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SCALING AND MULTIFRACTALITY IN ONE-DIMENSIONAL ASYMMETRIC MAPS

by

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ABSTRACT

We study two types of S-unimodal one-dimensional asymmetric maps. The asymmetries concern the amplitude and the exponent of the map $1 - a|x|^z$. In both cases the well known metric universality of the symmetric map is lost: the behavior of the scaling factors δ and α and of the multifractal function $f_k(\alpha)$ is oscillatory. Nevertheless, the road to chaos remains that of doubling-period bifurcations, and all its topological properties are preserved.

Key-words: Chaos; Dynamical systems; Asymmetric maps; Multifractals.

1 INTRODUCTION

The period-doubling road to chaos is nowadays very well established from both theoretical and experimental points of view. This road presents universal relations which have been studied in many physical systems. The period-doubling phenomenon was first observed in the logistic map, which, through a variable transformation, can be rewritten as

$$x_{t+1} = f(x_t) \equiv 1 - a x_t^{\,2} \qquad (1)$$

However, the period doubling road appears in any map which satisfies the following conditions [1]: (i) continuously differentiable; (ii) to map the interval $[-1,1]$ into itself, with a single maximum at $x=0$, strictly decreasing on $[0,1]$ and strictly increasing on $[-1,0]$; (iii) negative Schwarzian. Such maps are denominated S-unimodal maps.

As the parameter a in Eq. (1) is increased (starting from $a=0$) the attractors (or long time solutions) of the map show a sequence of periodic orbits with period $2^k (k = 0,1,2,\ldots)$; the k-th period appears at a_k through a pitchfork bifurcation of the (k-1)-th attractor, and the sequence $\{a_k\}$ accumulates $(k \to \infty)$ at $a^* \simeq 1.401155$, where the system enters into chaos. For every k-th periodic orbit there is one value $\tilde{a}_k$ of the control parameter for which the orbit includes the *critical point* (peak) of the map. At this value of a the cycle is called *superstable*.

The scaling factor defined through

$$\delta_k = \frac{\tilde{a}_{k+1}-\tilde{a}_k}{\tilde{a}_{k+2}-\tilde{a}_{k+1}} \tag{2}$$

converges to 4.6692... in the $k \to \infty$ limit. The same ratio of convergence is observed for the set $\{a_k\}$. A second scaling factor can be associated with the set $\{\tilde{a}_k\}$, namely α_k, defined through

$$\alpha_k = \frac{f_{\tilde{a}_k}^{2^{k-1}}(0)}{f_{\tilde{a}_{k+1}}^{2^k}(0)} \tag{3}$$

which converges, in the $k \to \infty$ limit, to 2.5029...

These universal metric constant were studied by Feigenbaum, Coullet and Tresser[2] and others. After these papers, a great amount of theoretical and experimental works have been dedicated to this problem. In the litterature the period-doubling road to chaos has been essentially associated with maps which are symmetric about the maximum. In fact, all proofs concerning the metric universality of this road are based on the symmetric logistic-like map.

A question aroused in recent papers, namely what could happen to the metric universality of the logistic-like maps if an asymmetry was introduced at the maximum of the map. Three types of asymmetry were studied: (i) an asymmetry in the amplitude[3,5], (ii) in the exponent[4,5], and (iii) a discontinuity[5]. In the two first cases the road to chaos still is through period-doubling bifurcations, but the metric universality of the logistic-like map is destroyed. When the asymmetry is a discontinuity the road to chaos is a completely new one (*gap road* to chaos[5]). Concerning the experimental standpoint, measurements in forced nonlinear oscillators described by asymmetric maps were done by Octavio et al [6]. They confirmed theoretical predictions [4,5].

It has been shown [4] that if a map is globally asymmetric, but with a maximum which is locally symmetric, the dynamical behavior of the map in the first bifurcations will be dominated by the asymmetries, until the successive iterates of the map localize the symmetric maximum, where the Feigenbaum scenario appears. In physical experiments only the first bifurcations are actually observed, and these might be dominated by the asymmetries. These remarks reinforce the importance of studying asymmetric maps.

It the present paper we will study maps with an asymmetry in the amplitude and in the exponent. In section 2 we study the asymmetry in the amplitude. In this case the scaling factors α_k and δ_k and the multifractal function $f_k(a)$ show an oscillatory behavior. Moreover, at the accumulation point of the bifurcations there is a function h(x) satisfying a functional equation which is used to introduce a renormalization group. Section 3 is dedicated to show the oscillatory (divergent) behavior of δ_k, α_k and $f_k(a)$ in maps with an asymmetry in the exponent. The conclusions are presented in the Section 4.

2 AMPLITUDE ASYMMETRY

2.1 Numerical results

The asymmetric map we shall consider is given by

$$x_{t+1} = f(x_t) \equiv \begin{cases} 1 - a_1|x_t|^z \;, & x_t \geq 0 \\ 1 - a_2|x_t|^z \;, & x_t \leq 0 \end{cases} \tag{4}$$

with $z > 1$. For $a_1 = a_2$ we recover the logistic-like symmetric map. In Fig. 1(a) we show $f(x)$ vs. x for a typical case, and in Fig. 1(b) we show the finite attractor of the map as a function of a_1 for $a_2 = a_1 + 0.2$ and $z = 2$. We observe that the road to chaos is via period-doubling bifurcations, since this map is S-unimodal. In Fig. 2 we show the critical lines $a_2^*(a_1)$ and $a_2^M(a_1)$, which represent the accumulation point of the bifurcations and the disappearance of the finite attractor, respectively.

