ISSN 0101-9201

CBPF - CENTRO BRASILEIRO DE PESQUISAS FÍSICAS
Rio de Janeiro

Notas de Física CBPF-NF-008/11
May 2011

An $\mathcal{N}=8$ superconformal particle in the half-plane

Olaf Lechtenfeld and Francesco Toppan

Ministério da
Ciência e Tecnologia

ITP–UH–02/11
CBPF-NF-008/11

An $\mathcal{N}\!=8$ superconformal particle in the half-plane

Olaf Lechtenfeld* and Francesco Toppan†

*Institut für Theoretische Physik, Leibniz Universität Hannover
Appelstraße 2, 30167 Hannover, Germany
email: lechtenf@itp.uni-hannover.de

†CBPF, Rua Dr. Xavier Sigaud 150, Urca
cep 22290-180, Rio de Janeiro (RJ), Brazil
email: toppan@cbpf.br

Abstract

By imposing global supersymmetry and scale invariance we construct an $\mathcal{N}\!=8$ superconformal mechanical system based on the inhomogeneous (2,8,6) linear multiplet. The unique action describes a special Kähler sigma model with a Calogero-type potential and Fayet-Iliopoulos terms. The classical dynamics of the two propagating bosons is restricted to a (warped) half-plane and bounded. We numerically inspect typical trajectories of this special particle.

1 Introduction and summary

For classical mechanics (field theory in 0+1 dimensions) there exists a rich landscape of $\mathcal{N}=8$ supersymmetric models, distinguished by the number b of propagating bosonic degrees of freedom and by the nature of the supersymmetry transformations (linear or non-linear) [1, 2, 3]. Restricting to the linear type, the notation $(b,\mathcal{N},\mathcal{N}-b)$ counts their propagating bosonic, fermionic and auxiliary components. As was already observed in [4, 5], an important role is played by a potential inhomogeneity in the supersymmetry transformation of the fermions. The parameters appearing there may be viewed as a constant shift of the auxiliary components and are introduced through the superfield constraints. Together with Fayet-Iliopoulos terms, they create a bosonic potential, lead to central charges and partial supersymmetry breaking.

To accomodate these inhomogeneous terms, we apply the techniques discussed in [6] and [7] and produce the most general inhomogeneous linear supermultiplets compatible with the ordinary supersymmetry algebra $\{Q_i, Q_j\} = \delta_{ij}H$ (without central extensions).

Here, we concentrate on the classical mechanics of a (2,8,6) particle. The Lagrangian and Hamiltonian of this model has been formulated for a general prepotential F in [8] (without inhomogeneity) and in [9] (with inhomogeneity). Here, we specialize to the conformal case and investigate the classical dynamics of the conformal (2,8,6) particle.

The inhomogeneous (2,8,6) $\mathcal{N}=8$ supermultiplet, under the requirement of scale-invariance for the action, defines a unique superconformal mechanical system. The only free parameters are the the scale-setting Fayet-Iliopoulos coupling and the dimensionless shift entering the inhomogeneous supersymmetry transformations.

We review the inhomogeneous supersymmetry transformations for $\mathcal{N}\leq 8$ and rederive the invariant conformal action for the inhomogeneous (2,8,6) multiplet including Fayet-Iliopoulos terms, without using superspace technology. After eliminating the auxiliary components we arrive at a very specific (non-isotropic and indefinite) Weyl factor and bosonic potential in the two-dimensional target space. It proves to be legitimate (at least classically) to restrict to a (positive-definite) half-space, where we present some typical particle trajectories.

The inhomogeneous supersymmetry transformations that we investigate here close the ordinary supersymmetry algebra without central extensions. This is the case because we work within the Lagrangian framework. Central extensions of the supersymmetry algebra can arise, both in the classical and quantum cases, as a consequence of the Hamiltonian formulation and the closure of the Noether-(super)charge algebra under the Poisson bracket structure [4].

It is tempting to push the idea of this paper to even higher-extended supersymmetry. For example, by coupling two inhomogeneous (2,8,6) multiplets linked by an extra, 9th, supersymmetry, one should be able to construct an $\mathcal{N}=9$ superconformal mechanics model with a four-dimensional target. This might be related with the standard reduction of $\mathcal{N}=4$ super Yang-Mills to an off-shell multiplet of type (9,16,7) in one dimension.

2 Inhomogeneous minimal linear supermultiplets

Minimal linear supermultiplets of extended supersymmetry in one dimension are usually formulated with homogeneous transformations for their component fields. However, in some cases it is possible to extend the supersymmetry transformations by the addition of an inhomogeneous term. This is admissible at

- $\mathcal{N}=2$ for the supermultiplet $(0,2,2)$
- $\mathcal{N}=4$ for the supermultiplets $(0,4,4)$ and $(1,4,3)$
- $\mathcal{N}=8$ for the supermultiplets $(0,8,8)$ and $(1,8,7)$ and $(2,8,6)$

The remaining $\mathcal{N}=2,4,8$ supermultiplets do not admit an inhomogeneous extension, as can be easily verified by investigating the closure of the ordinary $\mathcal{N}$-extended supersymmetry algebra.

