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# GRAVITATIONAL COUPLING OF SCALAR AND FERMION FIELDS TO MATTER VORTICITY: MICROSCOPIC ASYMMETRIES

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ABSTRACT

The gravitational coupling of scalar fields and spin-1/2 fermions to matter vorticity is examined; in the context of Einstein's theory of gravitation and for technical simplicity we have considered the Gödel model as the gravitational background whose matter content has a non-null vorticity. Scalar field equation and Dirac equation are solved by separation of the field amplitudes into invariant angular-momentum and energy modes. For each case (scalar or Dirac field) these modes provide two distinct complete bases of solutions, which are bases for two representations of the algebra of the total "angular momentum" of the system (one finite dimensional and the other infinite-dimensional). The presence of a vorticity field of matter generates, via gravitation, microscopic asymmetries in the physics of fermions. The "angular-momentum" vector space appears to be polarized along the direction determined by the local vorticity $\vec{\Omega}$ . Microscopic currents are asymmetric along the direction determined by the vorticity field: fermions (antifermions) currents are larger along the direction anti-parallel (parallel) to the vorticity field. This current asymmetry as well other parity violating effects (for example, a split of mass for fermions) can in principle be used in devising experiments to detect the presence of a cosmological rotation of the universe, its direction and intensity. In case of production of pairs under CP violation a net number asymmetry may be generated between fermions and antifermions.

## I. INTRODUCTION

Our purpose in the present paper is to describe the effect of matter vorticity in the physics of spin-1/2 fermions, the coupling of fermions to the vorticity field being realized obviously through gravitation. The problem is not purely academic because the observed anisotropy of the $3^{o}$K background radiation can possibly be due to a large scale primordial vorticity of the universe [1,2]. This fact and the present observed rotation of galaxies and nebulae could be an indication that the rotation of matter was a remarkable feature of earlier eras, playing an important role in the dynamics of the primordial universe. In this sense the results of our investigation could have some interesting applications in the realm of cosmology and theoretical astrophysics. The present paper continues a program [3,4] in which we have examined microscopic asymmetries (generated by matter vorticity) in neutrino physics.

In the context of the Einstein theory of gravitation and for technical simplicity, we take the Gödel universe [5] as the gravitational background. It is the simplest known solution of Einstein field equations with rotating incoherent matter. The vorticity field of matter is connected to the property that matter rotates with non-zero angular velocity, in the local inertial frames of its comoving observers. The model admits a global time-like Killing vector, a fact that is crucial for construction of invariant energy modes of the scalar and fermion fields. Spin-1/2 fermions are introduced as test fields over the background gravitational field, and are described by spinorial fields which satisfy Dirac's equation on the curved gravitational background.

The plan of the paper is the following. In section II we characterize the Gödel universe as the simply connected Lie group $H^3 \times R$ with a left-invariant metric, defined on it. This garantees that all vector fields over $H^3 \times R$ exist globally, and that the hyperbolic excitation modes in which we decompose the fields – are invariantly and globally defined over the manifold. In section III we solve the scalar field coupled to the rotating gravitational background by separating the scalar wave equation into invariant angular momentum modes. The scalar wave equation reduces then to an eigenvalue equation for the total angular-momentum of the scalar field. Two complete basis of solutions are obtained, one finite-dimensional and the other infinite-dimensional representation basis for the algebra of the angular-momentum. In sections IV and V the local dynamics of fermions and constants of motion are obtained with basis on Dirac's equation over the curved background. Two complete bases of fermion solutions are obtained, which are eigenstates of energy, total angular momentum, projection of the angular momentum along the axis determined locally by the vorticity field, and also eigenstates of a new constant of motion. They satisfy boundary and regularity conditions related to the test field character of the wave-functions. A suitable scalar product is defined over the Hilbert space of solutions and we construct the Fourier space associated to the above bases. In section VI we discuss the local microscopic asymmetry of neutrino emission which appears in the presence of a vorticity field; we also discuss the asymmetry between fermion and antifermion amplitudes which could appear due to CP violation and could produce a net asymmetry between the number of fermions and antifermions.

## II. GLOBAL STRUCTURE OF THE GÖDEL UNIVERSE AND THE EXCITATION MODES OF FERMION FIELDS

The Gödel universe is characterized here as the simply connected Lie group $H^3 \times R$ modulo identification of points, with a left-invariant metric introduced on $H^3 \times R$ and which is a solution of Einstein field equations for a perfect fluid. Since the invariant vector fields and forms are globally defined over the Lie group $H^3 \times R$, they are used to construct the invariant modes in which we expand spin-1/2 wave functions yielding a complete basis of solutions which exist globally. The methods used in this section are borrowed from Ozsvath and Schüking [6], and are present here concisely for completeness.

Let $E_4$ be the four-dimensional Euclidean space with Cartesian coordinates $a = (a^0, a^1, a^2, a^3)$. We define $H^3$ as the set of points of $E_4$ which satisfy

$$(a^0)^2 + (a^1)^2 - (a^2)^2 - (a^3)^2 = 1 \quad . \tag{2.1}$$

For any $a = (a^0, a^1, a^2, a^3)$ and $b = (b^0, b^1, b^2, b^3) \in H^3$ we define the multiplication law [7]

$$\begin{aligned} ab = (&a^0b^0 - a^1b^1 + a^2b^2 + a^3b^3 \, , \\ &a^0b^1 + a^1b^0 - a^2b^3 + a^3b^2 \, , \\ &a^0b^2 + a^2b^0 - a^1b^3 + a^3b^1 \, , \\ &a^0b^3 + a^3b^0 - a^2b^1 + a^1b^2 \, ) \quad . \end{aligned} \tag{2.2}$$

Under (2.2) $H^3$ becomes a group, acting on itself by left multiplication; namely for a given $v \in H^3$ a left motion of $H^3$ into itself is expressed by

$$a' = va \tag{2.3}$$

and from (2.2) we have $a' \in H^3$, for all $a \in H^3$. $H^3$ is simply transitive since for each $a \neq 0$ there exists only one left motion v from $a \in H^3$ to a given $a' \in H^3$.

$H^3$ acting on itself by left multiplication (2.3) is a Lie group, with the three independent left invariant vector fields [8] on $H^3$:

$$\begin{aligned} e_1^\mu(a) &= (-a^1,\ a^0,\ a^3,\ -a^2) \\ e_2^\mu(a) &= (\ a^2,\ a^3,\ a^0,\ a^1) \\ e_3^\mu(a) &= (\ a^3, -a^2, -a^1,\ a^0) \quad . \end{aligned} \tag{2.4}$$

They are obtained by an arbitrary left motion a of the three independent unit vectors (0,1,0,0), (0,0,1,0) and (0,0,0,1), which define the infinitesimal tangent space of $H^3$ at the identity (1,0,0,0).

We have the analogous picture for right-motions of the Lie group $H^3$ into itself, namely (cf. (2.3))

$$a' = av \tag{2.5}$$

with the corresponding independent right-invariant vectors fields on $H^3$

$$\begin{aligned} f_1^\mu(a) &= (-a^1,\ a^0,\ -a^3,\ a^2) \\ f_2^\mu(a) &= (\ a^2, -a^3,\ a^0, -a^1) \\ f_3^\mu(a) &= (\ a^3,\ a^2,\ a^1,\ a^0) \quad . \end{aligned} \tag{2.6}$$

We obviously have [9]

$$[e_i, f_j] = 0 \quad . \tag{2.7}$$

Bases (2.4) and (2.6) expressed as

$$X_0 = -\frac{1}{2} e_1^\mu \frac{\partial}{\partial a^\mu} \quad , \quad X_1 = -\frac{1}{2} e_3^\mu \frac{\partial}{\partial a^\mu} \quad , \quad X_2 = -\frac{1}{2} e_2^\mu \frac{\partial}{\partial a^\mu} \tag{2.8}$$

$$Y_0 = -\frac{1}{2} f_1^\mu \frac{\partial}{\partial a^\mu} \quad , \quad Y_1 = -\frac{1}{2} f_3^\mu \frac{\partial}{\partial a^\mu} \quad , \quad Y_2 = -\frac{1}{2} f_2^\mu \frac{\partial}{\partial a^\mu} \tag{2.9}$$

yield the representations of the algebra of $H^3$

$$[X_0, X_1] = X_2 \quad , \quad [X_1, X_2] = -X_0 \quad , \quad [X_2, X_0] = X_1 \tag{2.10}$$

$$[Y_0, Y_1] = -Y_2 \quad , \quad [Y_1, Y_2] = Y_0 \quad , \quad [Y_2, Y_0] = -Y_1 \tag{2.11}$$

We introduce on $H^3$ the coordinate system $(t,r,\phi)$ by the substitutions

$$\begin{aligned} a^0 &= \cosh r \cos \frac{\sqrt{2}}{2} t \\ a^1 &= \cosh r \sin \frac{\sqrt{2}}{2} t \\ a^2 &= -\sinh r \cos \left(\frac{\sqrt{2}}{2} t - \phi\right) \\ a^3 &= \sinh r \sin \left(\frac{\sqrt{2}}{2} t - \phi\right) \end{aligned} \tag{2.12}$$

where $0 \leq \frac{\sqrt{2}}{2} t, \ \phi \leq 2\pi$, $0 \leq r < \infty$, and the left-invariant vector fields (2.8) become

$$\begin{aligned} X_0 &= \sqrt{2} \frac{\partial}{\partial t} \\ X_1 &= -\sqrt{2} \cos(\sqrt{2}\, t - \phi) \frac{\sinh r}{\cosh r} \frac{\partial}{\partial t} - \sin(\sqrt{2}\, t - \phi) \frac{\partial}{\partial r} + \frac{\cos(\sqrt{2}\, t - \phi)}{\sinh r \cosh r} \frac{\partial}{\partial \phi} \\ X_2 &= -\sqrt{2} \sin(\sqrt{2}\, t - \phi) \frac{\sinh r}{\cosh r} \frac{\partial}{\partial t} + \cos(\sqrt{2}\, t - \phi) \frac{\partial}{\partial r} + \frac{\sin(\sqrt{2}\, t - \phi)}{\sinh r \cosh r} \frac{\partial}{\partial \phi} \end{aligned} \tag{2.13}$$

with dual invariant 1-forms

$$\sigma^0 = \frac{\sqrt{2}}{2}\,(dt + \sqrt{2}\,\sinh^2 r\, d\phi)$$

$$\sigma^1 = -\sin(\sqrt{2}\,t-\phi)\,dr + \cos(\sqrt{2}\,t-\phi)\,\sinh r\,\cosh r\,d\phi \qquad (2.14)$$

$$\sigma^2 = \cos(\sqrt{2}\,t-\phi)\,dr + \sin(\sqrt{2}\,t-\phi)\,\sinh r\,\cosh r\,d\phi\ .$$

Taking on the one-dimensional manifold R the coordinate $x^3$, with vector field $X_3 = \partial/\partial x^3$ and dual 1-form $\sigma^3 = dx^3$, the group $H^3\times R$ can be characterized by the left-invariant vector fields $(X_0, X_1, X_2, X_3)$ which satisfy (2.10) and

$$[X_i, X_3] = 0\ , \quad i = 0, 1, 2\ , \qquad (2.15)$$

and which are a basis for the vector fields on $H^3\times R$; correspondingl the invariant dual one-forms $(\sigma^0, \sigma^1, \sigma^2, \sigma^3)$ are a basis for the one-forms on $H^3\times R$. The manifold $H^3\times R$ is the covering group of the algebra (2.10), (2.15).