If we denote by $\tilde{a}_k$ the value of a_2 where the 2^k-cycle is superstable, *for fixed* a_1, we verify that the scaling factor δ_k defined by Eq. (2) presents an oscillatory behavior for increasing k. In the limit $k \to \infty$ there is a convergence of δ_k onto two constants c_1 and c_2, respectively given by

$$\lim_{k\to\infty} \frac{\tilde{a}_{2k} - \tilde{a}_{2k-1}}{\tilde{a}_{2k+1} - \tilde{a}_{2k}} = c_1 \tag{5.a}$$

and

$$\lim_{k\to\infty} \frac{\tilde{a}_{2k+1} - \tilde{a}_{2k}}{\tilde{a}_{2k+2} - \tilde{a}_{2k+1}} = c_2 \tag{5.b}$$

In Fig. 3.a we show c_1 and c_2 as functions of a_2^* for $z=2$. Observe that at $a_2^* \simeq 1.401155$, $c_1 = c_2 \equiv \delta$, thus recovering the universal scaling constant for the symmetric map. We find that $c_1 \simeq 3.3a_2^*$ and $c_2 \simeq 7/a_2^*$. Therefore, the product c_1c_2 is approximately constant and equal to $\delta^2 = 21.8014...$

A similar oscillatory behavior is observed for the scaling factor α_k defined by Eq. (3). There are two limit values for α_k as $k \to \infty$, respectively given by

$$d_1 = \lim_{k\to\infty} \frac{f_{\tilde{a}_{2k-1}}^{2^{2k-2}}(0)}{f_{\tilde{a}_{2k}}^{2^{2k-1}}(0)} \tag{6.a}$$

and

$$d_2 = \lim_{k\to\infty} \frac{f_{\tilde{a}_{2k}}^{2^{2k-1}}(0)}{f_{\tilde{a}_{2k+1}}^{2^{2k}}(0)} \tag{6.b}$$

In Fig. 3(b) we show the asymptotic values d_1 and d_2 as functions of a_2^*. Observe that at $a_2^* \simeq 1.401155$, $d_1 = d_2 \equiv \alpha$, thus recovering the universal scaling constant of the symmetric map for $z=2$. We find $d_1 \simeq 1.8a_2^*$ and $d_2 \simeq 3.5/a_2^*$. Therefore, the product d_1d_2 is approximately constant and equal to $\alpha^2 = 6.2645...$ In Table I we show the values of $\tilde{a}_k$, δ_k, α_k, $\delta_k\delta_{k+1}$ and $\alpha_k\alpha_{k+1}$ for $a_1 = 1.316461$. For $z \neq 2$ $(z>1)$ the qualitative behavior of these quantities is similar to the behavior for $z=2$.

2.2 The limiting function

The scaling factors δ and α are only two particular values of a set of universal quantities associated with the symmetric map. One of the most relevant elements of this set is the Feigenbaum-Cvitanovic equation.

Feigenbaum [2] showed that a S-unimodal function $f_a(x)$ with a symmetric maximum of order z (z > 1), when composed with itself at the superstable 2-cycle, will (roughly) reproduce itself reduced in scale by $-\alpha$. This selfsimilar structure is also observed for the successive 2^k (k > 1) superstable cycles. At the accumulation point of the bifurcations ($a = \tilde{a}^*$), we can verify the following relation (called Feigenbaum-Cvitanovic equation):

$$g(x) = -\alpha g^2(-x/\alpha) \tag{7}$$

where

$$g(x) = \lim_{k\to\infty} (-\alpha)^k f_{\tilde{a}^*}^{2^k} (x/(-\alpha)^k) \tag{8}$$

Equation (7) says nothing about absolute scales since it is invariant under the transformation $g(x) \to u\,g(x/u)$. Using this freedom one may set $g(0) = 1$, which leads to $g(1) = -1/\alpha$. The function g(x) is universal, i.e., it is the same for any S-unimodal map with a maximum of order z.

When there is an asymmetry in the amplitude of the map (Eq. (4)) relation (7) is not verified. However, in this case we observe that, at the accumulation point $a_2^*(a_1)$, the function $f^4(x)$

magnified by $d_1 d_2 = \alpha_k \alpha_{k+1}$ $(k \to \infty)$ is similar to f(x), for x small. This process of convergence will continue in the functions $f^{16}(x)$, $f^{64}(x)$, etc., when magnified by the appropriate factor.

Now, let us define the function

$$h_k(x) = P_k f^{2^{2k}}_{\tilde{a}_2^*}(x/P_k) \qquad (k = 1,2,3\ldots) \tag{9}$$

where $P_k = \alpha_1 \alpha_2 \ldots \alpha_{2k}$, the $\{\alpha_k\}$ being given by Eq. (3). We can rewrite Eq. (9) in the following form:

$$\frac{1}{P_k} h_k(P_k x) = f^{2^{2k}}_{\tilde{a}_2^*}(x) \tag{10}$$

Consequently we have

$$\frac{1}{P_{k+1}} h_{k+1}(P_{k+1}x) = f^{2^{2(k+1)}}_{\tilde{a}_2^*}(x) = \frac{1}{P_k} h_k^4(P_k x) \tag{11}$$

hence, by replacing x by x/P_{k+1},

$$h_{k+1}(x) = \frac{P_{k+1}}{P_k} h_k^4(x/(P_{k+1}/P_k)) \tag{12}$$

Since $P_{k+1}/P_k = \alpha_{2k+1}\alpha_{2k+2}$, we find

$$h_{k+1}(x) = \alpha_{2k+1}\alpha_{2k+2} h_k^4(x/(\alpha_{2k+1}\alpha_{2k+2})) \tag{13}$$

In the limit $k \to \infty$ we observe that $h_k(x)$ converges and consequently we verify the following relation

$$h(x) = \beta h^4(x/\beta) \tag{14}$$

where $\beta \equiv d_1 d_2$ (see also Kawai and Tye[3]). If we put $h(0)=1$ we obtain $\beta = 1/h^3(1)$. Note by eq. 9 that the function $h(x)$ depends on the value of a_2^*, which in turn depends on a_1. Consequently $h(x)$ is not universal.

2.3 Renormalization Group

The similarity shared by the functions $f(x)$, $\beta f^4(x/\beta)$, $\beta^2 f^{16}(x/\beta^2)$, etc., at the accumulation point of the bifurcations can be used to introduce a renormalization group. In a first approximation let us consider the renormalization group which consists in making equal $f(x)$ and $\alpha_1\alpha_2 f^4(x/(\alpha_1\alpha_2))$. We could improve this approximation by making equal $\alpha_1\alpha_2 f^4(x/(\alpha_1\alpha_2))$ and $\alpha_1\alpha_2\alpha_3\alpha_4 f^{16}(x/(\alpha_1\alpha_2\alpha_3\alpha_4))$, etc.

