Chern-Simons $AdS_5$ supergravity in a Randall-Sundrum background

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Abstract

Chern-Simons AdS supergravity theories are gauge theories for the super-AdS group. These theories possess a fermionic symmetry which differs from standard supersymmetry. In this paper, we study five-dimensional Chern-Simons AdS supergravity in a Randall-Sundrum scenario with two Minkowski 3-branes. After making modifications to the $D = 5$ Chern-Simons AdS supergravity action and fermionic symmetry transformations, we obtain a $\mathbb{Z}_2$-invariant total action $S = \tilde{S}_{\text{bulk}} + S_{\text{brane}}$ and fermionic transformations $\tilde{\delta}_c$. While $\tilde{\delta}_c \tilde{S}_{\text{bulk}} = 0$, the fermionic symmetry is broken by $S_{\text{brane}}$. Our total action reduces to the original Randall-Sundrum model when $\tilde{S}_{\text{bulk}}$ is restricted to its gravitational sector. We solve the Killing spinor equations for a bosonic configuration with vanishing $su(N)$ and $u(1)$ gauge fields.

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1 Introduction

Chern-Simons AdS supergravity [1, 2, 3] theories can be constructed only in odd spacetime dimensions. As the name implies, they are gauge theories for supersymmetric extensions of the AdS group.\(^1\) They have a fiber bundle structure and hence are potentially renormalizable [2]. The dynamical fields form a single adS superalgebra-valued connection and hence the supersymmetry algebra closes automatically off-shell without requiring auxiliary fields [4]. The Lagrangian in dimension \(D = 2n + 1\) is a Chern-Simons \((2n + 1)\)-form for the super-adS connection and is a polynomial of order \(n\) in the corresponding curvature. Unlike standard supergravity theories, there can be a mismatch between the number of bosonic and fermionic degrees of freedom.\(^2\) For this reason, the ‘supersymmetry’ of Chern-Simons AdS supergravity theories is perhaps better referred to as a fermionic symmetry.

\(D = 11, N = 1\) Chern-Simons AdS supergravity may correspond to an off-shell supergravity limit of M-theory [2, 3]. It has expected features of M-theory which are not shared by \(D = 11\) Cremmer-Julia-Scherk (CJS) supergravity [5]. These features include an \(osp(32|1)\) superalgebra [6] and higher powers of curvature [7]. Hořava-Witten theory [8] is obtained from CJS supergravity by compactifying on an \(S^1/\mathbb{Z}_2\) orbifold and requiring gauge and gravitational anomalies to cancel. This theory gives the low energy, strongly coupled limit of the heterotic \(E_8 \times E_8\) string theory. In light of the above discussion, it would be interesting to reformulate Hořava-Witten theory with \(D = 11, N = 1\) Chern-Simons AdS supergravity.

Reformulating Hořava-Witten theory as described above may prove to be difficult. It is simpler to compactify the five-dimensional version of Chern-Simons AdS supergravity on an \(S^1/\mathbb{Z}_2\) orbifold and ignore anomaly cancellation issues. Canonical sectors of \(D = 5\) Chern-Simons AdS supergravity have been investigated in locally \(AdS_5\) backgrounds possessing a spatial boundary with topology \(S^1 \times S^1 \times S^1\) located at infinity [9]. In this paper, as a preamble to reformulating Hořava-Witten theory, we will study \(D = 5\) Chern-Simons AdS supergravity in a Randall-Sundrum background with two Minkowski 3-branes [10]. We choose coordinates \(x^\mu = (x^a, x^5)\) to parameterize the five-dimensional spacetime manifold.\(^3\) In terms of these coordinates, the background metric takes the form

\[
g_{\mu \nu} dx^\mu dx^\nu = a^2(x^5) \eta^{(4)}_{\bar{\mu} \bar{\nu}} dx^\bar{\mu} dx^\bar{\nu} + (dx^5)^2, \tag{1.1}\]

where \(\eta^{(4)}_{\bar{\mu} \bar{\nu}} = \text{diag}(-1, 1, 1, 1)_{\bar{\mu} \bar{\nu}}, \ a(x^5) \equiv \exp(-|x^5|/\ell)\) is the warp factor, and \(\ell\) is the \(AdS_5\) curvature radius. The coordinate \(x^5\) parameterizes an \(S^1/\mathbb{Z}_2\) orbifold, where the circle \(S^1\) has radius \(\rho\) and \(\mathbb{Z}_2\) acts as \(x^5 \rightarrow -x^5\). We choose the range \(-\pi \rho \leq x^5 \leq \pi \rho\) with the endpoints identified as \(x^5 \sim x^5 + 2\pi \rho\). The Minkowski 3-branes are located at

\(^1\)The AdS group in dimension \(D \geq 2\) is \(SO(D - 1, 2)\). The corresponding super-AdS groups are given in [3]. For \(D = 5\) and \(D = 11\), the super-AdS groups are respectively \(SU(2, 2|N)\) and \(OSp(32|2N)\).

\(^2\)For example, in \(D = 5\) Chern-Simons AdS supergravity [1], the number of bosonic degrees of freedom \((N^2 + 15)\) is equal to the number of fermionic degrees of freedom \((8N)\) only for \(N = 3\) and \(N = 5\).