Let x and y be physical bosons, ψ, ψ_i, λ and λ_i denote fermions, and g, g_i, f and f_i describe auxiliary fields. Here, the isospin index i runs over a range depending on the number of supersymmetries. The presence of an inhomogeneous term requires the following mass dimension for the fields:

$$[t]=-1 \qquad \longrightarrow \qquad [x]=-1\ , \quad [\psi]=-\tfrac{1}{2}\ , \quad [g]=0\ . \tag{1}$$

In all the above cases, by a suitable R transformation, the inhomogeneous terms can be rotated to point only in a specific iso-direction. We choose the one with the highest iso-index, i.e. $i=2,3$ or 7, depending on the case. With this choice, let us list the various supersymmetry transformations Q_i for the six cases listed above.

(0,2,2). For the inhomogenous $\mathcal{N}=2$ $(0,2,2)$ supermultiplet, the two supersymmetry transformations, without loss of generality, can be expressed as $(j,k=1,2,\ \epsilon_{12}=1)$

$$\begin{aligned} Q_1\psi_j &= g_j\ , & Q_1 g_j &= \dot{\psi}_j\ ,\\ Q_2\psi_j &= \epsilon_{jk}\tilde{g}_k\ , & Q_2 g_j &= \epsilon_{jk}\dot{\psi}_k\ , \end{aligned} \tag{2}$$

where the inhomogeneous extension hides in

$$\tilde{g}_k \ :=\ g_k + c_k \qquad \text{with} \quad c_k\in\mathbb{R}\ , \tag{3}$$

and we rotate to $c_1=0$, $c_2\equiv c>0$.

(0,4,4). For the $\mathcal{N}=4$ $(0,4,4)$ multiplet, we have $(i,j,k=1,2,3,\ \epsilon_{123}=1)$

$$\begin{aligned} Q_0\psi &= g\ , & Q_0\psi_j &= g_j\ , & Q_0 g &= \dot{\psi}\ , & Q_0 g_j &= \dot{\psi}_j\ ,\\ Q_i\psi &= g_i\ , & Q_i\psi_j &= -\delta_{ij}g+\epsilon_{ijk}\tilde{g}_k\ , & Q_i g &= -\dot{\psi}_i\ , & Q_i g_j &= \delta_{ij}\dot{\psi}-\epsilon_{ijk}\dot{\psi}_k\ , \end{aligned} \tag{4}$$

and we may choose

$$\tilde{g}_1=g_1\ , \quad \tilde{g}_2=g_2 \quad \text{but} \quad \tilde{g}_3=g_3+c\ . \tag{5}$$

(1,4,3). The $\mathcal{N}{=}4$ $(1,4,3)$ multiplet looks slightly different,

$$\begin{array}{llll} Q_0 x = \psi\ , & Q_0 \psi = \dot{x}\ , & Q_0 \psi_j = g_j\ , & Q_4 g_j = \dot{\psi}_j\ , \\ Q_i x = \psi_i\ , & Q_i \psi = -g_i\ , & Q_i \psi_j = \delta_{ij}\dot{x} + \epsilon_{ijk}\tilde{g}_k\ , & Q_i g_j = -\delta_{ij}\dot{\psi} - \epsilon_{ijk}\dot{\psi}_k\ , \end{array} \tag{6}$$

with the same $\tilde{g}_k$ as in (0,4,4).

(0,8,8). Without loss of generality, we can generate the $\mathcal{N}{=}8$ multiplets from the $\mathcal{N}{=}4$ ones by replacing the quaternionic structure constants ϵ_{ijk} by the (totally antisymmetric) octonionic structure constants c_{ijk}, with $i,j,k = 1,\ldots,7$ and

$$c_{123} = c_{147} = c_{165} = c_{246} = c_{257} = c_{354} = c_{367} = 1\ , \tag{7}$$

together with $c_{ijk} = 0$ for all other index combinations. Therefore, the case of (0,0,8) yields

$$\begin{array}{llll} Q_0 \psi = g\ , & Q_0 \psi_j = g_j\ , & Q_0 g = \dot{\psi}\ , & Q_0 g_j = \dot{\psi}_j\ , \\ Q_i \psi = g_i\ , & Q_i \psi_j = -\delta_{ij} g + c_{ijk}\tilde{g}_k\ , & Q_i g = -\dot{\psi}_i\ , & Q_i g_j = \delta_{ij}\dot{\psi} - c_{ijk}\dot{\psi}_k\ , \end{array} \tag{8}$$

and we take

$$\tilde{g}_k = g_k + \delta_{k,7}\, c\ . \tag{9}$$

(1,8,7). In analogy with (1,4,3), we get

$$\begin{array}{llll} Q_0 x = \psi\ , & Q_0 \psi = \dot{x}\ , & Q_0 \psi_j = g_j\ , & Q_0 g_j = \dot{\psi}_j\ , \\ Q_i x = \psi_i\ , & Q_i \psi = -g_i\ , & Q_i \psi_j = \delta_{ij}\dot{x} + \epsilon_{ijk}\tilde{g}_k\ , & Q_i g_j = -\delta_{ij}\dot{\psi} - \epsilon_{ijk}\dot{\psi}_k\ , \end{array} \tag{10}$$

and again $\tilde{g}_k = g_k$ except for $\tilde{g}_7 = g_7 + c$ with $c > 0$.