Within the first approximation, i.e., by imposing (for small x)

$$f(x) = \alpha_1\alpha_2 f^4(x/(\alpha_1\alpha_2)) \tag{15}$$

we find the following relations

$$z^3 a_1^2 a_2^* (1-a_1)^{z-1} (1-a_2^*(1-a_1)^z)^{z-1} (1-a_1(1-a_2^*(1-a_1)^z)^z)^{z-1} + 1 = 0 \tag{16}$$

and

$$\alpha_1\alpha_2 = \frac{1}{1-a_1(1-a_2^*(1-a_1)^z)^z} \tag{17}$$

From Eq. (16) we obtain the critical line $a_2^*(a_1)$. For $z=2$ the curve obtained through the renormalization group coincides, within graphical resolution, with the exact (numerical) curve (Fig. 2). In Fig. 4 we show $a_2^*(a_1)$ for $z=2$ and $z=4$ obtained from Eq. (16). The Eq. (17) provides a good approximation for $\alpha_1\alpha_2$, differing at most by 10% from the numerically exact value.

By using Eq. (3) we can separately obtain α_1 and α_2. More explicitely, we obtain

$$\alpha_1 = \frac{1}{(1-a_1)} \tag{18}$$

and

$$\alpha_2 = \frac{(1-a_1)}{1-a_1(1-a_2^*(1-a_1)^z)^z} \tag{19}$$

These expressions provide only rough approximations for the asymptotic values of α_k and α_{k+1} in spite of the fact that the product $\alpha_k\alpha_{k+1}$ (Eq. (17)) is satisfactorily recovered.

2.4 The function $f(\alpha)$

The attractor of the symmetric map $(a_1=a_2)$ at the accumulation point of the bifurcations is a complex object. This object has different scale indices for different regions of the attractor and for this reason it is called a multifractal.

The formalism presented by Halsey et al[7] to study the multifractals consists in covering the attractor with boxes, indexed by i, of size ℓ_i, and assume that the probability density scales like $p_i \propto \ell_i^{\alpha}$ in the limit $\ell_i \to 0$. The next step is to define the normalized partition function.

$$\Gamma(q,\tau) \equiv \sum_i \frac{p_i^q}{\ell_i^{\tau}} = 1 \tag{20}$$

which determines the function $\tau(q)$ which in turn determines the function $f(\alpha)$ (frequently called $f(\alpha)$) through a Legendre transformation.

For a 2^k-cycle and $p_i \equiv p = \frac{1}{2^{k-1}}$ the partition function becomes

$$\Gamma_k = \left(\frac{1}{2^{k-1}}\right)^q \sum_{m=1}^{2^{k-1}} d_{k,m}^{-\tau} \tag{21}$$

where

$$d_{k,m} = |f_{\tilde{a}_k}^{(m+2^{k-1})}(0) - f_{\tilde{a}_k}^{(m)}(0)| \tag{22}$$

In this expression, $\tilde{a}_k$ represents the superstable 2^k-cycle. For the symmetric map $(a_1 = a_2)$, the function $f_k(\alpha)$ obtained by Eq. (21) converges, for k large enough, to a universal function $f(\alpha)$. The minimal and maximal values of α, which respectively characterize the most concentrated and most rarefied regions of the attractor, are given by

$$a_{min} = \frac{\ell n\ 2}{\ell n\ \alpha^{z}} \tag{23.a}$$

and

$$a_{max} = \frac{\ell n\ 2}{\ell n\ \alpha} \tag{23.b}$$

where z is the order of the maximum. Consequently $a_{max} = z a_{min}$. This relation is useful to determine the order of the maximum of the map in physical experiments where $f(a)$ can be determined.

For the asymmetric map ($a_1 \neq a_2$) the oscillatory behavior of the scaling factors δ_k and α_k is also present in the function $f_k(a)$. For k large enough, $f_k(a)$ oscillates between two limit functions. In Fig. 5 we show the two limit functions $f(a)$ for $a_1 = 2.411713$ and $z = 2$.

In spite of the fact that the unicity of $f(a)$ does not hold any-more, we verify that $a_{min}/a_{max} \simeq z$ in both limit curves of Fig. 5. Therefore, this relation remains useful for determining the order z of the maximum, even in asymmetric maps with $a_1 \neq a_2$.

3 EXPONENT ASYMMETRY

In this section we consider the S-unimodal asymmetric maps

$$x_{t+1} = f(x_t) \equiv \begin{cases} 1 - a|x_t|^{z_1}, & x \geq 0 \\ 1 - a|x_t|^{z_2}, & x \leq 0 \end{cases} \tag{24}$$

$(z_1, z_2 > 1)$. When $z_1 = z_2$ we recover the logistic-like symmetric map. As shown in Refs. [4] and [5] the metric universality observed in the symmetric map is completely modified when there is an asymmetry in the exponent. In this case, the scaling factors α_k and δ_k present an oscillatory behavior. In Table II we display the values of $\tilde{a}_k$, δ_k (calculated both for the superstable cycles and for the bifurcation points) and α_k for the case $(z_1, z_2) = (4,2)$. Note that the oscillatory behavior is present in both ways used to calculate δ_k; however, the oscillatory behavior is stronger for calculations based on the superstable cycles than for those based on the bifurcation points.

The multifractal function $f_k(\alpha)$ for this map also presents an oscillatory behavior. In Fig. 6 we show $f_k(\alpha)$ for $(z_1, z_2) = (4,2)$ and $k = 8,9,10$ and 11. For all the values of k we have studied, we observe that $\alpha_{k_{max}}/\alpha_{k_{min}}$ oscillates between the values z_1 and z_2. Therefore, oncemore we see that the dynamics of asymmetric maps is alternatively dominated by the right and left sides of the map. The maximum of $f_k(\alpha)$ oscillates, but it cannot be greater than 1, since the present map is one-dimensional.

4 CONCLUSIONS

We studied the behavior of the scaling factors δ and α and of the multifractal function $f(\alpha)$ associated with two types of S-unimodal asymmetric maps. The asymmetries were introduced at the maximum of the logistic-like map $x' = 1-a|x|^z$. The first case we considered is an asymmetry in the amplitude ($a_1 \neq a_2$) and the second one an asymmetry in the exponent ($z_1 \neq z_2$). In the first case the scaling factors present an oscillatory behavior between two limit convergent values. The function $f_k(\alpha)$ presents the same behavior, having two limit convergent functions. At the accumulation point of the bifurcations there is a limit function $h(x)$ which satisfies the relation $h(x) = \beta h^4(x/\beta)$ where $\beta = \lim_{k\to\infty} \alpha_k \alpha_{k+1}$. Therefore, we observe that this asymmetry preserves the behavior of the symmetric map if we consider, in the bifurcation tree, subtrees with k even or odd. Also in the case of the exponent asymmetry, the metric simplicity of the symmetric map is destroyed. In this case both scaling factors δ and α as well as the function $f(\alpha)$ exhibit an oscillatory behavior (between two branches one of which diverges).