\(^3\)We use indices \(\mu, \nu, \ldots = 0, 1, 2, 3, 5\) for local spacetime and \(a, b, \ldots = 0, 1, 2, 3\) for tangent spacetime. The corresponding metrics, \(g_{\mu \nu}\) and \(\eta_{ab}\) = \(\text{diag}(-1, 1, 1, 1)_{ab}\), are related by \(g_{\mu \nu} = e_\mu^a e_\nu^b \eta_{ab}\), where \(e_\mu^a\) is the fubinii. Barred indices \(\bar{\mu}, \bar{\nu}, \ldots = \bar{0}, \bar{1}, \bar{2}, \bar{3}\) denote the four-dimensional counterparts of \(\mu, \nu, \ldots\) and \(a, b, \ldots\), respectively.
the $\mathbb{Z}_2$ fixed points $x^5 = 0$ and $x^5 = \pi \rho$. These 3-branes have corresponding tensions $T^{(0)}$ and $T^{(\pi \rho)}$ and may support $(3 + 1)$-dimensional field theories.

This paper is organized as follows: In Section 2, we construct a $\mathbb{Z}_2$-invariant bulk theory. This bulk theory is obtained by making modifications to the $D=5$ Chern-Simons AdS supergravity action and fermionic symmetry transformations which allow consistent orbifold conditions to be imposed. The variation of the resulting bulk action $S_{\text{bulk}}$ under the resulting fermionic transformations $\delta_\epsilon$ vanishes everywhere except at the $\mathbb{Z}_2$ fixed points. We calculate $\delta_\epsilon S_{\text{bulk}}$ in Section 3. In Section 4, we modify $S_{\text{bulk}}$ and $\delta_\epsilon$ to obtain a modified $\mathbb{Z}_2$-invariant bulk theory. The modified bulk action $\tilde{S}_{\text{bulk}}$ is invariant under the modified fermionic transformations $\tilde{\delta}_\epsilon$. In Section 5, we complete our model by adding the brane action $S_{\text{brane}}$. We show in Section 6 that our total action

$$S = \tilde{S}_{\text{bulk}} + S_{\text{brane}}$$

(1.2)

reduces to the original Randall-Sundrum model [10] when $\tilde{S}_{\text{bulk}}$ is restricted to its gravitational sector. In Section 7, we solve the Killing spinor equations for a purely bosonic configuration with vanishing $su(N)$ and $u(1)$ gauge fields. Our concluding remarks are given in Section 8. Finally, in the Appendix, we work out the fünfbein, spin connection, curvature 2-form components, Ricci tensor, and Ricci scalar for our metric (1.1).

## 2 $\mathbb{Z}_2$-invariant bulk theory

In this section, we construct a $\mathbb{Z}_2$-invariant bulk theory. The bulk theory is obtained by making modifications to the $D=5$ Chern-Simons AdS supergravity [1] action and fermionic symmetry transformations which allow consistent orbifold conditions to be imposed.

The field content of $D=5$ Chern-Simons AdS supergravity is the fünfbein $e_\mu^a$, the spin connection $\omega_\mu^{ab}$, the $su(N)$ gauge connection $A_\mu = A_i^\mu \tau_i$, the $u(1)$ gauge connection $B_\mu$, and $N$ complex gravitini $\psi_{\mu r}$ which transform as Dirac spinors in a vector representation of $su(N)$. These fields form a connection for the adS superalgebra $su(2, 2|N)$. The action and fermionic symmetry transformations are given in [9] in terms of the $AdS_5$ curvature radius $\ell$. The only free parameter in the action is a dimensionless constant $k$. To allow consistent $\mathbb{Z}_2$ orbifold conditions to be imposed, we make the following modifications:

1. Rescale the $su(N)$ and $u(1)$ gauge connections:

$$A \rightarrow g_A A, \quad B \rightarrow g_B B.$$
2. Replace $g_A, g_B, \ell^{-1}$, and $k$ by the $\mathbb{Z}_2$-odd expressions\footnote{The signum function $\text{sgn}(x^5)$ is $+1$ for $0 < x^5 < \pi \rho$ and $-1$ for $-\pi \rho < x^5 < 0$. It obeys $\text{sgn}^2(x^5) = 1$ and $\partial_5 \text{sgn}(x^5) = 2[\delta(x^5) - \delta(x^5 - \pi \rho)]$.}

$$G_A \equiv g_A \text{sgn}(x^5), \quad G_B \equiv g_B \text{sgn}(x^5), \quad L^{-1} \equiv \ell^{-1} \text{sgn}(x^5), \quad K \equiv k \text{sgn}(x^5).$$

In this manner, we obtain the bulk action

$$S_{\text{bulk}} = S_{\text{grav}} + S_{\text{su}(N)} + S_{\text{u}(1)} + S_{\text{ferm}}, \quad (2.1)$$

where

\begin{align*}
S_{\text{grav}} &= \int \frac{1}{8} K \varepsilon_{abcd} \left( \frac{1}{4} R_{ab} R_{cd} e^e + \frac{2}{3 \ell^2} R_{ab} e^c e^d e^e + \frac{1}{5 \ell^2} e^a e^b e^c e^d e^e \right), \\
S_{\text{su}(N)} &= \int iK \text{str} \left( C_A^2 A^2 F - \frac{1}{2} G_A A^3 F + \frac{1}{10} G_A^3 A^5 \right), \\
S_{\text{u}(1)} &= \int K \left[ \left( \frac{1}{12} - \frac{1}{2} \right) C^3_B B(dB)^2 + \frac{3}{4 \ell^2} \left( T_a T_a - \frac{L}{2} R_{ab} R_{ab} \right) \right. \\
&\left. - R_{ab} e_a e_b \right] G_B B - \frac{3}{N} C^2_B G_B F_s \right] F_s B, \\
S_{\text{ferm}} &= \int \frac{3}{2 \ell^2} K \left( \bar{\psi} \Gamma^a \nabla \psi_r^a + \bar{\psi} \sigma^a \nabla \psi_s^a \right) + \text{c.c.,} \quad (2.2)
\end{align*}