The Gödel universe is obtained by introducing on $H^3\times R$ the left-invariant metric [10]

$$ds^2 = \frac{4}{\omega^2}\left[(\sqrt{2}\,\sigma^0)^2 - (\sigma^1)^2 - (\sigma^2)^2 - (\sigma^3)^2\right] \qquad (2.16)$$

where $\omega$ is a positive constant. (2.16) is a solution of Einstein field equations [11,12] with cosmological constant $\Lambda$ and incoherent matter whose density $\rho$ must satisfy

$$k\rho = \omega^2 = -2\Lambda\ . \qquad (2.17)$$

The four velocity of matter is $\partial/\partial t$. The model is stationary because (2.16) admits a timelike Killing vector. The velocity field of matter has zero expansion and shear but has a non-null vorticity

$$\Omega = \sqrt{2}\,\omega \frac{\partial}{\partial x^3} \quad . \tag{2.18}$$

We remark that Gödel universe is locally isometric to (2.16) but concerning connectivity-in-the-large the above model is obtained from the Gödel model by identification of the points $(\frac{\sqrt{2}}{2} t + 2n\pi,\ r,\phi,x^3)$, n = integer. In the Gödel universe any geodesic of the congruence determined by $\partial/\partial t$ is time-like and open.

From (2.7) and (2.16) it is obvious that Gödel's model admits the five Killing vectors

$$(Y_0\ ,\ Y_1\ ,\ Y_2\ ,\ \frac{\partial}{\partial x^3}\ ,\ \frac{\partial}{\partial t}) \quad . \tag{2.19}$$

All these vector fields are globally defined on the group manifold [13] . In the coordinate system $(t,r,\phi)$ on $H^3$, introduced by (2.12) the vector fields $Y_0$, $Y_1$, $Y_2$ are expressed

$$\begin{aligned} Y_0 &= (\frac{\sqrt{2}}{2}\frac{\partial}{\partial t} + \frac{\partial}{\partial \phi}) \\ Y_1 &= \frac{\sqrt{2}}{2}\sin\phi\,\frac{\sinh r}{\cosh r}\frac{\partial}{\partial t} - \frac{\cos\phi}{2}\frac{\partial}{\partial r} + \frac{\sin\phi}{2}\,\frac{\sinh^2 r + \cosh^2 r}{\sinh r\,\cosh r}\frac{\partial}{\partial \phi} \\ Y_2 &= -\frac{\sqrt{2}}{2}\cos\phi\,\frac{\sinh r}{\cosh r}\frac{\partial}{\partial t} - \frac{\sin\phi}{2}\frac{\partial}{\partial r} - \frac{\cos\phi}{2}\,\frac{\sinh^2 r + \cosh^2 r}{\sinh r\,\cosh r}\frac{\partial}{\partial \phi} \quad . \end{aligned} \tag{2.20}$$

We then select the Killing vector fields

$$(\frac{\partial}{\partial t}\ ,\ \frac{\partial}{\partial x^3}\ ,\ \frac{\partial}{\partial \phi} = Y_0 - \frac{\sqrt{2}}{2}\frac{\partial}{\partial t}) \tag{2.21}$$

to construct the invariant modes $\phi_{(i)}$ globally defined by [14]

$$\underset{\partial/\partial x^3}{\pounds} \phi_{(3)} = -i\, k_3\, \phi_{(3)} \qquad \underset{\partial/\partial\phi}{\pounds} \phi_{(2)} = -im\, \phi_{(2)} \tag{2.22}$$

and

$$\underset{\partial/\partial t}{\pounds} \phi_{(0)} = -i\varepsilon\phi_{(0)} \tag{2.23}$$

with respective solutions $\phi_{(3)} \sim e^{-ik_3x^3}$, $\phi_{(2)} \sim e^{-im\phi}$ and $\phi_{(0)} \sim e^{-i\varepsilon t}$. $\partial/\partial t$ is a globally defined timelike Killing vector generating time translations and we interpret (2.23) as the definition of invariant energy modes; $\partial/\partial t$ actually defines the Hamiltonian operator which describes the local dynamics of the field. We use the invariant modes $\phi_{(i)}$ to separate field amplitudes in the modes ($\varepsilon$, m, $k_3$).

## III. SCALAR FIELD COUPLED TO MATTER VORTICITY: HYPERBOLIC HARMONIC MODES

The equation for the scalar field $\phi$ coupled minimally to the gravitational background $g_{\alpha\beta}(x)$ is given by

$$\frac{1}{\sqrt{-g}}\, \partial_\alpha\, (\sqrt{-g}\; g^{\alpha\beta}\partial_\beta\phi) + \mu^2\phi = 0 \quad . \tag{3.1}$$

For the Gödel geometry (2.16), which can be expressed as

$$ds^2 = a^2\left\{dt^2 - dr^2 - (dx^3)^2 + (\sinh^4 r - \sinh^2 r)d\phi^2 + 2\sqrt{2}\,\sinh^2 r\; dt\; d\phi\right\} , \tag{3.2}$$

equation (3.1) reduces to

$$-\,4L^2\phi + \left\{\mu^2 a^2 - \frac{\partial^2}{(\partial x^3)^2} - \frac{\partial^2}{\partial t^2}\right\}\phi = 0 \tag{3.3}$$

where

$$L^2 = -\frac{1}{2\cosh^2 r}\frac{\partial^2}{\partial t^2} - \frac{\sqrt{2}}{2}\frac{1}{\cosh^2 r}\frac{\partial^2}{\partial\phi\partial t} + \frac{1}{4\sinh^2 r\,\cosh^2 r}\frac{\partial^2}{\partial\phi^2} +$$

$$+ \frac{1}{4}\frac{\sinh^2 r+\cosh^2 r}{\sinh r\,\cosh r}\frac{\partial}{\partial r} + \frac{1}{4}\frac{\partial^2}{\partial r^2} \quad . \qquad (3.4)$$

$L^2$ can be interpreted as the square of the total angular-momentum operator of the field. To see this let us start from the Killing vectors (2.20) and define the operators

$$\begin{aligned} L_3 &= i\, Y_0 \\ L_1 &= -\, Y_1 \\ L_2 &= \phantom{-}\, Y_2 \quad . \end{aligned} \qquad (3.5)$$

From (2.11) we can see that the ($L_a$, $a = 1,2,3$) satisfy the rule

$$[L_a, L_b] = i\, \varepsilon_{abc}\, L_c \qquad (3.6)$$

which is the algebra of the angular-momentum operators. From (3.5) we can see that the algebra of $H^3$ isometries and the algebra of the angular-momentum are related by the complexification of the generat $Y_0$. This implies that the positive-definite scalar product in the vector space of angular-momentum algebra has its correspondent as an indefinite product in the vector space of the algebra of $H^3$ isometries, with the consequent definitions of the square of the total angular momentum $\vec{L}$.

Indeed a straightforward calculation shows that $L^2$ is the square of the total angular-momentum $\vec{L}$,

$$L^2 = \vec{L}.\vec{L} \qquad (3.7)$$

where

$$\vec{L}.\vec{L} = (L_1)^2 + (L_2)^2 + (L_3)^2 = (Y_1)^2 + (Y_2)^2 - (Y_0)^2 \quad . \qquad (3.8)$$

Further properties of these operators will be examined later in this section.

To separate equation (3.1) we consider scalar wave functions which belong to the globally and invariantly defined set of modes $(\varepsilon, m, k_3)$

$$\phi = \phi(r)\ e^{-im\phi}\ e^{-ik_3x^3}\ e^{-i\varepsilon t} \quad . \qquad (3.9)$$

The field equation (3.3) reduces then to the eigenvalue equation

$$\vec{L}^2\phi = \frac{1}{4}\ (\mu^2a^2 + k_3{}^2 + \varepsilon^2)\phi = k\phi \quad . \qquad (3.10)$$

Introducing the variable $x = \cosh 2r$ and for the set of modes (3.9), equation (3.10) results

$$(x^2-1)\ \phi''(x) + 2x\phi'(x) + (\frac{Q}{x+1} - \frac{m^2}{x^2-1})\phi(x) - k\phi(x) = 0 \qquad (3.11)$$

where $Q = \varepsilon^2 + \sqrt{2}\ m\varepsilon$. We have denoted $\phi' = d\phi/dx$. By making the substitution $\phi(x) = (x^2-1)^{m/2}(x+1)^A g(x)$ we reduce equation (3.11) to

$$(1-x^2)g'' + \{2A(1-x)-2(m-1)x\}g' + \{k-m-m^2+A^2-Q\}g = 0 \qquad (3.12)$$

where the parameter A must satisfy $2A^2+2mA-Q = 0$. Without loss of generality, we choose the root $A = \frac{\sqrt{2}}{2}\ \varepsilon$. Equation (3.12) has then as first solution

$$g(x) = F(a,\ b,\ c;\ \frac{1-x}{2}) \qquad (3.13)$$

where $F(a,b,c;\ \frac{1-x}{2})$ is the hypergeometric function [15,16] with argument (1-x)/2 and parameters

$$a = m + \frac{\sqrt{2}}{2}\varepsilon + \frac{1}{2} \pm \frac{n}{2}$$
$$b = m + \frac{\sqrt{2}}{2}\varepsilon + \frac{1}{2} \mp \frac{n}{2} \qquad (3.14)$$
$$c = m + 1 \quad ,$$

and where

$$n^2 = 4k+1 = \varepsilon^2 + k_3^2 + \mu^2 a^2 + 1 \quad . \qquad (3.15)$$

The scalar field solutions have the form

$$\phi_{\varepsilon,m} = \phi_{\varepsilon,m}(x)\, e^{-ik_3x^3}\, e^{-im\phi}\, e^{-i\varepsilon t} \quad , \qquad (3.16)$$

$$\phi_{\varepsilon,m}(x) = (x^2-1)^{m/2}\, (x+1)^{\frac{\sqrt{2}}{2}\varepsilon}\, F(a,b,c;\tfrac{1-x}{2}) \quad . \qquad (3.17)$$

We remark that the other root $A = -m-\sqrt{2}\,\varepsilon$ yields a solution $\phi'_{\varepsilon,m}$ which is linearly dependent of (3.16) and related to it by $\phi'_{\varepsilon m} = 2^{-m-\sqrt{2}\,\varepsilon}\,\phi_{\varepsilon m}$.