CAPTION FOR FIGURES AND TABLES

Fig. 1 - (a) Asymmetric map with $a_1 = 1.8$, $a_2 = 2$ and $z = 2$; (b) a_1-evolution of the attractor with $a_2 = a_1 + 0.2$.

Fig. 2 - Critical lines representing the accumulation of the bifurcations $a_2^*(a_1)$ and the disappearance of finite attractor $a^M(a_1)$, for $z = 2$.

Fig. 3 - Scaling factors δ_k(a) and α_k(b) as function of a_2^* for $z = 2$.

Fig. 4 - The critical lines $a_2^*(a_1)$ obtained through the renormalization group for $z = 2$ and $z = 4$.

Fig, 5 - Asymptotic multifractal functions $f(\alpha)$ in the $k \to \infty$ limit for $a_1 = 2.411713$ and $z = 2$. The dashed line corresponds to the symmetric case $a_1 = a_2$.

Fig. 6 - Multifractal function $f_k(\alpha)$ for $k = 8, 9, 10$ and 11 and $(z_1, z_2) = (4, 2)$.

Table I - Values of $\tilde{a}_{2_k}$, δ_k, α_k, $\delta_k \delta_{k+1}$ and $\alpha_k \alpha_{k+1}$ for $a_1 = 1.316461$.

Table II - Values of $\tilde{a}_k$, δ_k (calculated through the bifurcation points and superstable cycles) and α_k for $(z_1, z_2) = (4, 2)$.

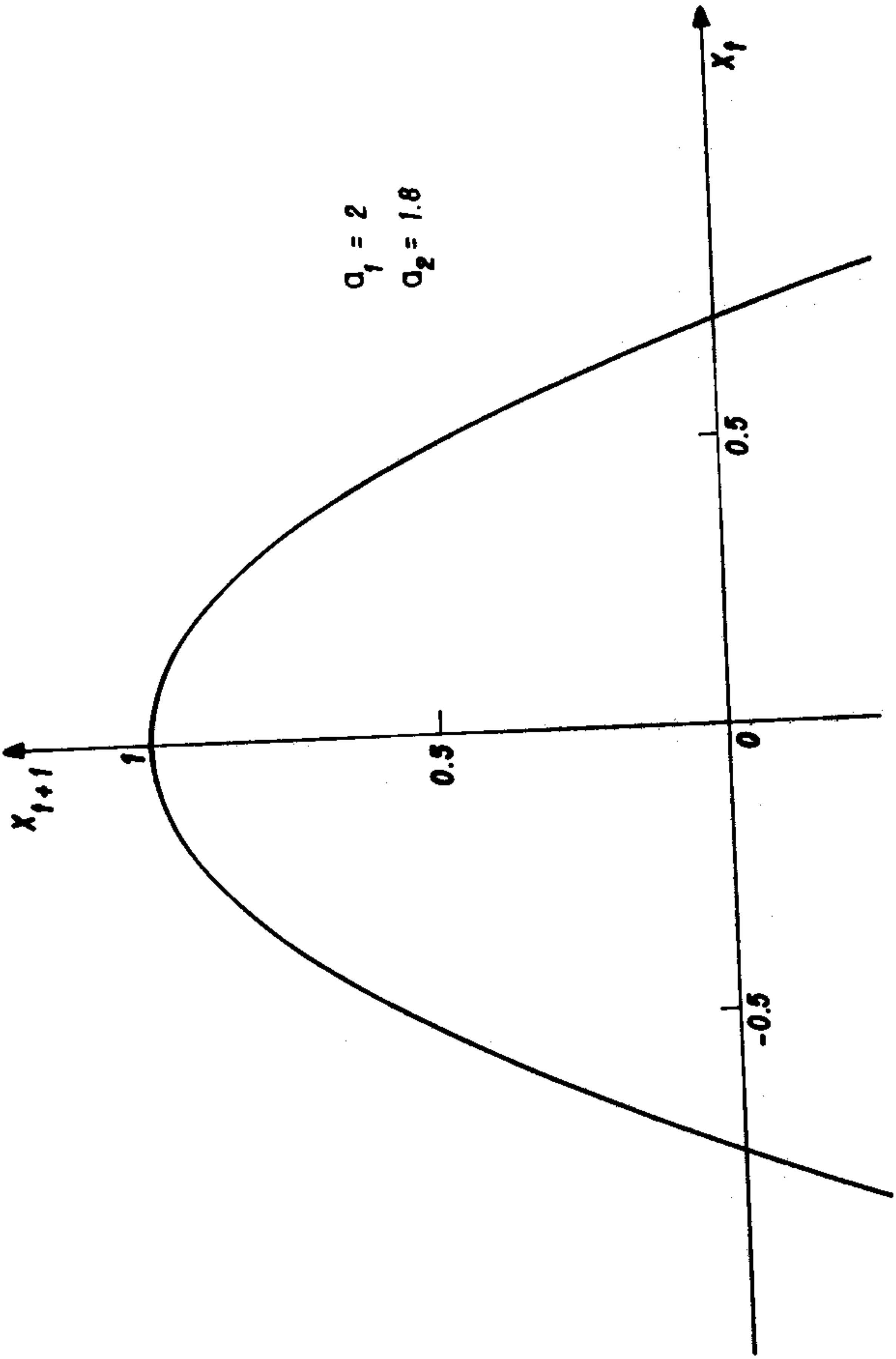

FIG. 1(a)

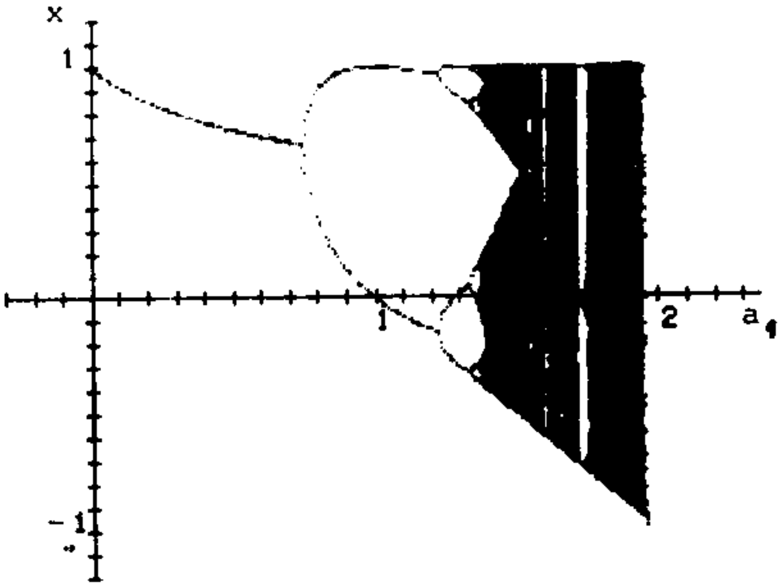

FIG. 1(b)