and the transformations

\begin{align*}
\delta e^a &= -\frac{1}{2} (\bar{\psi} \Gamma^a \epsilon_r - \bar{e} \Gamma^a \psi_r), & \delta \omega^{ab} &= \frac{1}{4} (\bar{\psi} \Gamma^{ab} \epsilon_r - \bar{e} \Gamma^{ab} \psi_r), \\
\delta \epsilon_r &= -\nabla \epsilon_r, & \delta \bar{\psi}^r &= -\nabla \bar{\psi}^r, \\
\delta \epsilon_s &= i (\bar{\psi} \epsilon_s - \bar{e} \epsilon_s), & \delta \epsilon_t &= i (\bar{\psi} \epsilon_t - \bar{e} \epsilon_t). \quad (2.3)
\end{align*}

In these expressions, $\Gamma^a$ are the Dirac matrices, $\Gamma^{ab} \equiv \frac{1}{2} (\Gamma^a \Gamma^b - \Gamma^b \Gamma^a)$, $R_{ab} = d \omega^{ab} + \omega^{ac} \omega_c^b$ is the curvature 2-form, $T^a = d e^a + \omega^{ab} \epsilon_b$ is the torsion 2-form, $F^a = d A + G_A A^2 = F^i \tau_i$ is the su($N$) curvature,

\begin{align*}
\mathcal{R}_\beta^\alpha &\equiv \frac{1}{2 \ell^2} T^a (\Gamma_\alpha^a)_{\beta}^a + \frac{1}{4} (R_{ab} + \frac{1}{2 \ell^2} e^a e^b) (\Gamma_{ab})_{\beta}^a + \frac{i}{4} G_B dB \delta_{\beta}^a - \frac{1}{2} \bar{\psi}_s \bar{\psi}_s^a, \\
\mathcal{F}_s^r &\equiv F_s^r + \frac{i}{N} G_B dB \delta_s^r - \frac{1}{2} \bar{\psi}_s \bar{\psi}_s^r, \quad (2.4)
\end{align*}

\text{str} is a symmetrized trace satisfying $\text{str} (\tau_i \tau_j \tau_k) \equiv \frac{1}{27} \text{tr} (\{\tau_i, \tau_j\} \tau_k)$, $\nabla$ is the ad$S_5 \times \text{su}(N) \times u(1)$ covariant derivative, and

\begin{align*}
\nabla \psi_r &\equiv (d + \frac{1}{2} \omega^{ab} \Gamma_{ab} + \frac{1}{4 \ell^2} e^a e^b) \psi_r - G_A \epsilon_s^a \psi_s + i \left( \frac{1}{4} - \frac{1}{N} \right) G_B B \psi_r, \\
\nabla \epsilon_r &\equiv (d + \frac{1}{2} \omega^{ab} \Gamma_{ab} + \frac{1}{4 \ell^2} e^a e^b) \epsilon_r - G_A \epsilon_s^a \epsilon_s + i \left( \frac{1}{4} - \frac{1}{N} \right) G_B B \epsilon_r. \quad (2.5)
\end{align*}

Note that the results in the Appendix can be used to show that the torsion vanishes for our metric.

We impose the following orbifold conditions:

\footnote{We choose a chiral basis for the $4 \times 4$ Dirac matrices

$$\Gamma^a = \left( \Gamma^{\bar{a}}, \Gamma^5 \right) = \begin{bmatrix}
0 & -i \sigma^a \\
-i \bar{\sigma}^\bar{a} & 0
\end{bmatrix}, \quad \begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix},$$

where $\sigma^a = (1, \sigma)$ and $\bar{\sigma}^\bar{a} = (1, -\sigma)$. These matrices satisfy $\text{tr} (\Gamma_a \Gamma_{\bar{a}} + \Gamma_5 \Gamma_{\bar{5}}) = -4i \varepsilon_{abcd}$, where $\varepsilon_{abcd}$ is the Levi-Civita tensor and $\varepsilon^{012345} = 1$.}
1. **Periodicity on $S^1$**. The fields and the fermionic parameters $\epsilon_r$, denoted generically by $\phi$, are required to be periodic on the circle $S^1$. That is,

$$\phi(x^\mu, x^5) = \phi(x^\mu, x^5 + 2\pi \rho).$$

(2.6)

2. **$\mathbb{Z}_2$ parity assignments**. The bosonic field components

$$\Phi = e_\mu^\mu, e_5^5, A_5^i, B_5^i, \quad \Theta = e_\mu^{\bar{\mu}}, e_5^{\bar{\mu}}, A_\bar{\mu}^i, B_{\bar{\mu}}$$

are chosen to satisfy

$$\Phi(x^\mu, x^5) = +\Phi(x^\mu, -x^5), \quad \Theta(x^\mu, x^5) = -\Theta(x^\mu, -x^5).$$

(2.7)

That is, the $\Phi$ components are $\mathbb{Z}_2$-even and the $\Theta$ components are $\mathbb{Z}_2$-odd. For the gravitini, we require

$$\Gamma^5 \psi_{\bar{\mu}r}(x^\bar{\mu}, x^5) = +\psi_{\bar{\mu}r}(x^\bar{\mu}, -x^5),$$

$$\Gamma^5 \psi_{5r}(x^\mu, x^5) = -\psi_{5r}(x^\mu, -x^5).$$

(2.8)

Finally, the fermionic parameters $\epsilon_r$ are required to satisfy

$$\Gamma^5 \epsilon_r(x^\bar{\mu}, x^5) = +\epsilon_r(x^\mu, -x^5).$$

(2.9)

These conditions imply that the $\mathbb{Z}_2$-odd quantities vanish at the orbifold fixed points. It is straightforward to check that $S_{\text{bulk}}$ is $\mathbb{Z}_2$-even and that the transformations (2.3) are consistent with the $\mathbb{Z}_2$ parity assignments.