On the space of solutions (3.16), (3.17) we introduce the operators

$$L_+ = L_1+iL_2 = e^{-i\phi}\{(x^2-1)^{1/2}\frac{\partial}{\partial x} - ix(x^2-1)^{-1/2}\frac{\partial}{\partial\phi} - i\frac{\sqrt{2}}{2}(\frac{x-1}{x+1})^{1/2}\frac{\partial}{\partial t}\}$$
$$(3.18)$$
$$L_- = L_1-iL_2 = e^{i\phi}\{(x^2-1)^{1/2}\frac{\partial}{\partial x} + ix(x^2-1)^{-1/2}\frac{\partial}{\partial\phi} + i\frac{\sqrt{2}}{2}(\frac{x-1}{x+1})^{1/2}\frac{\partial}{\partial t}\} \; .$$

They satisfy, from (3.5) and (3.6)

$$[L_+,L_-] = 2\,L_3$$
$$[L_+,L_3] = -\,L_+ \qquad (3.19)$$
$$[L_-,L_3] = L_-$$

and their effect on the set of solutions (3.16) is [17]

$$L_+ \phi_{\varepsilon,m} = -\frac{ab}{2c}\phi_{\varepsilon,m+1} \tag{3.20}$$

$$L_- \phi_{\varepsilon,m} = 2m\,\phi_{\varepsilon,m-1} \quad . \tag{3.21}$$

From the commutation relation (3.19) we have that if $\phi_{\varepsilon,m}$ is a solution – which is eigenstate of $L_3$ with eigenvalue $m+\frac{\sqrt{2}}{2}\varepsilon$ – then $L_+ \phi_{\varepsilon,m}$ is also a solution of the set (3.16) which is eigenstate of $L_3$ with eigenvalue $m+1+\frac{\sqrt{2}}{2}\varepsilon$. Also $L_- \phi_{\varepsilon,m}$ is a solution of the set (3.16) which is eigenstate of $L_3$ with eigenvalue $m-1+\frac{\sqrt{2}}{2}\varepsilon$. So given $\phi_{\varepsilon,m}$ it is possible to construct a sequence of solutions extending indefinitely in both directions or terminating if $L_+ \phi_{\varepsilon,m}$ or $L_- \phi_{\varepsilon,m}$ vanishes for some value of m.

We can now discuss the orthogonality and normalization of the set of scalar solutions (3.16). Before defining a scalar product on the set of functions (3.18) we must examine regularity and boundary conditions such that the functions $\phi_{\varepsilon,m}(x)$ be square-integrable over the whole domain $1 \leq x < \infty$. In fact, contrary to the case of the spherical harmonics it is not possible in the present case to use the properties of the operators $L_+$, $L_-$ and $L_3$ to set bounds on the range of m because $L_+$ and $L_-$ are not Hermitian with respect to any suitable scalar product defined for the set (3.19).

To proceed we impose boundary and regularity conditions connected to the character of test fields of the scalar solutions (3.16), namely that the scalar field solutions are finite perturbations at any space-time point. We assume [18]

$$\vec{L}.\vec{L} = (L_1)^2 + (L_2)^2 + (L_3)^2 = (Y_1)^2 + (Y_2)^2 - (Y_0)^2 \quad . \qquad (3.8)$$

Further properties of these operators will be examined later in this section.

To separate equation (3.1) we consider scalar wave functions which belong to the globally and invariantly defined set of modes $(\varepsilon, m, k_3)$

$$\phi = \phi(r)\, e^{-im\phi}\, e^{-ik_3x^3}\, e^{-i\varepsilon t} \quad . \qquad (3.9)$$

The field equation (3.3) reduces then to the eigenvalue equation

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Introducing the variable $x = \cosh 2r$ and for the set of modes (3.9), equation (3.10) results

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where $Q = \varepsilon^2 + \sqrt{2}\, m\varepsilon$. We have denoted $\phi' = d\phi/dx$. By making the substitution $\phi(x) = (x^2-1)^{m/2}(x+1)^A g(x)$ we reduce equation (3.11) to

$$(1-x^2)g'' + \{2A(1-x)-2(m+1)x\}g' + \{k-m-m^2+A^2-A-Q\}g = 0 \qquad (3.12)$$

where the parameter A must satisfy $2A^2+2mA-Q = 0$. Without loss of generality, we choose the root $A = \frac{\sqrt{2}}{2}\,\varepsilon$. Equation (3.12) has then as first solution

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The scalar field solutions have the form

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We remark that the other root $A = -m-\frac{\sqrt{2}}{2}\varepsilon$ yields a solution $\phi'_{\varepsilon,m}$ which is linearly dependent of (3.16) and related to it by $\phi'_{\varepsilon m} = 2^{-m-\sqrt{2}\,\varepsilon}\,\phi_{\varepsilon m}$.

On the space of solutions (3.16), (3.17) we introduce the operators

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$$(3.18)$$

$$L_- = L_1-iL_2 = e^{i\phi}\{(x^2-1)^{1/2}\frac{\partial}{\partial x} + ix(x^2-1)^{-1/2}\frac{\partial}{\partial\phi} + i\,\frac{\sqrt{2}}{2}\,(\frac{x-1}{x+1})^{1/2}\frac{\partial}{\partial t}\} \ .$$

They satisfy, from (3.5) and (3.6)

$$[L_+,L_-] = 2\,L_3$$

$$[L_+,L_3] = -\,L_+ \qquad (3.19)$$

$$[L_-,L_3] = L_-$$

The corresponding negative-energy solutions are obtained by taking the complex-conjugate solution $\phi^*$. Contrary to type I solutions, type II positive-energy solutions are associated to the negative exponential $e^{-im\phi}$ (cf. Ref. [22]). The eigenvalue of $L_3$ for this case is $\pm\,(m + \frac{\sqrt{2}}{2}\,|\varepsilon|)$ for positive/negative energy.

For both cases I and II, the eigenvalue of $\vec{L}^2$ is given by

$$k = (\ell - \frac{\sqrt{2}}{2}\,|\varepsilon|)(\ell - \frac{\sqrt{2}}{2}\,|\varepsilon| + 1) \tag{3.33}$$

which is always positive.

The basic difference between type I and type II solutions is that for type I solutions the values of m are bounded for a given $\ell$ (cf. (3.39)) while for type II solutions the range $m \geq 0$ is completely independent of the value of $\ell$. In other words, for a given $\ell$ = positive integer or zero type (I) solutions provide a finite-dimensional (dim = $\ell$ + 1) representation basis for the algebra of angular-momentum, while type II solutions provide an infinite-dimensional representation basis for the algebra of angular-momentum [23]. These functions are denoted hyperbolic harmonics of type I and type II, with general expression

<u>Type I basis</u>

$$\phi^{I}_{\varepsilon\ell m k_3} = (x^2-1)^{m/2}(x+1)^{-\frac{\sqrt{2}}{2}|\varepsilon|} F(a,b,c;\tfrac{1-x}{2})\, e^{im\phi}\, e^{ik_3x^3}\, e^{-i|\varepsilon|t} \tag{3.34a}$$

where

$$\begin{aligned} a &= m-\ell \\ b &= m+\ell-\sqrt{2}|\varepsilon|+1 \\ c &= m+1 \\ 0 &\leq m \leq \ell \quad . \end{aligned}$$

Type II basis

$$\phi^{II}_{\varepsilon\ell m k_3} = (x^2-1)^{m/2}(x+1)^{\frac{\sqrt{2}}{2}|\varepsilon|}F(a,b,c;\tfrac{1-x}{2})e^{-im\phi}\,e^{-ik_3x^3}\,e^{-i|\varepsilon|t} \tag{3.34b}$$

where

$$a = m-\ell+\sqrt{2}|\varepsilon|$$

$$b = m+\ell+1$$

$$c = m+1$$

$$0 \leq m < \infty\ , \quad 0 \leq \ell < \infty \qquad (m \text{ and } \ell \text{ independent}).$$

These two sets of functions (3.34) are not only bases of representation of the algebra of the angular-momentum of the system, and of the isometry algebra of the space-time, but also they provide two irreducible representations of the Lie group $H^3\times R$ (Gödel's manifold itself, cf. Sec. II). In particular type I solutions (3.34a) correspond to a finite dimensional irreducible representation of $H^3\times R$. Remarking that: (i) the functions (3.34) are eigenfunctions of the operator $\vec{L}^2$ with eigenvalues characterized by the positive integer or zero $\ell$ (cf. (3.10) and (3.33)), and $\vec{L}^2$ commutes with all elements of the algebra of the isometry group $H^3\times R$; (ii) $H^3\times R$ is a simply transitive group acting on itself by left-multiplication – we have that the eigenfunctions (3.34) for a given $\ell$ can in principle be given as a linear combination of the matrix elements of the representation characterized by $\ell$. Therefore [24] they constitute a complete basis for representing continuous functions defined over the Lie group $H^3 \times R$ (Gödel's manifold) and which satisfy the prescribed regularity conditions (3.22), (3.23). In other words, any continuous function defined over the simply transitive Lie group $H^3 \times R$ and satisfying (3.22), (3.23) can be expressed as the limit of a series constructed with the functions (3.34a) or (3.34b).

They have the important property that under the parity operator P: $(r \to r,\ z \to -z,\ \phi \to \phi+\pi)$, they transforms as

$$P\,\phi_{\varepsilon\ell m k_3}(t,r,\phi,z)=\phi_{\varepsilon,\ell,m,k_3}(t,r,\phi+\pi,-z)=(-1)^m\ \phi_{\varepsilon,\ell,m,-k_3}(t,r,\phi,z)\ .$$

We now discuss the orthogonality and normalization of the set of scalar functions (3.34). For this set of solutions we define the scalar product

$$<\phi',\phi> = \frac{4}{\omega^4}\int \sqrt{-g}\ d^4x\ \phi'^*\phi \tag{3.35}$$

where the integration is taken over the whole Gödel manifold. The normalization (3.34) is taken instead of the usual conserved normalization on a t = const. hypersurface because such hypersurfaces are not globally space-like (actually Gödel's universe does not admit any global space-like hypersurface). In cases when t = const. is a global space-like hypersurface the normalization defined in (3.34) is equivalent to the usual one for solutions which are eigenstates of the Hamiltonian of the system. Using the explicit expressions of $L_\pm$ , and the regularity and boundary conditions (3.22) and (3.23) we can demonstrate the adjointness properties

$$<L_+\phi',\phi> = -<\phi',L_-\phi> \tag{3.36}$$

$$<L_-\phi',\phi> = -<\phi',L_+\phi> \quad . \tag{3.37}$$

It then follows that $L_1$ and $L_2$ are anti-Hermitian with respect to (3.34), namely

$$<L_1\phi',\phi> = - <\phi',L_1\phi> \qquad (3.38)$$

$$<L_2\phi',\phi> = - <\phi',L_2\phi> \qquad (3.39)$$

$L_3$ and $\vec{L}^2$ are obviously Hermitian with respect to (3.35). We then see that the angular-momentum vector space associated to the complete basis (3.34) has a preferred direction determined by $L_3$. This corresponds to the physical fact that the vorticity field of matter $\vec{\Omega}$ polarizes the angular-momentum vector space along the direction determined by $\vec{\Omega}$ , and the allowed physical "rotations" in this space are the ones which maintain $L_3$ parallel to $\vec{\Omega}$ . The same result will show up for fermions, as we shall see later.

The scalar product defined above and the adjointness properties of the angular-momentum operators (3.36)-(3.39) will be helpful in the normalization of the hyperbolic harmonics (3.34). In fact starting from a given normalizable solution $\phi^I_{\varepsilon,\ell,m}$ for positive m we obtain for instance

$$<\phi^I_{\varepsilon,\ell,m-1}|\phi^I_{\varepsilon,\ell,m-1}> = \frac{k-(\sqrt{2}/2|\varepsilon|-m)^2+(\sqrt{2}/2|\varepsilon|-m)}{(2m)^2}<\phi^I_{\varepsilon,\ell,m}|\phi^I_{\varepsilon,\ell,m}> \qquad (3.40)$$

where we have also used (3.19) and (3.21).