(2,8,6). This is the most interesting multiplet. It is convenient to present it in quaternionic form, by fusing $(1,4,3)\oplus(1,4,3) = (2,8,6)$, with components labeled by $(x, \psi_{(i)}, g_{(i)})$ and $(y, \lambda_{(i)}, f_{(i)})$, respectively, where $i = 1,2,3$. It is convenient to present the supersymmetry transformations in the following table,

	x	g_1	g_2	g_3	y	f_1	f_2	f_3	ψ	ψ_1	ψ_2	ψ_3	λ	λ_1	λ_2	λ_3
Q_0	ψ	$\dot\psi_1$	$\dot\psi_2$	$\dot\psi_3$	λ	$\dot\lambda_1$	$\dot\lambda_2$	$\dot\lambda_3$	$\dot x$	g_1	g_2	g_3	$\dot y$	f_1	f_2	f_3
Q_1	ψ_1	$-\dot\psi$	$-\dot\psi_3$	$\dot\psi_2$	λ_1	$-\dot\lambda$	$\dot\lambda_3$	$-\dot\lambda_2$	$-g_1$	$\dot x$	$\tilde g_3$	$-\tilde g_2$	$-f_1$	$\dot y$	$-\tilde f_3$	$\tilde f_2$
Q_2	ψ_2	$\dot\psi_3$	$-\dot\psi$	$-\dot\psi_1$	λ_2	$-\dot\lambda_3$	$-\dot\lambda$	$\dot\lambda_1$	$-g_2$	$-\tilde g_3$	$\dot x$	$\tilde g_1$	$-f_2$	$\tilde f_3$	$\dot y$	$-\tilde f_1$
Q_3	ψ_3	$-\dot\psi_2$	$\dot\psi_1$	$-\dot\psi$	λ_3	$\dot\lambda_2$	$-\dot\lambda_1$	$-\dot\lambda$	$-g_3$	$\tilde g_2$	$-\tilde g_1$	$\dot x$	$-f_3$	$-\tilde f_2$	$\tilde f_1$	$\dot y$
Q_4	λ	$-\dot\lambda_1$	$-\dot\lambda_2$	$-\dot\lambda_3$	$-\psi$	$\dot\psi_1$	$\dot\psi_2$	$\dot\psi_3$	$-\dot y$	f_1	f_2	f_3	$\dot x$	$-g_1$	$-g_2$	$-g_3$
Q_5	λ_1	$\dot\lambda$	$\dot\lambda_3$	$-\dot\lambda_2$	$-\psi_1$	$-\dot\psi$	$\dot\psi_3$	$-\dot\psi_2$	$-f_1$	$-\dot y$	$-\tilde f_3$	$\tilde f_2$	g_1	$\dot x$	$-\tilde g_3$	$\tilde g_2$
Q_6	λ_2	$-\dot\lambda_3$	$\dot\lambda$	$\dot\lambda_1$	$-\psi_2$	$-\dot\psi_3$	$-\dot\psi$	$\dot\psi_1$	$-f_2$	$\tilde f_3$	$-\dot y$	$-\tilde f_1$	g_2	$\tilde g_3$	$\dot x$	$-\tilde g_1$
Q_7	λ_3	$\dot\lambda_2$	$-\dot\lambda_1$	$\dot\lambda$	$-\psi_3$	$\dot\psi_2$	$-\dot\psi_1$	$-\dot\psi$	$-f_3$	$-\tilde f_2$	$\tilde f_1$	$-\dot y$	g_3	$-\tilde g_2$	$\tilde g_1$	$\dot x$

Inspection shows that Q_0, Q_1, Q_2, Q_3 act within each of the two (1,4,3) submultiplets, while the additional supersymmetries Q_4, Q_5, Q_6, Q_7 mix the two. Having SO(3)-rotated inside each (1,4,3) submultiplet to

$$\tilde{g}_k = g_k + \delta_{k3}\, c \qquad \text{and} \qquad \tilde{f}_k = f_k + \delta_{k3}\, c' \tag{11}$$

we may employ a further SO(2) rotation, acting on the $\psi_3\lambda_3$ and $g_3 f_3$ planes, to remove the c' contribution and align the inhomogeneity with one of the two $\mathcal{N}{=}4$ submultiplets.