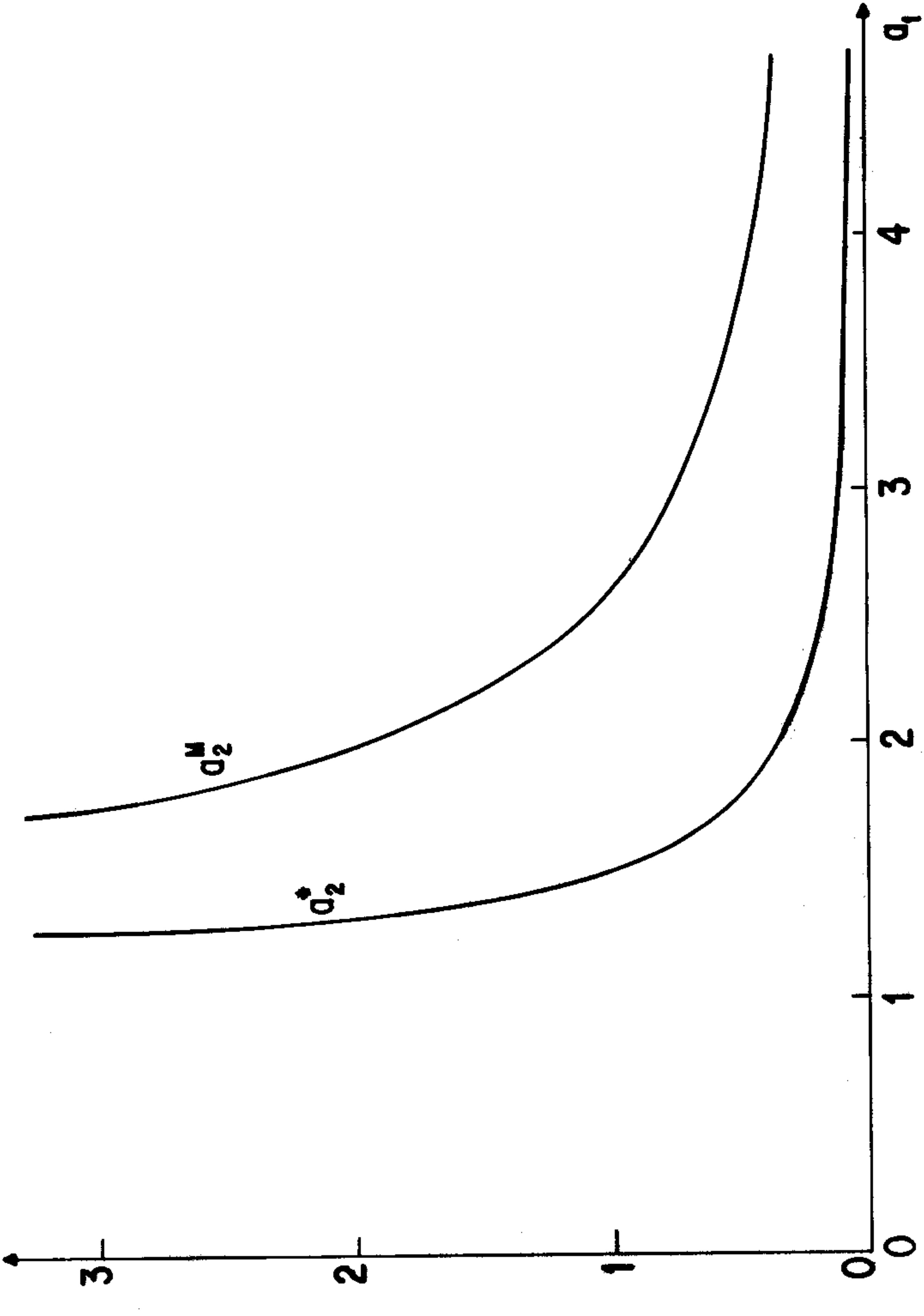

FIG. 2

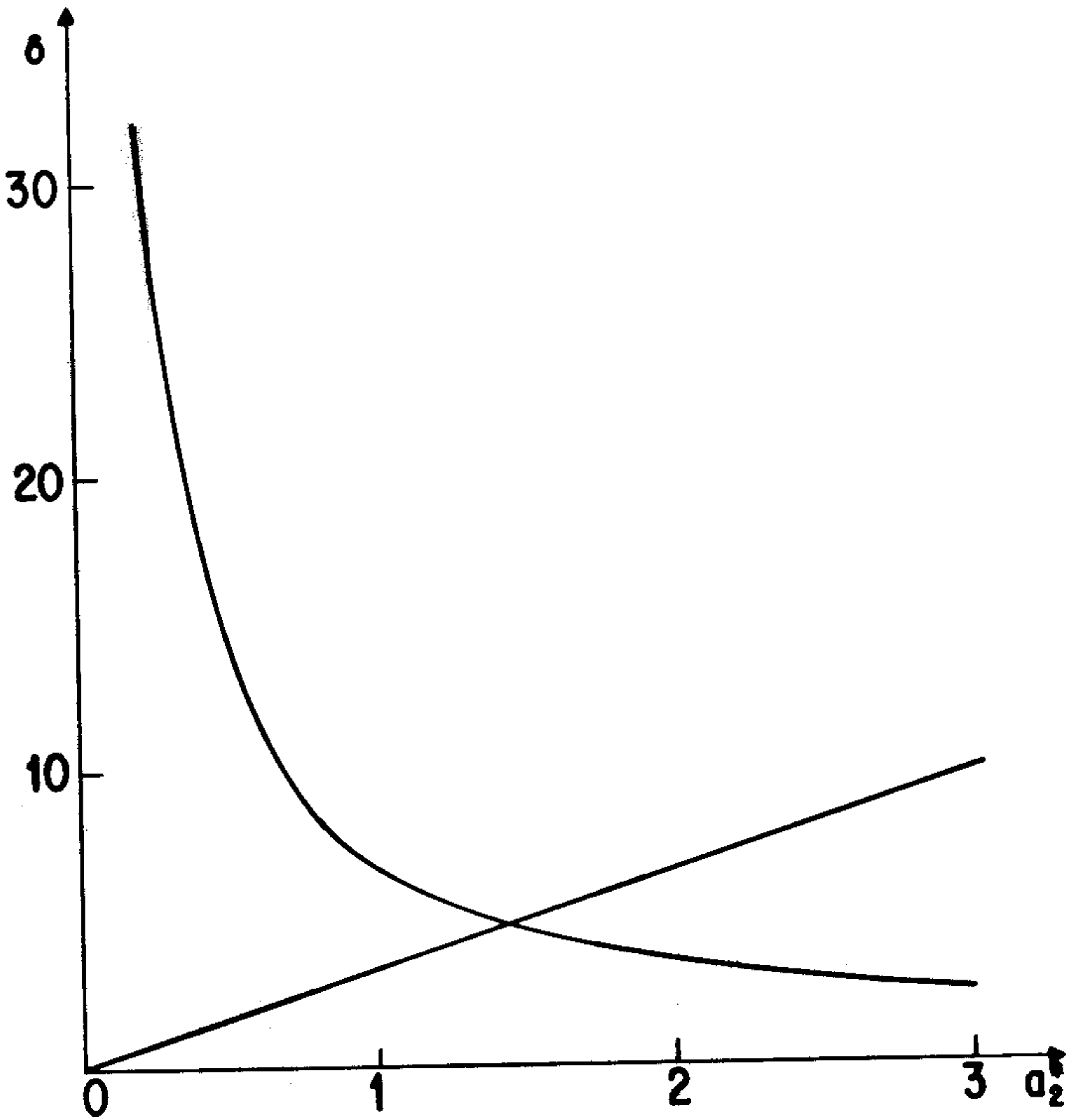