By using the adjointness properties (3.38) and (3.39) of $L_1$ and $L_2$ with respect to the scalar product (3.35) we can understand how the two distinct sets of solutions (I) and (II) appear. From the definition $\vec{L}^2 = L_1^2 + L_2^2 + L_3^2$ , we have

$$<\phi_{\varepsilon,\ell,m}|\vec{L}^2-L_3^2\ |\phi_{\varepsilon,\ell,m}> = <\phi_{\varepsilon\ell m}|L_1^2\ \phi_{\varepsilon\ell m}> + <\phi_{\varepsilon\ell m}|L_2^2\ \phi_{\varepsilon\ell m}>$$

or

$$\left\{(\frac{n^2-1}{4})-(m + \frac{\sqrt{2}}{2}\varepsilon)^2\right\}<\phi_{\varepsilon\ell m}|\phi_{\varepsilon\ell m}> = - <L_1\phi_{\varepsilon\ell m}|L_1\phi_{\varepsilon\ell m}> +$$

$$- <L_2\phi_{\varepsilon\ell m}|L_2\phi_{\varepsilon\ell m}> \quad .$$

Since $<\phi_{\varepsilon\ell m}|\phi_{\varepsilon\ell m}>$ and $<L_i\phi_{\varepsilon\ell m}|L_i\phi_{\varepsilon\ell m}>$ must be finite, by hypothesis, and positive definite, it then follows

$$\frac{n^2-1}{4} - (m + \frac{\sqrt{2}}{2}\varepsilon)^2 \leq 0$$

or

$$m^2 + \sqrt{2}\, m\varepsilon + \frac{\varepsilon^2}{2} - \frac{n^2-1}{4} \geq 0 \quad . \qquad (3.41)$$

The sign $\geq$ which appears in (3.41) (and therefore (3.41) does not impose bounds in the range of m) is a consequence of the fact that actually the square of the total angular-momentum $\vec{L}^2$ is not positive definite (cf. (3.8)) although its eigenvalue is always positive definite or equivalently, of the anti-Hermiticity of $L_1$ and $L_2$.

Equation (3.41) imposes that the domain of m lies in the two disconnected intervals $A = (m \leq m_1 = -\frac{\sqrt{2}}{2}\varepsilon - \frac{1}{2}\sqrt{\varepsilon^2+\mu^2a^2+k_3^2}$ and $B = (m \geq m_2 = -\frac{\sqrt{2}}{2}\varepsilon + \frac{1}{2}\sqrt{\varepsilon^2+\mu^2a^2+k_3^2})$. We remark that for normalizable solutions we have $m \geq 0$ and also $m_2-m_1 > 1$. The latter condition imposes that we cannot pass from A to B or from B to A by applying respectively $L_+$ or $L_-$. We then have the two distinct possibilities:

(i) $m_2 > 0$: The "point" $m = 0$ is contained in B and A is excluded. This is the case of type (II) solutions.

(ii) $m_1 > 0$: The interval B is excluded, and starting from the lower bound $m = 0$ and by successive application of $L_+$ we must reach a value $m = \ell < m_1$ for which $L_+ \phi_{\varepsilon\ell} = 0$. This is the case of type (I) solutions.

To proceed with the normalization we consider first type I solutions, which can be expressed in the form [25]

$$\phi^{I}_{\varepsilon\ell m k_3} = \phi_{\varepsilon\ell m k_3}(x)\ e^{-im\phi}\ e^{-ik_3 z}\ e^{i|\varepsilon|t}$$

where

$$\phi_{\varepsilon\ell m k_3}(x) = \frac{(-1)^{\ell} m!}{2^{\ell-m}\ \ell!}\ (x^2-1)^{-\frac{m}{2}} (x+1)^{\frac{\sqrt{2}}{2}|\varepsilon|}\ \frac{d^{\ell-m}}{dx^{\ell-m}}\ \{(1-x^2)^{\ell}(1-x)^{-\sqrt{2}|\varepsilon|}\} \tag{3.42}$$

with $0 \leq m \leq \ell$. From (3.35) we have the normalization [26]

$$<\phi^{(I)}_{\varepsilon'\ell' m' k_3'}|\phi^{(I)}_{\varepsilon\ell m k_3}> = N^{(I)}_{\varepsilon\ell m k_3}\ \delta_{\ell\ell'}\ \delta_{mm'}\ \delta(k_3-k_3')\delta(|\varepsilon|-|\varepsilon'|) \tag{3.43}$$

where

$$N^{(I)}_{\varepsilon\ell m k_3} = \int_1^{\infty} dx\ \phi^{(I)2}_{\varepsilon\ell m k_3}(x) \quad . \tag{3.44}$$

To calculate this integral we start from the case $\ell = m$. By using the expression (3.42) we obtain straightforwardly $N_{\varepsilon\ell\ell k_3} = \frac{\Gamma(\sqrt{2}|\varepsilon|-\ell)}{\Gamma(\sqrt{2}|\varepsilon|-2\ell-1)}\ \ell!\ 2^{2\ell+1-\sqrt{2}|\varepsilon|}$. By successive applications of the operator $L_-$ and from its Hermiticity properties with respect to the normalization integral, we finally have

$$N^{(I)}_{\varepsilon\ell m k_3} = \frac{(m!)^2(\ell-m)!}{\ell!} \cdot \frac{\Gamma(\sqrt{2}|\varepsilon|-\ell-m)}{\Gamma(\sqrt{2}|\varepsilon|-2\ell-1)}\ 2^{2m-\sqrt{2}|\varepsilon|+1} \quad . \tag{3.45}$$

As for type (II) solutions we can express them as

$$\phi_{\varepsilon\ell m k_3} = \phi_{\varepsilon\ell m k_3}(x)\ e^{-ik_3x^3}\ e^{-im\phi}\ e^{-i|\varepsilon|t} \tag{3.46}$$

where [15]

$$\phi_{\varepsilon\ell m k_3}(x) = \frac{(-1)^{\ell+m}\ 2^{m+\sqrt{2}|\varepsilon|-\ell}\ \Gamma(m+1)}{\Gamma(m+\ell+1)}\ (x-1)^{-\frac{m}{2}}\ (x+1)^{\frac{m+\sqrt{2}|\varepsilon|}{2}}\ \cdot$$

$$\cdot\ \frac{d^\ell}{dx^\ell}\ \{(1-x)^{\ell+m}\ (1+x)^{\ell-m-\sqrt{2}|\varepsilon|}\ \} \tag{3.47}$$

with $\ell, m$ = positive integers or zero, without any relation between them. We have the $\delta$ normalization

$$<\phi^{(II)}_{\varepsilon'\ell'm'k_3'}|\phi^{(II)}_{\varepsilon\ell m k_3}> = N^{(II)}_{\varepsilon\ell m k_3}\ \delta_{\ell\ell'}\delta_{mm'}\delta(k_3-k_3')\delta(|\varepsilon|-|\varepsilon'|) \tag{3.48}$$

where $N^{(II)}_{\varepsilon\ell m k_3} = \int_1^\infty \phi^{(II)^2}_{\varepsilon\ell m k_3}(x)dx$. We first calculate this integral for the case $\ell = 0$. Then by successive application of $L_+$ and from its Hermiticity properties with respect to the normalization integral we obtain

$$N^{II}_{\varepsilon\ell m k_3} = \frac{\ell!\ 2^{\sqrt{2}|\varepsilon|+2m+1}\ \Gamma^2(m+1)}{\Gamma(m+\ell+1)}\ \cdot\ \frac{1}{(\sqrt{2}|\varepsilon|-2\ell-1)}\ \frac{\Gamma(\sqrt{2}|\varepsilon|-\ell)}{\Gamma(\sqrt{2}|\varepsilon|-\ell+m)}\ . \tag{3.49}$$

The set of solutions (3.42) or (3.43) constitute two distinct complete bases for the space of normalizable scalar field solutions, respectively finite- and infinite-dimensional representation basis spaces for the angular-momentum of the system. The lowest energy modes occur for $\ell = 0 = k_3$, with corresponding $|\varepsilon| = \sqrt{\mu^2 a^2+2}$ .

In the case of conformally invariant scalar fields

$$\frac{1}{\sqrt{-g}}\, \partial_\alpha \left(\sqrt{-g}\, g^{\alpha\beta} \partial_\beta \phi\right) + \left(\mu^2 + \frac{R}{6}\right)\phi = 0 \quad ,$$

the complete bases of solutions are obtained from the above ones by the trivial substitution $\mu^2 \rightarrow \mu^2 + \frac{R}{6}$ since the Ricci curvature scalar R is constant for Gödel's geometry.

The mathematical properties of the scalar field solutions $\phi_{\varepsilon\ell m k_3}$ discussed in this section will be helpful in describing the complete solutions of Dirac's equation in Gödel's universe, for energy and total angular-momentum modes.

## IV. LOCAL DYNAMICS OF FERMIONS, CONSTANTS OF MOTION AND THE SOLUTIONS OF DIRAC'S EQUATION

Spin-1/2 particles in interaction with gravitation are described by spinorial fields in the curved space-time. For a general review of spinors on a Riemannian space-time, see Ref. [27]. Here we use four component spinors from the point of view of the tetrad formalism. We choose a tetrad field $e_\alpha^{(A)}(x)$ such that the line element is expressed [28] as

$$ds^2 = \eta_{AB}\, \theta^A \theta^B \tag{4.1}$$

where $\theta^A = e_\alpha^{(A)}\, dx^\alpha$. The definition of a fermion wave function $\psi$ in a curved space-time involves two group structures. Its spinor character is defined with respect to the local Lorentz structure (4.1), that is, it provides a spinorial representation of the local Lorentz group

$$\theta'^{A} = L^{A}{}_{B}(x)\theta^{B} \tag{4.2}$$

with

$$L^{A}{}_{D}(x)\eta_{AB}\, L^{B}{}_{F} = \eta_{DF} \quad . \tag{4.3}$$

These transformations, which can be made independently at each space-time point, leave (4.1) invariant. Under (4.2) and (4.3) the spinors $\psi$ transform as

$$\psi'(x) = S(x)\psi(x) \tag{4.4}$$

where 4×4 matrix S(x) must satisfy [29]

$$(L^{-1})^{A}{}_{B}(x)\gamma^{B} = S(x)\gamma^{A}\, S^{-1}(x) \quad . \tag{4.5}$$

On the other hand spinors $\psi$ transform as scalar functions with respect to general coordinate transformations of the space-time, and thus provide a scalar representation of the isometry group of the space-time.