3 Invariant action for a (2,8,6) particle

To investigate the dynamics of superconformal particles on a line, based on the various inhomogeneous supermultiplets, we shall need to construct invariant actions for them. For $\mathcal{N}\geq 4$ and the presence of at least one physical boson, there exists a canonical method [7] to generate such actions, by setting

$$\mathcal{S} \;=\; \int \mathrm{d}t\;\mathcal{L} \;=\; \int \mathrm{d}t\;Q_1Q_2Q_3Q_4\,F(x,y,\ldots)\;, \tag{12}$$

where $F(x,y,\ldots)$ is an unconstrained prepotential. In order to obtain conformally invariant mechanics, the action should not contain any dimensionful coupling parameter, and therefore, due to $[Q_i]=\frac{1}{2}$, we demand that $[F]=-1$. One can prove that the ensuing scale invariance extends to full conformal invariance.

Without the inhomogeneous extension, (12) yields only a kinetic term with some metric. It is the inhomogeneous term which will give rise to a Calogero-type potential. The action may be complemented by the addition of a Fayet-Iliopoulos term

$$\mathcal{S}_{\mathrm{FI}} \;=\; \int \mathrm{d}t\;\sum_i (q_ig_i + r_if_i) \qquad \text{with} \quad [q_i]=[r_i]=1\;, \tag{13}$$

introducing dimensionful couplings compatible with conformal invariance. These Fayet-Iliopoulos terms produce an oscillatorial damping, via the DFF trick of conformal mechanics [10].

For the (1,4,3) multiplet (only x and g_i, no y or f_i), the proper choice for the prepotential is

$$F(x) \;=\; \tfrac14\,x\ln x \qquad \longrightarrow \qquad \mathcal{L}+\mathcal{L}_{\mathrm{FI}} \;=\; F''(x)\big(\dot{x}^2+g_i^2\;+\;c\,g_3\big)+q_ig_i\;+\;\text{fermions}\;. \tag{14}$$

After eliminating the auxiliary components g_i via their equations of motion and putting the fermions to zero, one gets

$$\begin{aligned}\mathcal{L}'_{\mathrm{bos}} \;&=\; F''(x)\big(\dot{x}^2-\tfrac14 c^2\big)\;-\;\tfrac14 q_i^2/F''(x)\;-\;\tfrac12 c\,q_3\\ &=\; \tfrac14\big(\dot{x}^2-\tfrac14 c^2\big)/x\;-\;g_i^2x\;-\;\tfrac12 c\,q_3\\ &=\; \tfrac12\dot{w}^2-\tfrac18 c^2w^{-2}\;-\;\tfrac12 g_i^2w^2\;-\;\tfrac12 c\,q_3\;,\end{aligned} \tag{15}$$

and we have recovered the standard conformal action after the coordinate change $x=\frac12 w^2$.

Stepping up to $\mathcal{N}=8$, we change the iso-labelling to make Q_0,Q_1,Q_2,Q_3 manifest,

$$\mathcal{S} \;=\; \int \mathrm{d}t\;\mathcal{L} \;=\; \int \mathrm{d}t\;Q_0Q_1Q_2Q_3\,F(x,y,\ldots)\;. \tag{16}$$

Demanding invariance under the additional four supersymmetries by requiring

$$Q_l\mathcal{L} \;=\; \partial_t W_l \qquad \text{for} \quad l=4,5,6,7 \tag{17}$$

imposes severe constraints on F. In fact, for the (1,8,7) multiplet no action can be invariant under the inhomogeneous supersymmetry transformations.*

However, the situation is much more interesting for the (2,8,6) multiplet. Here, the constraint (17) says that, like in the homogeneous case [11], the prepotential $F(x,y)$ must be harmonic,

$$\Box F \;\equiv\; F_{xx}+F_{yy} \;=\; 0\ . \tag{18}$$

The general solution is encoded in a meromorphic function $H(z)$ via

$$F(x,y) \;=\; H(z)+\overline{H(z)} \;=\; 2\,\mathrm{Re}H(z)\ , \tag{19}$$

where it is convenient to pass to complex coordinates,

$$\begin{aligned} z&=x+\mathrm{i}y\ , \qquad \partial_z=\tfrac12(\partial_x-\mathrm{i}\partial_y)\ , \qquad h_i=g_i+\mathrm{i}f_i\ , \qquad \chi_{(i)}=\psi_{(i)}+\mathrm{i}\lambda_{(i)} \\ \bar z&=x-\mathrm{i}y\ , \qquad \partial_{\bar z}=\tfrac12(\partial_x+\mathrm{i}\partial_y)\ , \qquad \bar h_i=g_i-\mathrm{i}f_i\ , \qquad \bar\chi_{(i)}=\psi_{(i)}-\mathrm{i}\lambda_{(i)}\ . \end{aligned} \tag{20}$$