### 3 Calculation of $\delta_\epsilon S_{\text{bulk}}$

The $D=5$ Chern-Simons AdS supergravity action is invariant (up to a boundary term) under its fermionic symmetry transformations. In Section 2, we modified this action and its fermionic transformations to obtain a $\mathbb{Z}_2$-invariant bulk theory. Due to the signum functions introduced by the modifications, $\delta_\epsilon S_{\text{bulk}}$ contains terms which have no counterpart in the unmodified theory. More specifically, the extra terms arise from $\partial_5$ acting on the signum functions to yield delta functions. Such ‘delta function’ contributions to $\delta_\epsilon S_{\text{bulk}}$ can potentially spoil the fermionic symmetry only at the $\mathbb{Z}_2$ fixed points. Thus, $S_{\text{bulk}}$ is invariant under its fermionic transformations everywhere except perhaps at the $\mathbb{Z}_2$ fixed points. In this section, we will calculate $\delta_\epsilon S_{\text{bulk}}$. 
For our metric and $\mathbb{Z}_2$ parity assignments, the uncancelled variation $\delta_\epsilon S_{\text{bulk}}$ arises from the variation of the 4-Fermi terms. The 4-Fermi terms are

$$S_\psi = \frac{3i}{4} \int K \left( \bar{\psi}_\alpha \psi^\alpha \bar{\psi}_\beta \psi^\beta \nabla_\beta \psi^\alpha + \bar{\psi}_\alpha \bar{\psi}_\beta \psi^\beta \nabla_\beta \psi^\alpha \right) + \text{c.c.}$$

$$= \frac{3i}{2} \int K \bar{\psi}_\alpha \psi^\alpha \bar{\psi}_\beta \psi^\beta \nabla_\beta \psi^\alpha + \text{c.c.}$$

$$= \frac{3i}{2} \int K \left( \bar{\psi}_\alpha \psi^\alpha \right) \left( \bar{\psi}_\beta \psi^\beta \nabla_\beta \psi^\alpha \right) + \text{c.c.}$$

$$= \frac{3i}{2} \int d^5x \varepsilon^{\mu\nu\rho\sigma\lambda} K \left( \bar{\psi}_\mu \psi_{\nu s} \right) \left( \bar{\psi}_\rho \nabla_\sigma \psi_{\lambda r} \right) + \text{c.c.}$$

$$= \frac{3i}{2} \int d^5x K \left[ \varepsilon^{5\rho\bar{\sigma}\lambda} \left( \bar{\psi}_{5 \rho} \psi_{\bar{\sigma} \lambda} \right) \left( \bar{\psi}_{5 \rho} \nabla_{\bar{\sigma}} \psi_{\lambda r} \right) + \varepsilon^{\bar{\mu}5\rho\bar{\sigma}\lambda} \left( \bar{\psi}_{\bar{\mu} \rho} \psi_{5\rho} \right) \left( \bar{\psi}_{\bar{\mu} \rho} \nabla_{5\rho} \psi_{\lambda r} \right) + \varepsilon^{5\mu5\rho\lambda} \left( \bar{\psi}_{5 \mu} \psi_{\rho \lambda} \right) \left( \bar{\psi}_{5 \mu} \nabla_{\rho \lambda} \psi_{r} \right) + \varepsilon^{5\mu5\rho\bar{\sigma}\lambda} \left( \bar{\psi}_{5 \mu} \psi_{\rho \bar{\sigma}} \right) \left( \bar{\psi}_{5 \mu} \nabla_{\rho \bar{\sigma}} \psi_{r} \right) \right] + \text{c.c.}$$

(3.1)

Let us now compute $\delta_\epsilon S_{\text{bulk}}$ by applying $\delta_\epsilon$ to (3.1) and dropping all terms which contribute no delta functions. For our metric and $\mathbb{Z}_2$ parity assignments, we can drop all but

1. The $\partial_\mu$ part of $\nabla_\mu$.

2. The $-\partial_\mu \epsilon_r$ part of $\delta_\epsilon = -\nabla_\mu \epsilon_r$.

The only contributing terms are thus contained in the expression

$$Q \equiv -\frac{3i}{2} \int d^5x K \left\{ \varepsilon^{5\rho\bar{\sigma}\lambda} \left( \partial_{5 \rho} \bar{\psi}_{\bar{\sigma} \lambda} \right) \left( \bar{\psi}_{5 \rho} \partial_{\bar{\sigma}} \psi_{\lambda r} \right) + \varepsilon^{\bar{\mu}5\rho\bar{\sigma}\lambda} \left( \bar{\psi}_{\bar{\mu} \rho} \partial_{\bar{\sigma}} \psi_{\lambda r} \right) + \varepsilon^{5\mu5\rho\lambda} \left( \bar{\psi}_{5 \mu} \bar{\psi}_{\rho \lambda} \right) \left( \partial_{5 \rho} \bar{\psi}_{\lambda r} \right) + \varepsilon^{5\mu5\rho\bar{\sigma}\lambda} \left( \bar{\psi}_{5 \mu} \bar{\psi}_{\rho \bar{\sigma}} \right) \left( \partial_{5 \rho} \bar{\psi}_{\lambda r} \right) \right\} + \text{c.c.}$$