The Lagrangean for fermions is

$$i\,\sqrt{-g}\,\left(\overline{\psi}\gamma^{A}\nabla_{A}\psi - \nabla_{A}\overline{\psi}\gamma^{A}\psi\right) \quad . \tag{4.6}$$

In the above formalism $\overline{\psi} = \psi^{\dagger}\gamma^{0}$ where $\gamma^{0}$ is the constant Dirac matrix. The spinor covariant derivatives are given by

$$\begin{aligned} \nabla_{A}\psi &= e^{\alpha}_{(A)}\partial_{\alpha}\psi - \Gamma_{A}\psi \\ \nabla_{A}\overline{\psi} &= e^{\alpha}_{(A)}\partial_{\alpha}\overline{\psi} + \overline{\psi}\Gamma_{A} \end{aligned} \tag{4.7}$$

where the Fock-Ivanenko coefficients $\Gamma_A$ have the form

$$\Gamma_A = - \frac{1}{4} \gamma_{BCA}\gamma^B\gamma^C \quad . \tag{4.8}$$

The Ricci rotation coefficients $\gamma_{ABC}$ are defined by

$$\gamma_{ABC} = - e^{\alpha}_{(A)||\beta} \, e_{\alpha(B)} e^{\beta}_{(C)} \tag{4.9}$$

and Dirac equation for spin-1/2 fermions coupled to gravitation is expressed as

$$(i\gamma^A \nabla_A - M)\psi = 0 \quad . \tag{4.10}$$

For (2.16) we choose

$$\begin{aligned} \theta^0 &= a\ (dt + \sqrt{2}\ \sinh^2 r\ d\phi) \\ \theta^1 &= a\ dr \\ \theta^2 &= a\ \sinh r\ \cosh r\ d\phi \\ \theta^3 &= a\ dz \end{aligned} \tag{4.11}$$

where $a = 2/\omega$. With this choice the Fock-Ivanenko coefficients (4.8) have the expression

$$\begin{aligned} \Gamma_0 &= \frac{\sqrt{2}}{2a}\ \gamma^1\gamma^2 \\ \Gamma_1 &= \frac{\sqrt{2}}{2a}\ \gamma^0\gamma^2 \\ \Gamma_2 &= - \frac{\sqrt{2}}{2a}\ \gamma^0\gamma^1 + \frac{1}{2a}\ \frac{\cosh^2 r + \sinh^2 r}{\cosh r\ \sinh r}\ \gamma^2\gamma^1 \\ \Gamma_3 &= 0 \end{aligned} \tag{4.12}$$

For a fermion field in invariant energy excitation modes (2.23), using (4.11) and (4.12), Dirac equation (4.10) in Gödel's background can be expressed in the form

$$\epsilon\left[(-g)^{1/4}\,\psi\right] = (\gamma^5\vec{\Sigma}.\vec{\pi} + \mu\gamma^0)\left[(-g)^{1/4}\,\psi\right] \quad . \qquad (4.13)$$

Here we use the notation $\vec{A}.\vec{B} = \sum_{k=1}^{3} A^k B^k$, $\mu = aM$. $\vec{\Sigma}$ is the spin matrix $\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}$, g is the determinant of the metric and $\vec{\pi}$ is the generalized <u>local</u> momentum operator

$$\vec{\pi} = ia\vec{e}^{\alpha}\partial_{\alpha} + \gamma^5\vec{\Omega} \quad , \qquad (4.14)$$

where $\omega\vec{\Omega} = (0,0,\frac{\sqrt{2}}{2}\,\omega)$ is the vorticity of matter in the local Lorentz frame (4.11). Explicitly [30]

$$\pi^1 = -\,i\,\frac{\partial}{\partial r} \quad , \quad \pi^2 = i\,\left(\sqrt{2}\,\frac{\sinh r}{\cosh r}\,\frac{\partial}{\partial t} - \frac{1}{\sinh r\,\cosh r}\,\frac{\partial}{\partial \phi}\right)$$
$$\pi^3 = -\,i\,\frac{\partial}{\partial x^3} + \gamma^5\,\frac{\sqrt{2}}{2} \quad . \qquad (4.15)$$

Expression (4.14) is analogous to the case of the anomalous magnetic moment interaction of a charged spin-1/2 particle.

The operator $\gamma^5\vec{\Sigma}.\vec{\pi} + \mu\gamma^0$ acts on the Hilbert space of fermion wave functions [31] $\{(-g)^{1/4}\,\psi\}$ and determines the time development of any operator acting on this Hilbert space via the commutator rule. From (4.13) we have that

$$H = \gamma^5\vec{\Sigma}.\vec{\pi} + \mu\gamma^0 \qquad (4.16)$$

is the Hamiltonian of the system (expressed in terms of objects defined in the local frame (4.11)).

For $\mu \neq 0$, contrary to the flat space case, we have that $\vec{\Sigma}.\vec{\pi}$ and $H$ do not commute; it is not even possible to define another momentum $\vec{\pi}'$ (by properly subtracting terms in

(4.14)) such that $\vec{\Sigma}.\vec{\pi}'$ commute with the Hamiltonian. Therefore a stationary state of the system cannot have helicity as a good quantum number.

There is however a constant of motion of the system which in the limit of high momenta can be interpreted as the helicity. Let us express (4.16) as

$$H = \gamma^5 (\Sigma^1\pi^1 + \Sigma^2\pi^2) + \Sigma^3(S + \frac{\sqrt{2}}{2}) \tag{4.17}$$

where

$$S = \mu\gamma^3\gamma^5 + i\gamma^5 e^{(3)\alpha}\partial_\alpha \quad . \tag{4.18}$$

We have that

$$\left[S, H\right] = 0 \quad . \tag{4.19}$$

For fermion fields in the modes $e^{-ik_3x^3-i\varepsilon t}$, the constant of motion (4.18) becomes

$$S = \mu\gamma^3\gamma^5 - k_3\gamma^5 \quad . \tag{4.20}$$

We shall then select a complete basis of simultaneous eigenstates of $H$ and $S$ ; for large $k_3$, S is proportional to $\gamma^5$ and these stationary states tend to eigenstates of $\vec{\Sigma}.\vec{\pi}/\varepsilon$. From S we can define a conserved projection operator into states which are left- or right-polarized in the large $x^3$-momentum limit.

For neutrinos ($\mu = 0$) the helicity $\vec{\Sigma}.\vec{\pi}/\varepsilon$ is a constant of motion, and eigenstates of S are also eigenstates of $\vec{\Sigma}.\vec{\pi}/\varepsilon$. For this complete set of neutrino helicity eigenfunctions (note that $\gamma^5\psi = L\psi$ , $L^2 = 1$) the motion of the local momentum $\vec{\pi}$ is calculated $\dot{\vec{\pi}} = i\ [\vec{\pi}, L\vec{\Sigma}.\vec{\pi}]$ and we have

$$\dot{\vec{\pi}} = \sqrt{2}\ \varepsilon L\ \vec{\Sigma} \wedge \vec{\Omega} \qquad (4.21)$$

where L is the helicity of neutrinos. Since $\vec{\Sigma}.\vec{\pi}$ is conserved we have from (4.21) that (for a given sign of $\varepsilon$ ) the spin $\vec{\Sigma}$ precesses about the direction determined by $\vec{\Omega}$ ($x^3$-direction) with angular velocity proportional to $\frac{\sqrt{2}\,\varepsilon}{\omega}\vec{\Omega}$ and independent of the sign of L, that is, independent of being neutrino or antineutrino.

To separate Dirac equation (4.13) for a massive fermion in Gödel's background we consider fermion wave functions which belong to the globally defined set of modes $(\varepsilon, k_3, m, m')$ described by

$$\psi = A \begin{pmatrix} \phi(r)\ e^{-im\phi} \\ \eta(r)\ e^{-im'\phi} \end{pmatrix} e^{-ik_3x^3 - i\varepsilon t} \quad . \qquad (4.22)$$

and eigenstates of (4.20), namely

$$S\psi = (\mu\gamma^3\gamma^5 - k_3\gamma^5)\psi = -e\sqrt{\mu^2+k_3^2}\ \psi \qquad (4.23)$$

where $e = \pm 1$. The unitary matrix $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ has the role of interchanging the second and third components of the four-spinor in (4.22). Using (4.22) and (4.23), Dirac equation (4.13) reduces to

$$\left\{i\pi^1+\pi^2 + \frac{1}{2}\,\frac{\sinh^2 r+\cosh^2 r}{\sinh r\,\cosh r}\right\}\phi = -i(-\varepsilon - \frac{\sqrt{2}}{2} + e\sqrt{\mu^2+k_3^2})\sigma^1\eta$$

$$\left\{i\pi^1-\pi^2 + \frac{1}{2}\,\frac{\sinh^2 r+\cosh^2 r}{\sinh r\,\cosh r}\right\}\eta = -i(-\varepsilon + \frac{\sqrt{2}}{2} - e\sqrt{\mu^2+k_3^2})\sigma^1\phi \qquad (4.24)$$

or explicitly

$$\frac{d\phi}{dr} - \left[-\sqrt{2}\,\varepsilon\,\frac{\sinh r}{\cosh r} + \frac{m}{\sinh r\,\cosh r} - \frac{1}{2}\,\frac{\sinh^2 r+\cosh^2 r}{\sinh r\,\cosh r}\right]\phi =$$

$$= -i\,(-\varepsilon - \frac{\sqrt{2}}{2} + e\sqrt{\mu^2+k_3^2})\,\sigma^1\eta \qquad (4.24a)$$

$$\frac{d\eta}{dr} + \left[-\sqrt{2}\,\varepsilon\,\frac{\sinh r}{\cosh r} + \frac{m'}{\sinh r\,\cosh r} + \frac{1}{2}\,\frac{\sinh^2 r+\cosh^2 r}{\sinh r\,\cosh r}\right]\eta =$$

$$= -i\,(-\varepsilon + \frac{\sqrt{2}}{2} - e\sqrt{\mu^2+k_3^2})\,\sigma^1\phi \qquad (4.24b)$$

satisfied by the two component spinors $\phi$ and $\eta$ . By using a constant Foldy-Wouthuysen transformation we shall discuss later that our choice of the above set of solutions (4.22), (4.23) corresponds to a hybrid representation which mixes the advantages of the usual Foldy-Wouthuysen representation of Dirac solutions (which has a good non-relativistic limit), and of the Cini-Toushek representation which is most convenient for large momenta (or massless) particles.

Introducing the variable $x = \cosh 2r$, Dirac equation (4.13) does not change in form, with $\pi^1$ and $\pi^2$ given now by

$$\pi^1 = -2i\,(x^2-1)^{1/2}\,\frac{\partial}{\partial x}$$

$$\pi^2 = i\left\{\sqrt{2}\,\left(\frac{x-1}{x+1}\right)^{1/2}\frac{\partial}{\partial t} - \frac{2}{(x^2-1)^{1/2}}\,\frac{\partial}{\partial\phi}\right\} \qquad (4.15')$$

The second-order equations resulting from (4.24) take the eigen-value equation form

$$J^2\psi = k\psi \tag{4.25}$$

where

$$J^2 = (x^2-1)\frac{d^2}{dx^2} + \left[2x + m' - m\right]\frac{d}{dx} + \frac{1}{(x+1)}\left[\varepsilon^2 + \frac{\sqrt{2}}{2}\varepsilon(m'+m) + \right.$$

$$\left. + \frac{m'+m+\sqrt{2}\,\varepsilon}{2}\Sigma^3\right] - \frac{1}{(x^2-1)} A \begin{pmatrix} (m-1/2)(m'-1/2)\mathbb{1} & 0 \\ 0 & (m+1/2)(m'+1/2)\mathbb{1} \end{pmatrix} \tag{4.26}$$

is the square of the total angular-momentum operator of the system, as we shall show, with eigenvalue

$$k = \frac{1}{4}\left\{\varepsilon^2 + \left(e\sqrt{\mu^2+k_3^2} - \frac{\sqrt{2}}{2}\right)^2 - 1\right\} \quad . \tag{4.27}$$