Inserting (19) into (16), we obtain

$$\begin{aligned} \mathcal{L} \;=\;& 2\,\mathrm{Re}\big\{H_{zz}(\dot{\bar z}\dot z\,+\,\bar h_i h_i\,+\,c\,h_3\,+\,\tfrac12\dot{\bar\chi}\chi-\tfrac12\bar\chi\dot\chi\,+\,\tfrac12\dot{\bar\chi}_i\chi_i-\tfrac12\bar\chi_i\dot\chi_i) \\ &+\; H_{zzz}(\chi\chi_i h_i\,+\,\tfrac12\epsilon_{ijk}\chi_i\chi_j h_k\,+\,c\,\chi\chi_3)\;+\;\tfrac16 H_{zzzz}\epsilon_{ijk}\chi\chi_i\chi_j\chi_k\big\}\ , \end{aligned} \tag{21}$$

where the inhomogeneous extension is clearly visible in the terms containing the parameter c. The bosonic metric $g_{z\bar z}=H_{zz}{+}\bar H_{\bar z\bar z}$ is special Kähler of rigid type [12]. Reverting to real notation and introducing the Weyl factors

$$\Phi \;=\; 2\,\mathrm{Re}H_{zz} \;=\; \tfrac12(F_{xx}{-}F_{yy}) \qquad\text{and}\qquad \widetilde\Phi \;=\; -2\,\mathrm{Im}H_{zz} \;=\; F_{xy}\ , \tag{22}$$

the Lagrangian reads

$$\begin{aligned} \mathcal{L} \;=\;& \Phi\big(\dot x^2+\dot y^2+{g_i}^2+{f_i}^2-\psi\dot\psi-\lambda\dot\lambda-\psi_i\dot\psi_i-\lambda_i\dot\lambda_i\big) \\ &+\;\Phi_x\big(\psi\psi_i g_i-\psi\lambda_i f_i-\lambda\psi_i f_i-\lambda\lambda_i g_i\,+\,\epsilon_{ijk}(\tfrac12 g_i\psi_j\psi_k-\tfrac12 g_i\lambda_j\lambda_k-f_i\lambda_j\psi_k)\big) \\ &+\;\Phi_y\big(\lambda\psi_i g_i-\lambda\lambda_i f_i+\psi\psi_i f_i+\psi\lambda_i g_i\,+\,\epsilon_{ijk}(\tfrac12 f_i\psi_j\psi_k-\tfrac12 f_i\lambda_j\lambda_k+g_i\lambda_j\psi_k)\big) \\ &+\;\tfrac12(\Phi_{xx}{-}\Phi_{yy})\epsilon_{ijk}\big(\tfrac16\psi\psi_i\psi_j\psi_k+\tfrac16\lambda\lambda_i\lambda_j\lambda_k-\tfrac12\psi\psi_i\lambda_j\lambda_k-\tfrac12\lambda\lambda_i\psi_j\psi_k\big) \\ &+\;\Phi_{xy}\,\epsilon_{ijk}\big(\tfrac16\lambda\psi_i\psi_j\psi_k-\tfrac16\psi\lambda_i\lambda_j\lambda_k+\tfrac12\psi\lambda_i\psi_j\psi_k-\tfrac12\lambda\psi_i\lambda_j\lambda_k)\big) \\ &+\;c\,\big(\Phi g_3+\widetilde\Phi f_3+\Phi_x(\psi\psi_3-\lambda\lambda_3)+\Phi_y(\lambda\psi_3+\psi\lambda_3)\big)\ , \end{aligned} \tag{23}$$

to which we add the Fayet-Iliopoulos terms

$$\mathcal{L}_{\mathrm{FI}} \;=\; q_i g_i+r_i f_i\ . \tag{24}$$

The harmonic prepotential with the correct scaling dimension $[H]=-1$ is †

$$H(z) \;=\; \tfrac18\,z\ln z \qquad\longleftrightarrow\qquad F(x,y) \;=\; \tfrac18\,x\ln(x^2{+}y^2)-\tfrac14\,y\arctan\tfrac{y}{x}\ , \tag{25}$$

*In the homogeneous case the constraint reads $F''''(x)=0$, which produces $\mathcal{L}=(ax{+}b)\,\dot x^2+\ldots$.

†Multiplying H with a phase corresponds to an irrelevant rotation in the complex plane.

and the corresponding Weyl factors read

$$\Phi \;=\; \tfrac14\,\mathrm{Re}\frac{1}{z} \;=\; \tfrac14\frac{x}{x^2{+}y^2} \qquad\text{and}\qquad \widetilde{\Phi} \;=\; -\tfrac14\,\mathrm{Im}\frac{1}{z} \;=\; \tfrac14\frac{y}{x^2{+}y^2}\ . \tag{26}$$

Note that the corresponding metric is an indefinite one, as it must be for any harmonic Weyl factor.