(3.2)

More specifically, the delta function terms contained in $Q$ are obtained by integrating by parts and keeping only the terms in which $\partial_5$ acts on $K$. Thus,

$$\delta_\epsilon S_{\text{bulk}} = \frac{3i}{2} \int d^5x \partial_5 K \left\{ \varepsilon^{5\rho\bar{\sigma}\lambda} \left( \bar{\psi}_{5 \rho} \partial_{\bar{\sigma}} \psi_{\lambda r} \right) + \varepsilon^{\bar{\mu}5\rho\bar{\sigma}\lambda} \left( \bar{\psi}_{\bar{\mu} \rho} \partial_{\bar{\sigma}} \psi_{\lambda r} \right) + \varepsilon^{5\mu5\rho\lambda} \left( \bar{\psi}_{5 \mu} \bar{\psi}_{\rho \lambda} \right) \left( \partial_{5 \rho} \bar{\psi}_{\lambda r} \right) + \varepsilon^{5\mu5\rho\bar{\sigma}\lambda} \left( \bar{\psi}_{5 \mu} \bar{\psi}_{\rho \bar{\sigma}} \right) \left( \partial_{5 \rho} \bar{\psi}_{\lambda r} \right) \right\} + \text{c.c.},$$

(3.3)

where

$$\partial_5 K = 2k \left[ \delta(x^5) - \delta(x^5 - \pi \rho) \right].$$

(3.4)
4 Modified $\mathbb{Z}_2$-invariant bulk theory

The result (3.3) for $\delta S_{\text{bulk}}$ demonstrates that $S_{\text{bulk}}$ is not invariant under the fermionic transformations $\delta \epsilon$. In this section, we will modify $S_{\text{bulk}}$ and $\delta \epsilon$ by replacing the $adS_5 \times su(N) \times u(1)$ covariant derivative $\nabla$ with $\tilde{\nabla}$, where

$$\tilde{\nabla}_\sigma \psi_{\lambda r} \equiv \nabla_\sigma \psi_{\lambda r} + 2\delta_\sigma^\lambda \delta_\lambda^5 \left[ \delta(x^5) - \delta(x^5 - \pi \rho) \right] \text{sgn}(x^5) \Gamma_5 \psi_{\lambda r},$$

$$\tilde{\nabla}_\sigma \epsilon_r \equiv \nabla_\sigma \epsilon_r + 2\delta_\sigma^5 \delta^\lambda_5 \left[ \delta(x^5) - \delta(x^5 - \pi \rho) \right] \text{sgn}(x^5) \Gamma_5 \epsilon_r. \quad (4.1)$$

We will show that the modified bulk action

$$\tilde{S}_{\text{bulk}} \equiv S_{\text{bulk}}(\nabla \to \tilde{\nabla}) \equiv S_{\text{bulk}} + \Delta S_{\text{bulk}} \quad (4.2)$$

is invariant under the modified transformations

$$\tilde{\delta} \epsilon \equiv \delta \epsilon(\nabla \to \tilde{\nabla}) \equiv \delta \epsilon + \Delta \delta \epsilon. \quad (4.3)$$

That is, we will show that

$$\tilde{\delta} \epsilon \tilde{S}_{\text{bulk}} = \delta \epsilon S_{\text{bulk}} + (\Delta \delta \epsilon) S_{\text{bulk}} + \tilde{\delta} \epsilon (\Delta S_{\text{bulk}}) \quad (4.4)$$

vanishes. It is straightforward to check that $\tilde{S}_{\text{bulk}}$ is $\mathbb{Z}_2$-invariant and the transformations $\tilde{\delta} \epsilon$ are consistent with our $\mathbb{Z}_2$ parity assignments.

We begin by computing $(\Delta \delta \epsilon) S_{\text{bulk}}$. For our metric and $\mathbb{Z}_2$ parity assignments, the only part of $S_{\text{bulk}}$ which is not invariant under $\Delta \delta \epsilon$ is $S_{\psi^4}$ (given by (3.1)). Note that

$$(\Delta \delta \epsilon) \psi_{\lambda r} = -2\delta_5^\lambda \left[ \delta(x^5) - \delta(x^5 - \pi \rho) \right] \text{sgn}(x^5) \Gamma_5 \psi_{\lambda r}. \quad (4.5)$$

Thus, after using $K \text{sgn}(x^5) = k$, (2.9), and (3.4), we obtain

$$(\Delta \delta \epsilon) S_{\text{bulk}} = -\frac{3i}{2} \Gamma_5 \int d^5x \partial_5 K \left[ \varepsilon^5 \rho \sigma \lambda \left( \bar{\psi}_{\rho}^r \partial_\sigma \psi_{\lambda r} \right) \left( \bar{\psi}_\mu^s \partial_\mu \psi_{\lambda r} \right) \right. \quad (4.6)$$

