach space whose closed balls have extreme points and whose collection of closed balls has the weak binary intersection property, meaning that if any two members of a finite subcollection intersect then this finite subcollection has a non empty intersection, is then F metrically isomorphic to some $\mathcal{C}(K, R)$ with K compact ?

(4) Let F be a real Banach space such that for any Banach space X and any closed vector subspace Y of X the natural mapping $F \check{\otimes} X \longrightarrow F \check{\otimes} X/Y$ is a metric homomorphism of the first space into the second one. This is equivalent to the dual Banach space F' being metrically isomorphic to some $L^1(\mu)$; or to the bidual Banach space F" having the extension property of type ∞; or to any continuous linear transformation φ of a closed vector subspace S of a real Banach space E into F having a continuous linear extension Φ to E with values in F", so that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\;\Phi\;} & F'' \\ \uparrow & & \uparrow \\ S & \xrightarrow[\;\varphi\;]{} & F \end{array}$$

is commutative, and with the same norm $\|\Phi\| = \|\varphi\|$; or to any compact linear transformation φ of a closed vector subspace S of a real Banach space E into F having a compact linear extension

ϕ_ε to E with values in F and norm $\|\phi_\varepsilon\| < \|\varphi\| + \varepsilon$ for any given $\varepsilon > 0$. The question of a suitable functional representation for such a space F as insinuated by Grothendieck is open.

(5) Given a linear transformation T on the Banach space X with values in the Banach space Y, when do there exist Banach spaces E and F, a closed vector subspace S of E and metric isomorphisms α and β of $\mathcal{L}(E, F)$ and $\mathcal{L}(S, F)$ into X and Y, respectively, such that $\beta^{-1}T\alpha$ is the natural restriction mapping of $\mathcal{L}(E, F)$ into $\mathcal{L}(S, F)$? The assumptions that come immediately into mind, after our previous discussions, are those according to which T is a metric homomorphism of X into Y, or a strict metric homomorphism of X into Y. Their role in this problem is nor clear. Dually for the lifting problem.

(6) Let E and F be Banach spaces, S a closed vector subspace of E and φ a continuous linear transformation of S into F. In order that there should exist a continuous linear extension ϕ of φ to E with values in F and the same norm $\|\phi\| = \|\varphi\|$, is it necessary and sufficient that, for every vector subspace E_1 of E containing S as a vector subspace of codimension 1 (or finite codimension) there should exist a continuous linear extension ϕ_1 to E_1 with values in F and the same norm $\|\phi_1\| = \|\varphi\|$? It is known that if a real Banach space $F \neq 0$ is such that every real Banach space E containing F as a closed vector subspace of codimension 1 has a projection into F of norm 1, then the same is true without any restriction on the codimension. A similar remark holds,

a fortiori, in the case of the extension property, where F is given and E, S and φ are arbitrary and we impose or not a codimension restriction on S in E; and analogously in the case of the extension property of type 2. Dual problem for the lifting property.

(7) Study all cases in which (E, F, S, φ) has the extension property, either in the strict metric sense, i.e. $\|\phi\| = \|\varphi\|$, or in the ample metric sense, i.e. $\|\phi_\varepsilon\| < \|\varphi\| + \varepsilon$, or in the topological sense, by holding some of the E, F, S and φ as given data and the remaining as arbitrary. For instance, given a continuous linear transformation φ of a Banach space S into a Banach space F, when does it have a continuous linear extension ϕ to E with values in F and the same norm $\|\phi\| = \|\varphi\|$, for an arbitrary Banach space E containing S as a Banach subspace ? This is known to be true if either S or F has the extension property of type ∞, a condition however which does not involve φ itself. In the case in which all E, F, S and φ are given data, a theory of metric or topological obstruction to extension is lacking. Same questions with some of the spaces restricted to important categories (reflexive spaces, Hilbert spaces, L^p spaces, C(K) spaces, etc) and some of the transformations restricted to important categories (compact transformations, integral transformations, trace-class transformations, etc). In applications of Functional Analysis to Partial Differential Equations one encounters situations in which certain continuous linear transformations should be extended, but usually one notices a lack of general theorems to cover the specific situations at hand. A study of such concrete situation

might lead to new interesting extension theorems. Some of the extension problems have been studied only in the real case and often the complex case requires additional effort. Most of the known results concerning extension wait a dual study as to what concerns lifting.

(8) Let F be a finite dimensional real vector space and K a compact convex set in F. Assume that the collection of all sets in the family $\{\lambda K + a\}$ for $\lambda \in R, \lambda \neq 0$ and $a \in F$ has the n-ary intersection property, where $n \geqslant 2$, namely that given any subcollection such that any n members of it intersect, there results that all members of the subcollection have a non empty intersection. Is it true that there exists a direct sum vector space decomposition $F = F_1 + \ldots + F_s$ into vector subspaces of dimensions at most equal to $n - 1$ and compact convex sets $K_i \subset F_i$ $(i = 1, \ldots, s)$ such that $K = K_1 + \ldots + K_s$? It is known, as a consequence of a theorem of Helly, that, conversely, if such a decomposition exists, then $\{\lambda K + a\}$ has the n-ary intersection property. The answer to the above question is affirmative for $n = 2$, as shown by the author as a by-product in the study of the extension problem, provided K has a center of symmetry and, more generally, by Nagy, without such restriction and by assuming only $\lambda = 1$. This leads to a characterization of parallelepipeds by their translations in finite dimensional real vector spaces. A thorough discussion of the case $n = 2$ was done by Hanner.

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