In the bosonic limit, obtained by setting all fermions equal to zero, we obtain

$$\mathcal{L}_{\rm bos}+\mathcal{L}_{\rm FI} \;=\; \Phi\,(\dot{x}^2+\dot{y}^2+{g_i}^2+{f_i}^2)+c\,(\Phi\,g_3+\widetilde{\Phi}f_3)+q_ig_i+r_if_i\ . \tag{27}$$

We eliminate the auxiliary fields via their algebraic equations of motion,

$$\begin{array}{lll} g_1=-\frac{q_1}{2\Phi}\ , & g_2=-\frac{q_2}{2\Phi}\ , & g_3=-\frac{q_3+c\Phi}{2\Phi} \\ f_1=-\frac{r_1}{2\Phi}\ , & f_2=-\frac{r_2}{2\Phi}\ , & f_3=-\frac{r_3+c\widetilde{\Phi}}{2\Phi}\ , \end{array} \tag{28}$$

and arrive at

$$\begin{aligned} \mathcal{L}'_{\rm bos} \;&=\; \Phi\left(\dot{x}^2+\dot{y}^2\right) \;-\; \tfrac{1}{4\Phi}\big(q_1^2+q_2^2+(q_3{+}c\Phi)^2+r_1^2+r_2^2+(r_3{+}c\widetilde{\Phi})^2\big) \\ &=\; \frac{x}{x^2{+}y^2}\frac{\dot{x}^2+\dot{y}^2}{4} \;-\; \frac{(q_i^2{+}r_i^2)(x^2{+}y^2)}{x} \;-\; c\,\frac{q_3x{+}r_3y}{2x} \;-\; \frac{c^2}{16x} \\ &=:\; K \;-\; V\ , \end{aligned} \tag{29}$$

making explicit the effect of both the inhomogeneous supersymmetry transformation (c) and the Fayet-Iliopoulos terms (q_i, r_i) on the potential V.

It is tempting to perform the same coordinate change as for the (1,4,3) multiplet, $x=\frac12 w^2$, which yields

$$\mathcal{L}'_{\rm bos} \;=\; \tfrac12(1{+}\gamma^2)^{-1}\Big(\dot{w}^2+\frac{\dot{y}^2}{w^2}\Big) \;-\; \tfrac12(1{+}\gamma^2)(q_i^2{+}r_i^2)w^2 \;-\; \tfrac12\,c\,(q_3{+}r_3\gamma) \;-\; \frac{c^2}{8w^2}\ , \tag{30}$$

where $\gamma\;=\;2y/w^2$. This form reveals both the oscillator and Calogero terms, but also shows the added complexity in two dimensions (mostly hidden in γ). Putting $y\equiv0$ (also $\gamma{=}0$) brings back the (1,4,3) result (15).

4 Trajectories of a (2,8,6) particle

Without loss of generality, let us drop inessential Fayet-Iliopoulos terms and put

$$q_1=q_2=r_1=r_2=0 \qquad\text{and}\qquad q_3=:q\ ,\quad r_3=:r\ ,\quad q{+}\mathrm{i}r=:s\ . \tag{31}$$

In complex coordinates, the kinetic and potential energies then read

$$K \;=\; \Phi\,\dot{z}\dot{\bar{z}} \;=\; \tfrac18\frac{z{+}\bar{z}}{z\bar{z}}\,\dot{z}\dot{\bar{z}}\ , \tag{32}$$

$$V \;=\; \big((q{+}c\Phi)^2+(r{+}c\widetilde{\Phi})^2\big)/4\Phi \;=\; \tfrac18\frac{1}{z{+}\bar{z}}\big(4s\bar{z}+c\big)\big(4\bar{s}z+c\big)\ . \tag{33}$$

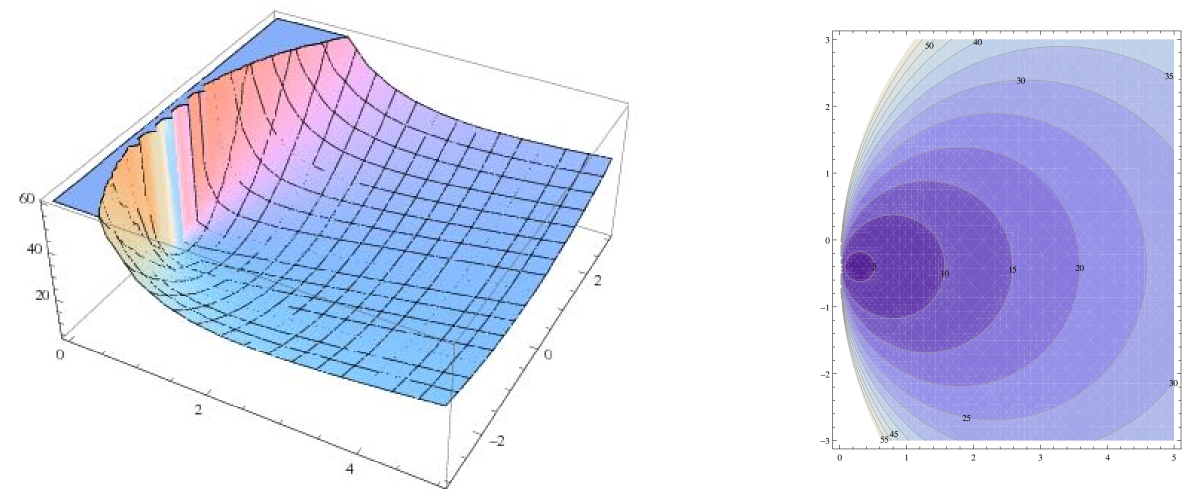

Figure 1: Potential V and its level curves for $(c,q,r)=(4,1,2) \quad\longrightarrow\quad z_{\min}=\frac{1}{5}(1{-}2\mathrm{i})$.