Now, let us compute $\tilde{\delta} \epsilon (\Delta S_{\text{bulk}})$. For our metric and $\mathbb{Z}_2$ parity assignments, the only part of $S_{\text{bulk}}$ which is changed by the replacement $\nabla \to \tilde{\nabla}$ is $S_{\psi^4}$. After using $K \text{sgn}(x^5) = k$, (2.8), and (3.4), we obtain

$$\Delta S_{\text{bulk}} = \frac{3i}{2} \Gamma_5 \int d^5x \partial_5 K \varepsilon^\mu \rho \sigma \lambda \left( \bar{\psi}_\mu^r \psi_{\rho}^s \right) \left( \bar{\psi}_\mu^s \psi_{\rho}^r \right) + c.c. \quad (4.7)$$

Applying $\tilde{\delta} \epsilon$ to (4.7) yields

$$\tilde{\delta} \epsilon (\Delta S_{\text{bulk}}) = -\frac{3i}{2} \Gamma_5 \int d^5x \partial_5 K \varepsilon^\mu \rho \sigma \lambda \left[ \left( \partial_\mu \bar{\psi}_\sigma^r \psi_{\rho}^s \right) \left( \bar{\psi}_\mu^s \psi_{\rho}^r \right) + \left( \bar{\psi}_\mu^r \partial_\sigma \psi_{\rho}^s \right) \left( \bar{\psi}_\mu^s \psi_{\rho}^r \right) \right. \quad (4.8)$$

Using the results (3.3), (4.6), and (4.8) in (4.4) yields

$$\tilde{\delta} \epsilon \tilde{S}_{\text{bulk}} = 0. \quad (4.9)$$
5 Brane action

To complete our model, we add the brane action \( S_{\text{brane}} \). In the absence of particle excitations, the brane action consists of brane tensions. That is,

\[
S_{\text{brane}} = - \int d^5 x \ e^{(4)} \left[ T^{(0)} \delta(x^5) + T^{(\pi \rho)} \delta(x^5 - \pi \rho) \right] + \text{excitations},
\]

where \( e^{(4)} \equiv \det(e_{\mu}^{\ a}) \). As discussed in Section 2, \( \mathbb{Z}_2 \)-odd quantities vanish at the \( \mathbb{Z}_2 \) fixed points. Thus, it is clear that \( S_{\text{brane}} \) is \( \mathbb{Z}_2 \)-even. Further discussion of 3-brane actions can be found in [11].

6 Connection with original RS model

In this section, we will show that our total action \( S = \tilde{S}_{\text{bulk}} + S_{\text{brane}} \) reduces to the original Randall-Sundrum model [10] when \( \tilde{S}_{\text{bulk}} \) is restricted to its gravitational sector.

The gravitational sector of \( \tilde{S}_{\text{bulk}} \) is \( S_{\text{grav}} \), given by the first equation of (2.2). \( S_{\text{grav}} \) consists of three terms:

1. The ‘Gauss-Bonnet’ term \( \int \frac{1}{8} K \varepsilon_{abcde} R^{ab} R^{cd} e^e / L \).
2. The ‘Einstein-Hilbert’ term \( \int \frac{1}{8} \cdot \frac{2}{3} K \varepsilon_{abcde} R^{ab} e^c e^d e^e / L^3 \).
3. The ‘cosmological constant’ term \( \int \frac{1}{8} \cdot \frac{1}{5} K \varepsilon_{abcde} e^a e^b e^c e^d e^e / L^5 \).

For our metric, the first term can be expressed as a linear combination of the other two. Summing the three terms yields an effective Einstein-Hilbert term and an effective cosmological constant term. To demonstrate this explicitly, let us evaluate \( S_{\text{grav}} \) for our metric. Using the results in the Appendix, we obtain

\[
\begin{align*}
\varepsilon_{abcde} R^{ab} R^{cd} e^e &= d^5 x \ e \left( -\frac{120}{\ell^4} + \frac{192}{\ell^3} \left[ \delta(x^5) - \delta(x^5 - \pi \rho) \right] \right), \\
\varepsilon_{abcde} R^{ab} e^c e^d e^e &= d^5 x \ e \left( -6R \right), \\
\varepsilon_{abcde} e^a e^b e^c e^d e^e &= d^5 x \ e \left( -5! \right),
\end{align*}
\]

(6.1)
where \( e \equiv \det(e^a_\mu) \). Thus,

\[
S_{\text{grav}} = \int d^5x \frac{1}{8} \left\{ \frac{k}{\ell} \left( -\frac{120}{\ell^4} + \frac{192}{\ell^3} \left[ \delta(x^5) - \delta(x^5 - \pi \rho) \right] \right) \right.
+ \frac{2k}{3\ell^3} (-6R) + \frac{k}{\ell^5} (-5! \right\} \\
= \int d^5x \frac{k}{\ell^3} \left\{ -\frac{15}{\ell^2} + \frac{24}{\ell} \left[ \delta(x^5) - \delta(x^5 - \pi \rho) \right] - \frac{1}{2} R - \frac{3}{\ell^2} \right\} \\
= \int d^5x \frac{k}{\ell^3} \left\{ \frac{3}{2} \left( -\frac{20}{\ell^2} + \frac{16}{\ell} \left[ \delta(x^5) - \delta(x^5 - \pi \rho) \right] \right) - \frac{1}{2} R + \frac{12}{\ell^2} \right\} \\
= \int d^5x \frac{k}{\ell^3} \left( R + \frac{12}{\ell^2} \right) \\
= \int d^5x \ e (2M^3 R - \Lambda), \tag{6.2}
\]

where \( M \) is the five-dimensional gravitational mass scale\(^7\), \( \Lambda \) is the bulk cosmological constant, and

\[
M^3 = \frac{k}{2\ell^3}, \quad \Lambda = -\frac{24M^3}{\ell^2}. \tag{6.3}
\]