Equations (4.25) are completely decoupled for each component; the condition (4.23) however requires that $\phi$ and $\eta$ must have the form

$$\phi(x) = f(x)\binom{1}{\gamma_+} \quad , \quad \eta(x) = g(x)\binom{1}{\gamma_-} \tag{4.28}$$

where

$$\gamma_\pm = \frac{\pm\mu + e\sqrt{\mu^2+k_3^2}}{k_3} \quad . \tag{4.29}$$

For consistency, taking a solution $\phi$ of the form (4.28), the corresponding solution $\eta$ is obtained by using (4.24a); conversely taking a solution $\eta$ (4.28) the corresponding solution $\phi$ is obtained by using (4.24b). The second-order equations (4.25) can be solved by substitutions analogous as done to the scalar field in Section III. We distinguish the set of solutions

$$\psi(\varepsilon,k_3,m,m') = \begin{pmatrix} \Lambda\alpha\ e^{-im\phi} \\ \beta e^{-im'\phi} \\ \Lambda\gamma_+\alpha\ e^{-im\phi} \\ \gamma_-\beta\ e^{-im'\phi} \end{pmatrix} e^{-ik_3x^3} e^{-i\varepsilon t} \tag{4.30}$$

where

$$\alpha = (m+m'+1)\left(\frac{x+1}{x-1}\right)^{\frac{m'-m}{4}} (x^2-1)^{\frac{m+m'-1}{4}} (x+1)^{\frac{\sqrt{2}}{2}\varepsilon+\frac{1}{2}} F(a,b,c;\tfrac{1-x}{2}) \tag{4.31}$$

$$\beta = \left(\frac{x+1}{x-1}\right)^{\frac{m'-m}{4}} (x^2-1)^{\frac{m+m'+1}{4}} (x+1)^{\frac{\sqrt{2}}{2}\varepsilon-\frac{1}{2}} F(a,b,c+1;\tfrac{1-x}{2}) \tag{4.32}$$

$$\Lambda = -i\left[k_3 + \left(\varepsilon - \frac{\sqrt{2}}{2} + \mu\right)\gamma_+\right]^{-1} . \tag{4.33}$$

$F(a,b,c;\frac{1-x}{2})$ is the hypergeometric function [15,16] with argument $\frac{1-x}{2}$ and parameters

$$\begin{aligned} a &= \frac{m+m'}{2} + \frac{\sqrt{2}}{2}\varepsilon + \frac{1}{2} \pm \frac{n}{2} \\ b &= \frac{m+m'}{2} + \frac{\sqrt{2}}{2}\varepsilon + \frac{1}{2} \mp \frac{n}{2} \\ c &= \frac{m+m'}{2} + \frac{1}{2} \end{aligned} \tag{4.34}$$

with

$$n = \sqrt{4k+1} = \sqrt{\varepsilon^2 + \left(\sqrt{\mu^2+k_3^2} - e\sqrt{2}/2\right)^2} . \tag{4.35}$$

Analogous to the case of scalar solutions, we now define on the space of solutions (4.30) the operators

$$J_- = L_- + S_- \tag{4.36}$$

$$J_+ = L_+ + S_+ \tag{4.37}$$

where $L_+$ and $L_-$ are the operators (3.18) of the algebra of angular momentum associated to the scalar field, and

$$S_- = e^{i\phi} \left\{ \frac{\sigma}{2} \frac{1}{(x^2-1)^{1/2}} + \frac{(\sigma x-1)}{2(x^2-1)^{1/2}} \Sigma^3 \right\} \tag{4.38}$$

$$S_+ = e^{-i\phi} \left\{ \frac{\sigma}{2} \frac{1}{(x^2-1)^{1/2}} - \frac{(\sigma x-1)}{2(x^2-1)^{1/2}} \Sigma^3 \right\} . \tag{4.39}$$

We have denoted

$$m' = m + \sigma \quad . \tag{4.40}$$

We define $J_3$ by the relation $[J_+, J_-] = 2J_3$ and obtain

$$J_3 = i \left(\frac{\partial}{\partial\phi} + \frac{\sqrt{2}}{2} \frac{\partial}{\partial t}\right) + \frac{\sigma}{2} \Sigma^3 \tag{4.41}$$

The effect of the operators (4.36), (4.37) and (4.41) on the set of solutions (4.30) is

$$J_- \psi(m,m') = 2 \left(\frac{m+m'+1}{2}\right) \psi(m-1,m'-1) \tag{4.42}$$

$$J_+ \psi(m,m') = - \frac{ab}{m+m'+1} \psi(m+1,m'+1) \tag{4.43}$$

$$J_3 \psi(m,m') = \left(m + \frac{\sqrt{2}}{2} \varepsilon + \frac{\sigma}{2}\right) \psi(m,m') \tag{4.44}$$

From the definition of $J_3$ and from the relations

$$[J_+, J_3] = -J_+ \qquad (4.45)$$

$$[J_-, J_3] = J_- \qquad (4.46)$$

we see that $J_+$, $J_-$ and $J_3$ generate the algebra of angular momentum. Actually introducing $J_1 = J_+ + J_-/2$ and $J_2 = J_+ - J_-/2i$ it results $[J_i, J_k] = i\varepsilon_{ijk}J_k$, and a straightforward calculation gives

$$(J_1{}^2 + J_2{}^2 + J_3{}^2)\psi(m,m') = J^2\ \psi(m,m') = k\ \psi(m,m') \qquad (4.26)$$

as we have mentioned earlier, where k is given in (4.27).

In a procedure analogous to the scalar field case discussed in Section 3, by using (4.42)-(4.44) and the commutation relations (4.45) and (4.46), we start from a given solution $\psi(m,m')$ to construct a sequence of solutions – by successive applications of $J_+$ or $J_-$ – which are eigenstates of $J_3$, for increasing or decreasing values of (m,m'). The sequence extends indefinitely in both directions or terminates if $J_+\ \psi(m,m')$ or $J_-\ \psi(m,m')$ vanishes for some value of $m+m'/2$.

Also in the present case of spin-1/2 Dirac fields it is not possible to use the properties of the operators $J_1$, $J_2$ and $J_3$ to set bounds on the range of $m+\sigma/2$ because $J_1$ and $J_2$ lack any hermiticity property (for $\sigma \neq 0$) with respect to the normalization scalar product to be defined in Section V for the functions (4.30). For $\sigma = 0$, $J_1$ and $J_2$ are anti Hermitian. $J_3$ in all cases is obviously Hermitian, as in the scalar case. Angular momentum space has thus a preferred direction defined locally by the vorticity vector $\vec{\Omega}$. This is characterized by the fact that the projection of $\vec{J}$ along $\vec{\Omega}$ is Hermitian while any of its components along a direction orthogonal to $\vec{\Omega}$ is not. The <u>allowed</u> "rotations" in this space maintain the direction $\vec{\Omega}$ invariant.

To proceed we shall then make use of regularity and boundary conditions on the wave functions, and obtain two distinct sets of solutions, one infinite dimensional and the other finite dimensional representation basis of the algebra of angular momentum. On the set of solutions (4.30) we now impose boundary and regularity conditions, namely that Dirac fields (which are test fields and do not contribute to the curvature of the cosmological background) are finite perturbations at any space-time point. We impose similarly to (3.22), (3.23), that

$$\lim_{x \to 1} \psi^\dagger \psi = \text{finite} \tag{4.47}$$

$$\lim_{x \to \infty} \sqrt{-g}\; \psi^\dagger \psi = 0 \quad . \tag{4.48}$$

The quantity $\psi^\dagger\psi$ is the component of the Dirac current $\bar{\psi}\gamma^\mu(x)\psi$ along the four-velocity of the matter content of the model. By using (4.31) and (4.32), the regularity condition (4.47) implies

$$m \geq -1/2 \quad . \tag{4.49}$$

The lower bound $m = -1/2$ is not in contradiction with the regularity condition (4.47) because we have [33] $\lim_{m \to -1/2} \alpha$ = finite for all x. So starting from a given regular solution $\psi(m,m')$ and by successively applying $J_{(-)}$ we necessarily arrive at a solution which does not satisfy (4.49) unless $J_-\psi = 0$ for some value $(m,m')$. From (4.42) we have that the sequence finishes on the left for $m+m'/2 = -1/2$, and we must then have

$$\frac{m+m'}{2} \geq -1/2 \qquad (4.50)$$

that is, m+m'/2 takes half-integer values greater or equal to -1/2. On the right the sequence could in principle extend to infinite values of m+m'/2 by successive application of $J_+$. Condition (4.48) will nevertheless impose an upper bound on the values of m+m'/2.

From (4.48) two distinct possibilities arise [20,21]. Either (cf. the scalar field case)

(I) a = negative integer or zero (4.51)

or

(II) c-b = negative integer or zero (4.52)

with

$$a = \frac{m+m'}{2} + \frac{\sqrt{2}}{2}\,\varepsilon + \frac{1}{2} + \frac{n}{2} \qquad (4.53)$$

$$b = \frac{m+m'}{2} + \frac{\sqrt{2}}{2}\,\varepsilon + \frac{1}{2} - \frac{n}{2} \qquad (4.54)$$

for both cases (I) and (II) and we obtain the two distinct sets of solutions:

Type I solutions

We denote any negative integer or zero by m+m'/2 - j, with j = half-integer $\geq$ m+m'/2, that is,

$$-1/2 \leq \frac{m+m'}{2} \leq j \quad . \qquad (4.55)$$

From (4.51) and (4.53) we then have $j + \frac{\sqrt{2}}{2}\varepsilon + \frac{1}{2} + \frac{n}{2} = 0$ which implies

$$\varepsilon = -\left[\sqrt{2}(2j+1) + \sqrt{(2j+1)^2 + (\sqrt{\mu^2+k_3^2} - e\sqrt{2}/2)^2}\right] . \quad (4.56)$$

The corresponding positive-energy solutions of type (I) are obtained from the symmetry $\psi \to i\gamma^2\psi^*$ of Dirac equation (4.10), where $*$ denotes complex-conjugation. We remark that the eigenvalues of $J_3$ and $J^2$ for this case are given by eigenv $(J_3) = = m + \frac{\sigma}{2} + \frac{\sqrt{2}}{2}\varepsilon$ and

$$\text{eigenv}(J^2) = (j + \frac{\sqrt{2}}{2}\varepsilon)(j + \frac{\sqrt{2}}{2}\varepsilon + 1) , \quad (4.57)$$

respectively. Also from (4.43) we obviously have

$$J_+ \psi_{\varepsilon,j} = 0$$

as expected.

Type II solutions

We here denote any negative integer or zero by $-(j + 1/2)$, where

$$j = \text{half-integer} \geq -1/2 . \quad (4.58)$$

From (4.52) and (4.54) we have $j - \frac{\sqrt{2}}{2}\varepsilon + \frac{1}{2} + \frac{n}{2} = 0$ which implies

$$\varepsilon = \sqrt{2}\,(2j+1) + \sqrt{(2j+1)^2 + (\sqrt{\mu^2+k_3^2} - e\sqrt{2}/2)^2} . \quad (4.59)$$

The corresponding negative-energy states of type II are obtained from the symmetry $\psi \to i\gamma^2\psi^*$ of Dirac equation (4.10).

We note that for type I solutions the values of $\frac{m+m'}{2}$ are bounded for a given j (cf. (4.55)), and for type II solutions the range $\frac{m+m'}{2} \geq -1/2$ is completely independent of the value of j. In other words, for a given j = half-integer $\geq -1/2$ type I solutions provide a finite-dimensional (dim = $j + \frac{3}{2}$) representation basis for the algebra of angular momentum, while type (II) solutions provide an infinite-dimensional representation basis for the algebra of angular momentum. In the above discussion we have discarded normalizable solutions which could not constitute a basis of representation for the algebra of angular--momentum [32]. We should mention that some of these solutions have interesting features as zero energy and eigenvalue of $J_3$ equal to an integer.