FIG. 3(a)

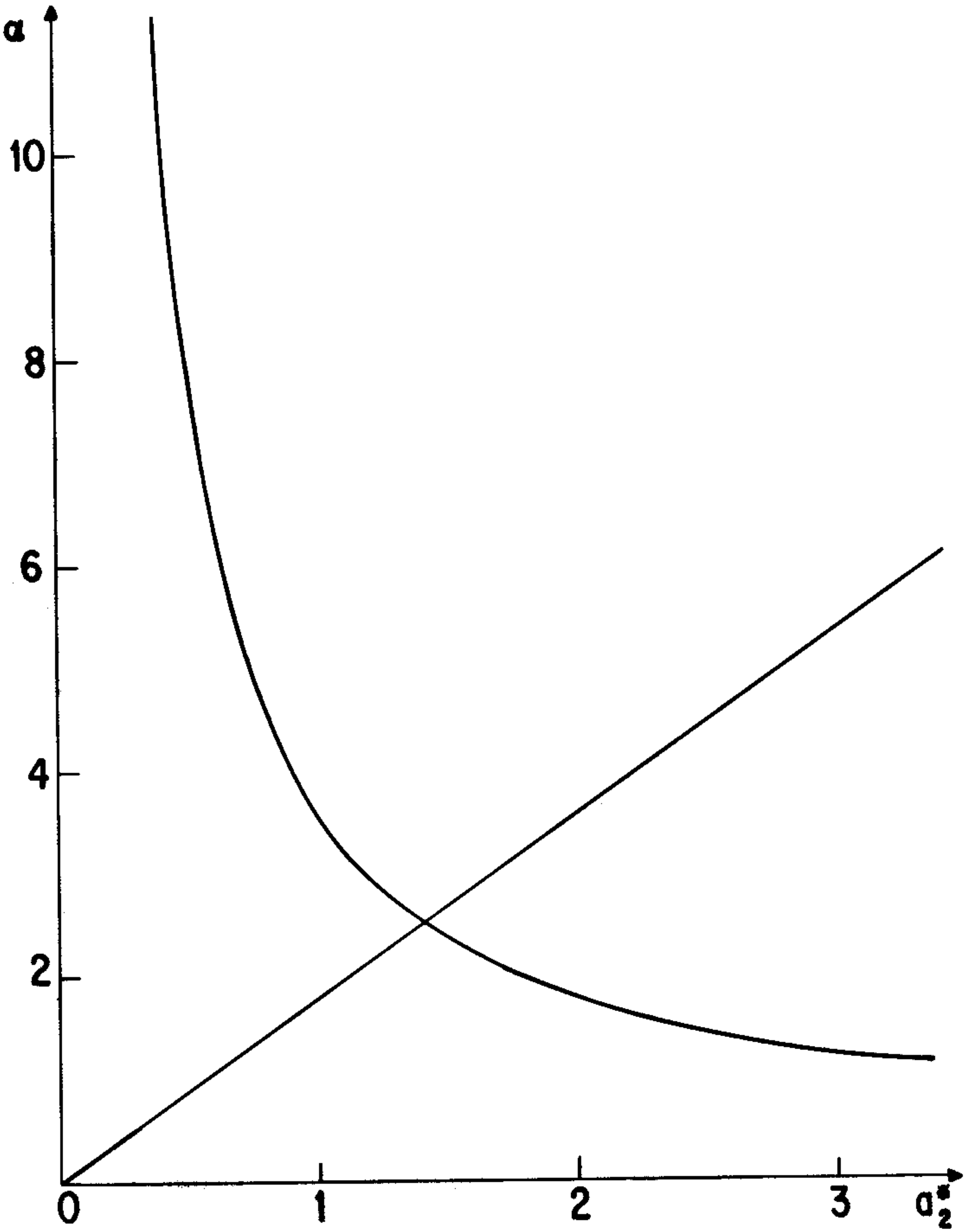

FIG. 3(b)