The level curves of this potential are circles of center and radius

$$z_0(V)=\frac{2V-c\,s}{4(q^2{+}r^2)} \qquad\text{and}\qquad r(V)=\frac{\sqrt{V(V{-}c\,q)}}{2(q^2{+}r^2)}\ , \tag{34}$$

respectively, and its only minimum $V_{\min}=cq$ is located at

$$z_{\min}=z_0(cq)=\frac{c\,\bar{s}}{4(q^2{+}r^2)}\ . \tag{35}$$

The parameter r governs the asymmetry under $y\to -y$. The reflection $x\to -x$ flips the sign of $V{-}\frac{1}{2}cq_3$. Due to the factor of $z{+}\bar{z}=2x$, both the Weyl factor and the potential are strictly positive on the right half-space $x{>}0$ and strictly negative for $x{<}0$. Therefore, the (2,8,6) particle is a reasonable dynamical system only if its trajectories do not cross the $x{=}0$ dividing line. Seen from the right half-space, the potential barrier for $x{\to}0$ has a hole at $y{=}0$ if $c{=}0$, but the Weyl factor explodes precisely there. For large coordinate values, the potential grows linearly with x and quadratically with y, so the $x{>}0$ trajectories remain bounded.

The equation of motion takes the form

$$\begin{aligned} 0 \;&=\; \Phi^3\ddot{z} \;+\; \Phi^2\Phi_z\dot{z}^2 \;-\; \tfrac{1}{4}\Phi_{\bar{z}}\big(q^2+(r+2\mathrm{i}cH_{zz})^2\big) \\ &\propto\; (z{+}\bar{z})^3 z\bar{z}\,\ddot{z} \;-\; (z{+}\bar{z})^2\bar{z}^2\dot{z}^2 \;+\; z^2\bar{z}^2\big((4qz)^2+(4rz{+}\mathrm{i}c)^2\big)\ , \end{aligned} \tag{36}$$

which in real coordinates reads

$$\begin{aligned} 0 \;&=\; \ddot{x}-\frac{1}{2x}\frac{x^2{-}y^2}{x^2{+}y^2}(\dot{x}^2{-}\dot{y}^2)-\frac{2y}{x^2{+}y^2}\,\dot{x}\,\dot{y}+\frac{x^2{+}y^2}{x^3}\big(2(q^2{+}r^2)(x^2{-}y^2)-cr\,y-\tfrac{1}{8}c^2\big)\ , \\ 0 \;&=\; \ddot{y}+\frac{y}{x^2{+}y^2}(\dot{x}^2{-}\dot{y}^2)-\frac{1}{x}\frac{x^2{-}y^2}{x^2{+}y^2}\,\dot{x}\,\dot{y}+\frac{x^2{+}y^2}{x^3}\big(4(q^2{+}r^2)\,x\,y+cr\,x\big)\ . \end{aligned} \tag{37}$$

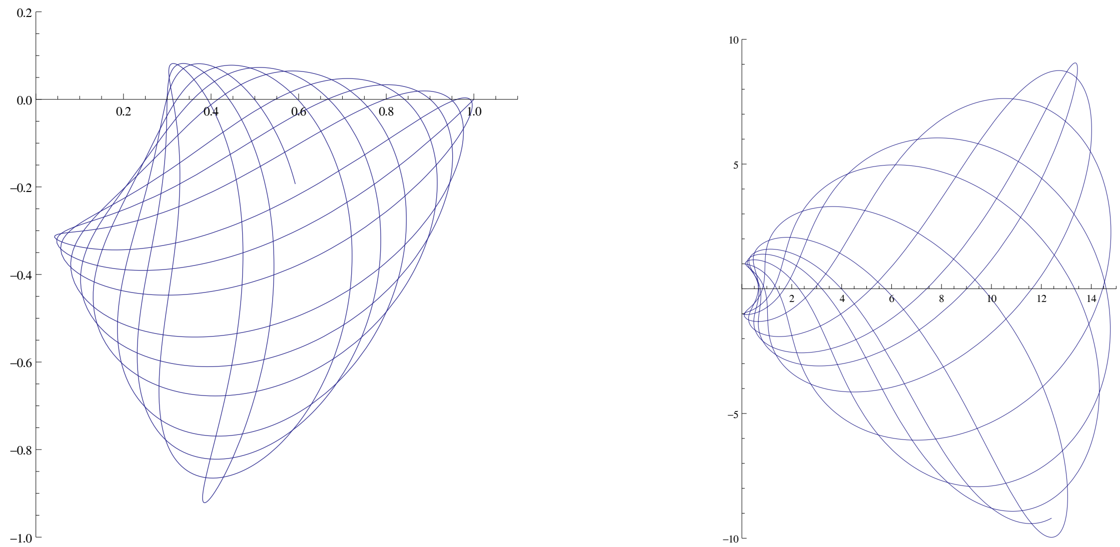

Figure 2: Trajectories for $(c, q, r) = (4, 1, 2)$ with initial conditions $(z, \dot{z})(0) = (1, 0)$ (left) and $(z, \dot{z})(0) = (\frac{1}{10}+\mathrm{i}, 0)$ (right).