We finally remark that (similar to the scalar field solutions) the above two sets of continuous spinor solutions provide two bases of irreducible representations of the simply transitive Lie group $H^3 \times R$ (Gödel's manifold), the representation associated to type I solutions being finite dimensional (dim = $= j + \frac{3}{2}$). By arguments analogous to the scalar field case we have that these bases are two complete bases, for continuous spinorial functions defined over the group manifold $H^3 \times R$ and which satisfy the prescribed regularity conditions (4.47) and (4.48).

## V. COMPLETE SET OF SOLUTIONS AND NORMALIZATION OF FERMION AMPLITUDES

We restrict ourselves to the complete basis of type

(I) solutions for two reasons. Firstly the use of type I basis is physically more satisfactory because it corresponds to a finite-dimensional representation of the angular momentum algebra of the system, that is, for a fixed energy $|\varepsilon|$ and for a given value of the total angular momentum $\sqrt{(j - \frac{\sqrt{2}}{2}|\varepsilon|)(j - \frac{\sqrt{2}}{2}|\varepsilon|+1)}$, where j = half-integer $\geq -1/2$, we have $j + \frac{3}{2}$ eigenstates of the angular momentum projection on the local axis $\vec{\Omega}$; secondly for simplicity, because all the following results are analogous to the ones obtained if we considered also type II basis. Without loss of generality in what follows we consider only the case $\sigma = 0$.

If we examine carefully expressions (4.29) and (4.30)-–(4.33) we observe that we have a problem in the low-momentum limit $k_3 \to 0$. The constants $\gamma_+$ and $\gamma_-$ which appear multiplicatively in the solutions have a different behaviour at this limit for distinct values of e, namely for $k_3 \to 0$

$$\begin{aligned} e = +1 : \gamma_+ \to \infty \quad &\text{and} \quad \gamma_- \to 0 \\ e = -1 : \gamma_+ \to 0 \quad &\text{and} \quad \gamma_- \to \infty \quad . \end{aligned} \tag{5.1}$$

We remark that $\gamma_+\gamma_- = 1$. In order that the solutions are bounded for all values of the momentum $k_3$, they must be normalized differently for different values of e, that is, they must differ by a factor linear in $\gamma_+$ or $\gamma_-$ for different values of e. A suitable choice for the complete basis of solutions is

positive energy solution, e = +1

$$\psi_{(+),e=+1} = \begin{pmatrix} \gamma_-\beta \\ -\gamma_+\Lambda\alpha \\ \beta \\ -\Lambda\alpha \end{pmatrix} e^{im\phi} \, e^{-ik_3x^3} \, e^{-i|\varepsilon|t} \tag{5.2}$$

positive energy solution, e = -1

$$\psi_{(+),e=-1} = \begin{pmatrix} \beta \\ -\gamma_+^2\Lambda\alpha \\ \gamma_+\beta \\ -\gamma_+\Lambda\alpha \end{pmatrix} e^{im\phi}\, e^{-ik_3x^3}\, e^{-i|\varepsilon|t} \qquad (5.3)$$

negative energy solution, e = +1

$$\psi_{(-),e=+1} = \begin{pmatrix} \Lambda\alpha \\ \beta \\ \gamma_+\Lambda\alpha \\ \gamma_-\beta \end{pmatrix} e^{-im\phi}\, e^{-ik_3x^3}\, e^{i|\varepsilon|t} \qquad (5.4)$$

negative energy solution, e = -1

$$\psi_{(-),e=-1} = \begin{pmatrix} \gamma_+\Lambda\alpha \\ \gamma_+\beta \\ \gamma_+^2\Lambda\alpha \\ \beta \end{pmatrix} e^{-im\phi}\, e^{-ik_3x^3}\, e^{i|\varepsilon|t} \qquad (5.5)$$

where $\Lambda = -i[k_3 + (-|\varepsilon| - \frac{\sqrt{2}}{2} + \mu)\gamma_+]^{-1}$. For all cases $\alpha$ and $\beta$ are given by (cf. eqs. (4.31) and (4.32))

$$\alpha = (2m+1)(x^2-1)^{\frac{2m-1}{4}}(x+1)^{-\frac{\sqrt{2}}{2}|\varepsilon|+\frac{1}{2}}\, F(a,b,c;\tfrac{1-x}{2}) \qquad (5.6)$$

$$\beta = (x^2-1)^{\frac{2m+1}{4}}(x+1)^{-\frac{\sqrt{2}}{2}|\varepsilon|-\frac{1}{2}}\, F(a,b,c+1;\, \tfrac{1-x}{2}) \qquad (5.7)$$

where

$$a = m - j \tag{5.8}$$

$$b = m + j - \sqrt{2}|\varepsilon|+1 \tag{5.9}$$

$$c = m + 1/2 \tag{5.10}$$

$$-1/2 \leq m \leq j \quad , \quad j = \text{half-integer} \geq -1/2 \tag{5.11}$$

and

$$|\varepsilon| = \sqrt{2}\ (2j+1) + \sqrt{(2j+1)^2+(\sqrt{k_3^{\,2}+\mu^2} - e\sqrt{2}/2)^2} \ . \tag{5.12}$$

We remark that the lower bound $m = -1/2$ in (5.11) is not in contradiction with the regularity condition (4.47) because we have [33] $\lim_{m\to -1/2} \alpha = \text{finite}$ for all x. The positive-energy solutions are orthogonal to all negative energy ones by $\phi$-integration only (cf. Ref. [22] for the scalar field case), except for $m,m' = \pm 1/2$. In this case they are orthogonal either by $(r,x^3)$ integration only or by t integration.

We finally note that from the first order equations (4.24) we obtain the useful differential relation between the functions $\alpha$ and $\beta$ ,

$$\left\{ i\pi^1 + \pi^2 + \frac{x}{(x^2-1)^{1/2}} \right\} \Lambda\alpha = -i(|\varepsilon| - \frac{\sqrt{2}}{2} + e\sqrt{\mu^2+k_3^{\,2}}\,)\ \gamma_-\beta$$

$$\left\{ i\pi^1 - \pi^2 + \frac{x}{(x^2-1)^{1/2}} \right\} \beta = -i(|\varepsilon| + \frac{\sqrt{2}}{2} - e\sqrt{\mu^2+k_3^{\,2}}\,)\gamma_+\Lambda\alpha \tag{5.13}$$

where here $\quad \pi^2 = -\sqrt{2}|\varepsilon|(\frac{x-1}{x+1})^{1/2} - \frac{2m}{(x^2-1)^{1/2}} \quad .$

We now examine two important limiting cases of the above set of solutions. The first is the <u>low-momentum limit</u>

$k_3 \to 0$ which gave origin to the necessity of the distinct normalization for each value of e. Having in mind (5.1) we obtain

for e = +1:

$$\psi_{(+),e=+1} = \begin{pmatrix} 0 \\ \dfrac{\alpha}{i(|\varepsilon|+\frac{\sqrt{2}}{2}-\mu)} \\ \beta \\ 0 \end{pmatrix} e^{im\phi}\ e^{-i|\varepsilon|t} \quad ,$$

$$\psi_{(-),e=+1} = \begin{pmatrix} 0 \\ \beta \\ \dfrac{-\alpha}{i(|\varepsilon|+\frac{\sqrt{2}}{2}-\mu)} \\ 0 \end{pmatrix} e^{-im\phi}\ e^{i|\varepsilon|t} \qquad (5.14a)$$

and for e = -1

$$\psi_{(+),e=-1} = \begin{pmatrix} \beta \\ 0 \\ 0 \\ \dfrac{\alpha}{i(|\varepsilon|+\frac{\sqrt{2}}{2}+\mu)} \end{pmatrix} e^{im\phi}\ e^{-i|\varepsilon|t} \quad ,$$

$$\psi_{(-),e=-1} = \begin{pmatrix} \dfrac{-\alpha}{i(|\varepsilon|+\frac{\sqrt{2}}{2}+\mu)} \\ 0 \\ 0 \\ \beta \end{pmatrix} e^{-im\phi}\ e^{i|\varepsilon|t} \quad . \qquad (5.14b)$$

This set of states (5.14) has the symmetry $\psi(-e,\mu) = \gamma^5\psi(e,-\mu)$. In other words, for $k_3 = 0$ the mass reversal substitution symmetry [34] of Dirac equation $(\psi(\mu) \to \gamma^5\psi(-\mu))$ corresponds to a change in the sign of e.

An interesting class of solutions are the lowest-energy modes $j = m = -1/2$ with corresponding $|\varepsilon| = |\mu - e\sqrt{2}/2|$ and square of the total angular momentum $J^2 = \frac{\mu}{2}(\mu - \sqrt{2}\, e)$. For this case $\alpha \equiv 0$. The non-relativistic Minkowski limit is obtained here by considering $\sqrt{2}/2$ infinitesimal – the energy is then the rest mass of the particle and the sign of the energy is given by the eigenvalue of $\Sigma^3$, namely

$$\psi_{(+),e=+1} = k(x)\begin{pmatrix}0\\0\\1\\0\end{pmatrix} e^{-i|\varepsilon|t} \quad , \quad \psi_{(+),e=-1} = k_{(x)}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} e^{-i|\varepsilon|t}$$

$$\psi_{(-),e=+1} = k(x)\begin{pmatrix}0\\1\\0\\0\end{pmatrix} e^{i|\varepsilon|t} \quad , \quad \psi_{(-),e=-1} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} e^{i|\varepsilon|t} \tag{5.15}$$

where $k(x) = (x+1)^{-\frac{\sqrt{2}|\varepsilon|+1}{2}}$.

Besides the trivial solution $\psi = 0$, two <u>zero-energy modes</u> occur for $e = +1$ and $\mu = \sqrt{2}/2$:

$$\psi \sim \frac{1}{(x+1)^{1/2}}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} e^{-i\phi/2} \quad \text{and} \quad \psi \sim \frac{1}{(x-1)^{1/2}}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} e^{i\phi/2}$$

which go to the solution $\psi = 0$ in the limit $x \to \infty$. If we require that solutions must have $J^2$ always greater or equal to zero – like the scalar field solutions have – asymmetry with respect to e occurs because for $e = +1$ only solutions with $\mu > \sqrt{2}$ are allowed, and the zero-energy modes must be excluded.

Also the modes $j = m = -1/2$ are eigenstates of $\Sigma^3$ and $J_3$ with respective eigenvalues $\pm 1$ and $\pm (\frac{1+\sqrt{2}|\varepsilon|}{2})$, for positive/ /negative energy solutions. In the special case $\mu = 0$, the lowest energy and angular momentum modes $j = m = -1/2$ have the total angular-momentum projection $J_3 = \pm 1$ for positive/negative energy solutions. Thus due to the gravitational coupling to matter vorticity these massless fermions are converted to bosons polarized along the direction $\vec{\Omega}$. This latter case and the case $j = m = -1/2$, $\mu = \sqrt{2}/2$ are the only ones in which the eigenvalues of the total angular momentum $J_3$ are definitely integer or half-integer.

We finally remark that in the limit $k_3 \to 0$ there is a representation – defined by the unitary transformation

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

– where the operator S (cf. (4.20) and (4.23)) can be interpreted as the projection of the spin $\vec{\Sigma}$ along the local direction determined by $\vec{\Omega}$. Indeed in this representation ($\psi' = A\psi$) the eigenfunctions (5.14) of S are eigenfunctions of $\Sigma^3$ with eigenvalue e, as can be easily verified [35].