Combining the result (6.2) with (5.1), we obtain the action of the original Randall-Sundrum model. It is shown in [10] that the five-dimensional vacuum Einstein’s equations for this system,

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{1}{4M^3} \left\{ g_{\mu
u} \Lambda + \frac{e^{(4)}}{e} \delta^\mu_\nu \delta^\rho_\sigma g_{\mu\rho} \left[ T^{(0)} \delta(x^5) + T^{(\pi \rho)} \delta(x^5 - \pi \rho) \right] \right\}, \tag{6.4}
\]

are solved by our metric provided that the relations

\[
T^{(0)} = -T^{(\pi \rho)} = 24M^3/\ell, \quad \Lambda = -24M^3/\ell^2
\]

are satisfied.

### 7 Killing spinors

In this section, we will solve the Killing spinor equations for a purely bosonic configuration with vanishing \( su(N) \) and \( u(1) \) gauge fields. In this case, the Killing spinor equations reduce to

\[
0 = \delta_\epsilon \psi_{\mu r} = -\partial_\mu \epsilon_r - \frac{1}{2} \alpha' \Gamma_\mu (\Gamma_5 - 1) \epsilon_r, \\
0 = \delta_\epsilon \psi_{5r} = -\partial_5 \epsilon_r + \frac{1}{2} \alpha' \Gamma_5 \epsilon_r - 2 \left[ \delta(x^5) - \delta(x^5 - \pi \rho) \right] \text{sgn}(x^5) \Gamma_5 \epsilon_r. \tag{7.1}
\]

\(^7\)\( M \) is related to the four-dimensional gravitational mass scale \( M_{(4)} = 2.43 \times 10^{18} \text{ GeV} \) by \( M_{(4)}^2 = M^3 \int_{-\pi \rho}^{+\pi \rho} dx^5 a^2(x^5) = M^3 \ell [1 - \exp(-2\pi \rho/\ell)]. \) The effective mass scales on the 3-branes at \( x^5 = 0 \) and \( x^5 = \pi \rho \) are respectively \( M_{(4)} \) and \( M_{(4)} e^{-\pi \rho/\ell}. \) If the Standard Model fields live on the 3-brane at \( x^5 = \pi \rho, \) then \( M_{(4)} e^{-\pi \rho/\ell} \) can be associated with the electroweak scale.
To solve these equations, split $\epsilon_r$ into $\mathbb{Z}_2$-even ($\epsilon^+_r$) and $\mathbb{Z}_2$-odd ($\epsilon^-_r$) pieces:

$$\epsilon_r = \epsilon^+_r + \epsilon^-_r,$$

(7.2)

where

$$\epsilon^\pm \equiv \frac{1}{2} (\epsilon_r \pm \Gamma_5 \epsilon_r) = \pm \Gamma_5 \epsilon^\pm_r.$$

(7.3)

We obtain the following system of equations:

$$
\partial_\mu \epsilon^+_r = -(a'/a) \Gamma_\mu \Gamma_5 \epsilon^-_r,
\partial_\mu \epsilon^-_r = 0,
\partial_5 \epsilon^+_r = \frac{1}{2} (a'/a) \epsilon^+_r - 2 [\delta(x^5) - \delta(x^5 - \pi \rho)] \text{sgn}(x^5) \epsilon^+_r,
\partial_5 \epsilon^-_r = -\frac{1}{2} (a'/a) \epsilon^-_r + 2 [\delta(x^5) - \delta(x^5 - \pi \rho)] \text{sgn}(x^5) \epsilon^-_r.
$$

(7.4)

These equations are solved by

$$
\epsilon^+_r = a^{1/2} \left[ -(a'/a) x^5 \Gamma_\mu \Gamma_5 \text{sgn}(x^5) \chi_r^{-(0)} + \chi_r^{+(0)} \right],
\epsilon^-_r = a^{-1/2} \text{sgn}(x^5) \chi_r^{-(0)},
$$

(7.5)

where $\chi_r^{+(0)}$ and $\chi_r^{-(0)}$ are constant (projected) spinors. Thus, our solution for the Killing spinors is

$$\epsilon_r = a^{1/2} \chi_r^{+(0)} + a^{-1/2} \text{sgn}(x^5) \left( 1 - \frac{a'}{a} x^5 \Gamma_\mu \Gamma_5 \right) \chi_r^{-(0)}.
$$

(7.6)

## 8 Conclusion

We have constructed a Randall-Sundrum scenario from $D=5$ Chern-Simons AdS supergravity. Our total action $S = \tilde{S}_{\text{bulk}} + S_{\text{brane}}$ is $\mathbb{Z}_2$-invariant. $\tilde{S}_{\text{bulk}}$ is invariant under the fermionic transformations $\tilde{\delta}_\epsilon$. However,

$$\tilde{\delta}_\epsilon S_{\text{brane}} = - \int d^5x \tilde{\delta}_\epsilon e^{(4)} \left[ T^{(0)} \delta(x^5) + T^{(\pi \rho)} \delta(x^5 - \pi \rho) \right] + \cdots,$$

(8.1)

where

$$\tilde{\delta}_\epsilon e^{(4)} = e^{(4)} \left[ -\frac{1}{2} \left( \bar{\psi}_\mu \Gamma^\mu \epsilon_r - \bar{\epsilon}^5 \Gamma^\mu \psi_r \right) \right].
$$

(8.2)

Thus, the fermionic symmetry is broken by $S_{\text{brane}}$. Nevertheless, the Killing spinors of Section 7 are globally defined.