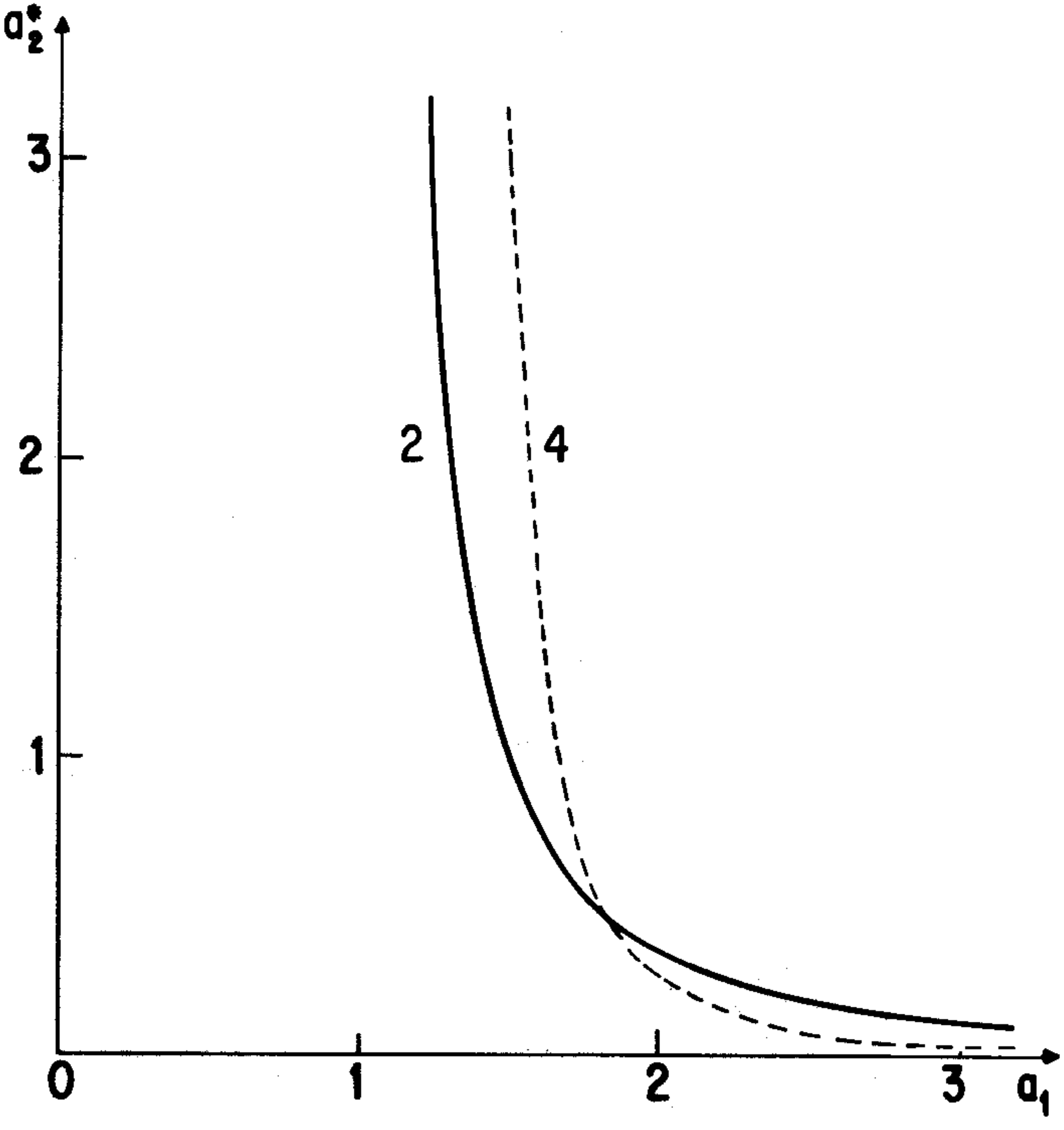

FIG. 4

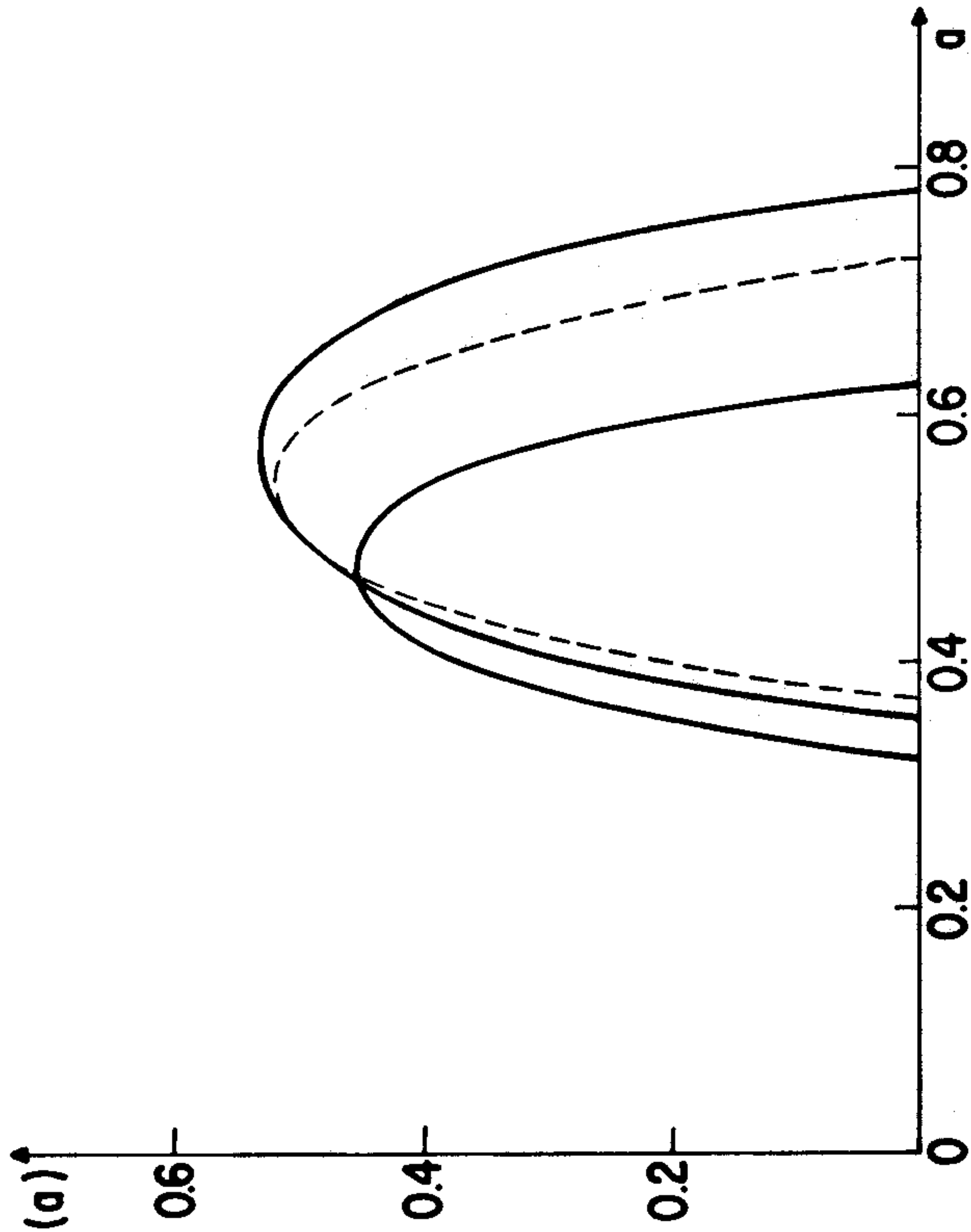

FIG. 5

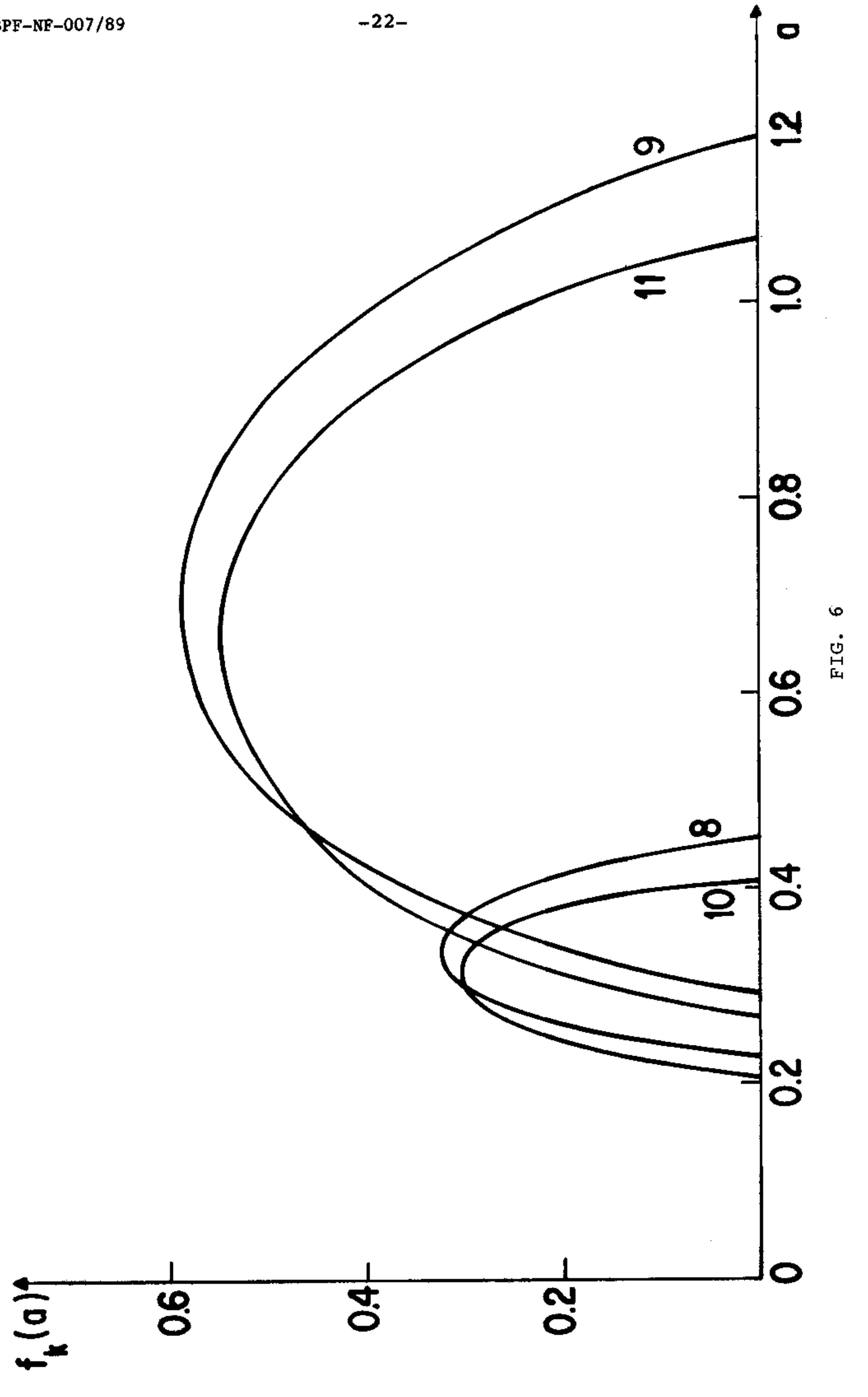

FIG. 6

TABLE I

k	$\tilde{a}_{2_k}$	δ_k	α_k	$\delta_k\delta_{k+1}$	$\alpha_k\alpha_{k+1}$
1	1.000000000	0.51113	3.15994	2.32986	8.13026
2	1.282533305	4.55825	2.57291	15.3855	9.08135
3	1.835293608	3.37531	3.52960	21.9198	6.32448
4	1.956559371	6.49417	1.79184	21.7948	6.42833
5	1.992486639	3.35665	3.58756	22.1307	6.30935
6	1.998018868	6.59427	1.75867	22.1190	6.30794
7	1.999667300	3.35427	3.58989	22.1328	6.30810
8	1.999917279	6.59839	1.75714	22.1316	6.30784
9	1.999991805	3.35409	3.58998	-	-

TABLE II

k	$\tilde{a}_k$	δ_k superst.	δ_k bifurc.	α_k
1	1.000000000000	1.12375	3.643	5.30261
2	1.188585861842	118.088	6.625	0.51899
3	1.356403706313	0.37241	7.588	29.0440
4	1.357824831910	800.653	6.548	0.09612
5	1.361640892733	0.13862	18.01	204.042
6	1.361645658918	6337.59	6.202	0.01657
7	1.361680041889	0.05078	-	1980.78
8	1.361680047315	64367.1	-	0.00219
9	1.361680154136	0.01699	-	28186.5
10	1.361680154137	-	-	0.00020
11	1.361680154235	-	-	-

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