The only constant of motion of this system is the energy $E = T+V$, so the generic particle motion is not integrable. Figure 2 shows the trajectory for the (c, q, r)-value chosen in Figure 1 and a couple of initial conditions.

One sees that the curve does not fill out the region $V(x, y) \leq E$, on effect of the position-dependent effective mass $M = 2\Phi(x, y)$. It is also clear that the x=0 barrier is impenetrable. Therefore, it makes sense to substitute $w = \sqrt{2x}$ and introduce the dynamics in the wy-plane according to (30). The trajectories of Figure 2 get somewhat distorted in these variables, but their qualitative behavior is unchanged.

Acknowledgements

O.L. thanks CBPF for warm hospitality. This work was partially supported by CNPq and by DFG grant Le-838/9-2.

References

[1] S. Bellucci, E. Ivanov, S. Krivonos and O. Lechtenfeld,
N=8 superconformal mechanics,
Nucl. Phys. B **684** (2004) 321-350, `arXiv:hep-th/0312322`.

[2] S. Bellucci, E. Ivanov, S. Krivonos and O. Lechtenfeld,
ABC of N=8, d=1 supermultiplets,
Nucl. Phys. B **699** (2004) 226-252, `arXiv:hep-th/0406015`.

[3] E. Ivanov, O. Lechtenfeld and A. Sutulin,
Hierarchy of N=8 mechanics models,
Nucl. Phys. B **790** [PM] (2008) 493-523, `arXiv:0705.3064 [hep-th]`.

[4] E. Ivanov, S. Krivonos and A. Pashnev,
Partial supersymmetry breaking in N=4 supersymmetric quantum mechanics,
Class. Quant. Grav. **8** (1991) 19-40.

[5] E.E. Donets, A. Pashnev, J. Juan Rosales and M.M. Tsulaia,
N=4 supersymmetric multidimensional quantum mechanics, partial susy breaking and superconformal quantum mechanics,
Phys. Rev. D**61** (2000) 043512, `arXiv:hep-th/9907224`.

[6] A. Pashnev and F. Toppan,
On the classification of N-extended supersymmetric quantum mechanical systems,
J. Math. Phys. **42** (2001) 5257-5271, `arXiv:hep-th/0010135`.

[7] Z. Kuznetsova, M. Rojas and F. Toppan,
Classification of irreps and invariants of the N-extended supersymmetric quantum mechanics,
JHEP **0603** (2006) 098, `arXiv:hep-th/0511274`.

[8] S. Bellucci, S. Krivonos and A. Nersessian,
$\mathcal{N}=8$ supersymmetric mechanics on special Kähler manifolds,
Phys. Lett. B **605** (2005) 181-184, `arXiv:hep-th/0410029`.

[9] S. Bellucci, S. Krivonos, A. Nersessian and A. Shcherbakov,
2k-dimensional N=8 supersymmetric quantum mechanics,
`arXiv:hep-th/0410073`.

[10] V. de Alfaro, S. Fubini and G. Furlan,
Conformal invariance in quantum mechanics,
Nuovo Cim. A**34** (1976) 569.

[11] M. Gonzales, M. Rojas and F. Toppan,
One-dimensional sigma-models with N=5,6,7,8 off-shell supersymmetries,
Int. J. Mod. Phys. A **24** (2009) 4317-4326, `arXiv:0812.3042 [hep-th]`.

[12] P. Fre,
Lectures on special Kahler geometry and electric-magnetic duality rotations,
Nucl. Phys. Proc. Suppl. BC **45** (1996) 59, `arXiv:hep-th/9512043`.

NOTAS DE FÍSICA é uma pré-publicação de trabalho original em Física.
Pedido de cópias desta publicação deve ser enviado aos autores ou ao:

Centro Brasileiro de Pesquisas Físicas
Área de Publicações
Rua Dr. Xavier Sigaud, 150 – 4º andar
22290-180 – Rio de Janeiro, RJ
Brasil
E-mail: socorro@cbpf.br/valeria@cbpf.br
http://www.biblioteca.cbpf.br/index_2.html

NOTAS DE FÍSICA is a preprint of original unpublished works in Physics.
Request for copies of this report should be addressed to:

Centro Brasileiro de Pesquisas Físicas
Área de Publicações
Rua Dr. Xavier Sigaud, 150 – 4º andar
22290-180 – Rio de Janeiro, RJ
Brazil
E-mail: socorro@cbpf.br/valeria@cbpf.br
http://www.biblioteca.cbpf.br/index_2.html