Our model reduces to the original Randall-Sundrum model [10] when $\tilde{S}_{\text{bulk}}$ is restricted to its gravitational sector. The original Randall-Sundrum model involves the fine-tuning relations

$$T^{(0)} = - T^{(\pi \rho)} = 24 M^3 / \ell, \quad \Lambda = -24 M^3 / \ell^2.$$

---

8It is straightforward to check that (7.5) satisfies the first, second, and fourth equations of (7.4). There is, however, a subtlety in checking that (7.5) satisfies the third equation of (7.4). Unlike $\epsilon^+_r, \epsilon^-_r$ is a smooth function of $x^5$. Thus, the second term on the right side of the third equation of (7.4) contributes nothing.
Randall-Sundrum scenarios constructed from standard $D=5$ supergravity theories yield these relations (up to an overall normalization factor) as a consequence of local supersymmetry (some examples are given in [12]). In our case, the relation $\Lambda = -24M^3/\ell^2$ follows from our metric choice. We do not obtain the relations $T^{(0)} = -T^{(\pi \rho)} = 24M^3/\ell$ as a consequence of local fermionic symmetry. These are fine-tuning relations in our model.

A Appendix

In this Appendix, we work out the fünfbein, spin connection, curvature 2-form components, Ricci tensor, and Ricci scalar for our metric (1.1). For the fünfbein, we obtain

$$e_\mu^\bar{a} = a\delta_\mu^\bar{a}, \quad e_{\mu\bar{a}} = e_\mu^\bar{b} \eta_{\bar{b}\bar{a}}, \quad e^{\bar{a}\bar{a}} = g^{\bar{a}\bar{b}} e_\mu^\bar{a},$$

$$e_\mu^\bar{a} = a^{-1} \delta_\mu^\bar{a}, \quad e_{\bar{a}\bar{a}} = e_\bar{a}^\bar{b} g_{\bar{b}\bar{a}}, \quad e^{\bar{a}\bar{a}} = \bar{\eta}^{\bar{a}\bar{b}} e_\bar{b}^{\bar{a}},$$

$$e_5^\bar{a} = e_{\bar{a}\bar{a}} = e_5^\bar{a} = e_5^{\bar{a}} = 1.$$

(A.1)

Our conventions for the spin connection, curvature 2-form components, Ricci tensor, and Ricci scalar are respectively

$$\omega_{\mu}^{ab} = \frac{1}{2} e^{\nu c}(\partial_\mu e_\nu^b - \partial_\nu e_\mu^b) - \frac{1}{2} e^{\sigma b}(\partial_\mu e_\nu^a - \partial_\nu e_\mu^a) - \frac{1}{2} e^{\sigma b}(\partial_\mu e_\nu^a - \partial_\nu e_\mu^a)\epsilon_\mu,$$

$$R_{\mu\nu}^{ab} = \delta_\mu \omega_\nu^{ab} - \delta_\nu \omega_\mu^{ab} + \omega_\mu^{ac} \omega_\nu^{bc} - \omega_\nu^{ac} \omega_\mu^{bc},$$

$$R_{\nu\sigma} = R_{\mu\nu}^{ab} e^{a}_\nu e^{b}_\sigma, \quad R = e^{a}_\mu e^{a\nu} R_{\mu\nu}^{ab}.$$

For the metric (1.1), the nonzero quantities here are

$$\omega_{\mu}^\bar{a} = -\omega_{\bar{a}}^\mu = a' \delta_\mu^\bar{a} = -e_\mu^\bar{a}/L,$$

(A.2)

$$R_{\bar{a}\bar{b}} = -a'^2 (\delta_{\mu}^\bar{a} \delta_{\nu}^\bar{b} - \delta_{\nu}^\bar{a} \delta_{\mu}^\bar{b}) = -(e_\mu^\bar{a} e_\nu^\bar{b} - e_\mu^\bar{b} e_\nu^\bar{a})/L^2;$$

$$R_{\bar{a}\bar{a}} = a'' \delta_\mu^\bar{a} = e_\mu^\bar{a} \left\{ 1/L^2 - 2[\delta(x^5) - \delta(x^5 - \pi \rho)]/L \right\},$$

(A.3)

$$R_{\bar{a}\bar{a}} = - (a a'' + 3 a'^2) \eta_{\bar{a}\bar{a}} = - \left\{ 4/L^2 - 2[\delta(x^5) - \delta(x^5 - \pi \rho)]/L \right\} g_{\bar{a}\bar{a}},$$

$$R_{\bar{5}\bar{5}} = - 4 a^{-1} a'' = - \left\{ 4/L^2 - 8[\delta(x^5) - \delta(x^5 - \pi \rho)]/L \right\},$$

(A.4)

$$R = -8 a^{-1} a'' - 12 a^{-2} a'^2 = -20/L^2 + 16[\delta(x^5) - \delta(x^5 - \pi \rho)]/L,$$

(A.5)

and those related to (A.3) by $R_{\mu\nu}^{ab} = -R_{\nu\mu}^{ab} = -R_{\mu\nu}^{ba}$. The prime symbol $'$ denotes partial differentiation with respect to $x^\nu$.

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References