## Neutrino Amplitudes

The other important limit of the complete basis (5.2)–(5.5) is the high-momentum limit $k_3^2 \gg \mu^2$. In this case all mass terms are neglected as compared to $k_3$ and the solutions assume the form of neutrino solutions [4], which correspond in (5.2)–(5.5) to take $\mu = 0$. Denoting $s(k_3)$ = sign of $k_3$ and having in mind that $\gamma_{\pm} = e\, s(k_3)$ for $\mu = 0$ we obtain [36] from (5.2)–(5.5)

$$\psi_{(+)}(L,k_3) = \begin{pmatrix} \phi_{(+)}(L,k_3) \\ L\ \phi_{(+)}(L,k_3) \end{pmatrix} e^{im\phi}\, e^{-ik_3x^3}\, e^{-i|\varepsilon|t} \qquad (5.16)$$

where

$$\phi_{(+)}(L,k_3) = \begin{pmatrix} \beta \\ \dfrac{iL}{(Lk_3-|\varepsilon|-\sqrt{2}/2)}\,\alpha \end{pmatrix} \qquad (5.17)$$

and

$$\psi_{(-)}(L,k_3) = \begin{pmatrix} \phi_{(-)}(L,k_3) \\ L\phi_{(-)}(L,k_3) \end{pmatrix} e^{-im\phi}\, e^{-ik_3x^3}\, e^{i|\varepsilon|t} \qquad (5.18)$$

where

$$\phi_{(-)}(L,k_3) = \begin{pmatrix} \dfrac{-i}{Lk_3-|\varepsilon|-\sqrt{2}/2}\,\alpha \\ L\beta \end{pmatrix} . \qquad (5.19)$$

In the above expressions $\alpha$ and $\beta$ are given by (5.6)–(5.11), with

$$|\varepsilon| = \sqrt{2}\,(2j+1) + \sqrt{(2j+1)^2 + (k_3 - \tfrac{\sqrt{2}}{2}L)^2} . \qquad (5.20)$$

We have made the identification

$$L = e\, s(k_3) = \gamma_{\pm}\Big|_{\mu=0} \qquad (5.21)$$

where L is the helicity (or chiralty) of the neutrino amplitude.

We remark the invariant character of (5.21) since $s(k_3)$ is invariantly defined (with respect to coordinate and local Lorentz transformations) in (2.22) through the use of the global Killing vector field $\partial/\partial x^3$ and the left-invariant tetrad component $e^{\alpha}_{(3)} = \delta^{\alpha}_{3}$.

Solutions (5.16) and (5.18) are related by

$$\psi_{(+)}(L,k_3) = -iL\gamma^5\gamma^2\psi^*_{(-)}(-L,-k_3) \quad . \qquad (5.22)$$

Foldy-Wouthuysen and Cini-Toushek Representations of the Solutions

As we have mentioned already, the set of solutions characterized by (4.22) and (4.23) corresponds to a hybrid representation which mixes the advantages of the Foldy-Wouthuysen (FW) representation [37] of Dirac solutions (with its good non-relativistic limit) and the Cini-Toushek (CT) representation [38] which is very convenient for large momenta (or massless) particles. To see this let us start from the original Hamiltonian

$$H = \gamma^5 (\Sigma^1\pi^1 + \Sigma^2\pi^2) + \Sigma^3 (S + \frac{\sqrt{2}}{2}) \quad , \qquad (4.17)$$

where $S = \mu\gamma^3\gamma^5 - k_3\gamma^5$ (cf. (4.20)) and $\pi^1$ and $\pi^2$ are given by (4.15'), and make the following unitary transformations characterized by a unitary matrix of the form

$$U = e^{\sigma\gamma^3} \qquad (5.23)$$

where $\sigma$ is a real parameter to be specified for each case:

(i) FW transformation corresponding to a generalized Lorentz rotation in the plane $(\mu,k_3)$, or equivalently in the plane $(\gamma^0,\gamma^0\gamma^3)$ (cf. Ref. [39]), which makes the new components of $\Sigma^3 S$ along the axis $\gamma^0\gamma^3$ (or the axis $k_3$) equal to zero. This is acomplished by taking the parameter $\sigma$ in (5.23) determined by $\cos 2\sigma = \mu/\sqrt{\mu^2+k_3^2}$, $\sin 2\sigma = -k_3/\sqrt{\mu^2+k_3^2}$. The transformed Hamiltonian is given by

$$H' = UHU^{-1} = P + \gamma^0 \left(\sqrt{\mu^2+k_3^2} + \frac{\sqrt{2}}{2}\gamma^0\Sigma^3\right) \tag{5.24a}$$

or equivalently

$$H' = P + \Sigma^3 \left(\sqrt{\mu^2+k_3^2}\,\gamma^0\Sigma^3 + \frac{\sqrt{2}}{2}\right) \tag{5.24b}$$

where we have denoted $P = \gamma^5(\Sigma^1\pi^1 + \Sigma^2\pi^2)$. By the above transformation the constant of motion S is diagonalized, namely

$$S' = USU^{-1} = \sqrt{\mu^2+k_3^2}\,\gamma^0\Sigma^3 \tag{5.25}$$

and (4.23) implies that

$$\gamma^0\Sigma^3\psi' = -\,e\,\psi' \tag{5.26}$$

where $\psi' = U\psi$. Since $\gamma^0$ or $\Sigma^3$ anticommute with P and $\gamma^0\Sigma^3$ is a constant of motion, by a further FW transformation we can reduce the Hamiltonians (5.24a,b) to the following ones

$$H'' = \sqrt{P^2 + \left(\sqrt{\mu^2+k_3^2} + \frac{\sqrt{2}}{2}\gamma^0\Sigma^3\right)^2}\;\gamma^0 \tag{5.27a}$$

$$\tilde{H}'' = \sqrt{P^2 + (\sqrt{\mu^2+k_3^2}\ \gamma^0\Sigma^3 + \frac{\sqrt{2}}{2})^2}\ \Sigma^3 \qquad (5.27b)$$

where $P^2 = (\pi^1)^2 + (\pi^2)^2 - i\Sigma^3[\pi^1,\pi^2]$. We remark that the constant of motion (5.23) commutes with the latter FW transformations. We than have from (5.27a,b) and (5.26) the Hamiltonian equations

$$H''\psi'' = \sqrt{P^2 + (\sqrt{\mu^2+k_3^2} - e\frac{\sqrt{2}}{2})^2}\ \gamma^0\psi'' = \varepsilon\psi'' \qquad (5.28a)$$

$$\tilde{H}''\tilde{\psi}'' = \sqrt{P^2 + (\sqrt{\mu^2+k_3^2} - e\frac{\sqrt{2}}{2})^2}\ \Sigma^3\tilde{\psi}'' = \varepsilon\psi'' \quad . \qquad (5.28b)$$

For (5.28a) the sign of the energy is given by the eigenvalue of $\gamma^0$, as usual, and for (5.28b) the sign of the energy is given by the eigenvalues of $\Sigma^3$ (cf. the NR limit (5.15)). Representations (5.28a) and (5.28b) are equivalent by the unitary constant transformation

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad .$$

(ii) CT transformation which corresponds to a generalized Lorentz rotation in the plane $(\mu,k_3)$, or equivalently in the plane $(\gamma^0,\gamma^0\gamma^3)$ (cf. Ref. [39]), and which brings $\Sigma^3 S$ to the axis $\gamma^0\gamma^3$ (or the axis $k_3$). This is acomplished by the unitary matrix (5.23) with the parameter $\sigma$ given by $\sin 2\sigma = s(k_3)\mu/\sqrt{\mu^2+k_3^2}$, $\cos 2\sigma = s(k_3)k_3/\sqrt{\mu^2+k_3^2}$, where $s(k_3)$ stands for the sign of $k_3$. It then results that

$$(\Sigma^3 S)' = U\Sigma^3 S U^{-1} = -s(k_3)\sqrt{\mu^2+k_3^2}\ \gamma^0\Sigma^3 \tag{5.29}$$

and from (4.17) we have the transformed Hamiltonian

$$H' = UHU^{-1} = P + \Sigma^3(-s(k_3)\sqrt{\mu^2+k_3^2}\,\gamma^5 + \frac{\sqrt{2}}{2}) \ . \tag{5.30}$$

In the new representation $\gamma^5$ is a constant of motion, and we choose $\psi' = U\psi$ such that

$$\gamma^5\psi' = L\psi' \quad , \qquad L^2 = 1 \tag{5.31}$$

where L is the chirality of $\psi'$ (CT or chiral representation). The operator $\gamma^5$ corresponds in the old representation to

$$U^{-1}\gamma^5 U = \frac{-s(k_3)}{\sqrt{\mu^2+k_3^2}}\, S \quad , \tag{5.32}$$

and from (5.31), (5.32) and (4.23) we derive

$$L = e\ s(k_3) \tag{5.21}$$

which is a result we have obtained already for the high momenta ($k_3^2 >> \mu^2$), or neutrino ($\mu = 0$) amplitudes.

Since $\Sigma^3$ anticommutes with P, by a further CT transformation we can reduce (5.30) to

$$H'' = \sqrt{P^2 + (-s(k_3)\sqrt{\mu^2+k_3^2}\ \gamma^5 + \frac{\sqrt{2}}{2})^2}\ \Sigma^3 \ . \tag{5.33}$$

We remark that the constant of motion $\gamma^5$ (cf. (5.31) and (5.32)) commutes with the latter CT transformation. We then have from

(5.31) the Hamiltonian equation

$$H''\psi'' = \sqrt{P^2 + (\sqrt{\mu^2 + k_3^2} - e\sqrt{2}/2)^2}\ \Sigma^3\psi'' \tag{5.34}$$

where we have used (5.21).

We finally note that for the solutions in the representations (5.27a,b) and (5.34), obtained from our solutions (5.2)–(5.5) by unitary FW or CT transformations, the operator $P^2$ has eigenvalue

$$P^2 = 4(j + \tfrac{1}{2})(\sqrt{2}|\varepsilon| - j - \tfrac{1}{2}) \quad .$$

We now discuss the normalization of the set of modes $(j, m, \varepsilon, k_3, e)$ defined in (5.2)–(5.5). Let us consider the local classical Dirac current

$$j^{(A)} = \overline{\psi}\gamma^A\psi = e_\alpha^{(A)}(x)\ \overline{\psi}\gamma^\alpha(x)\psi \quad . \tag{5.35}$$

The component $j^{(0)} = \psi^\dagger\psi$ of (5.35) is the <u>local number density</u> of fermions. As expected $j^{(0)}$ transforms as the zeroth component of a Lorentz vector with respect to local Lorentz transformations (4.2) and it is a scalar function with respect to coordinate transformations (and/or point transformations) of the space--time. The local number $j^{(0)}\ \sqrt{-g}\ d^4x$ is thus a scalar and integrated over a given volume of the manifold

$$\int \sqrt{-g}\ \ j^{(0)} d^4x \equiv \langle\psi|\psi\rangle \tag{5.36}$$

yields a positive definite quantity which is coordinate invariant.

Fermions amplitudes are normalized according to the integral (5.22) taken over the whole Gödel manifold [40], and for the set (5.2)–(5.5) we have the $\delta$ normalization [41]

$$<\psi_{(r)}(j',m',k_3',\varepsilon',e')|\psi_{(s)}(j,m,k_3,\varepsilon,e)> = (2\pi)^3 N^2(e) .$$

$$\cdot \delta_{rs}\delta_{jj'}\delta_{mm'}\delta(|\varepsilon|-|\varepsilon'|)\delta(k_3-k_3')\delta_{ee'} , \quad (5.37)$$

where $r,s = +,-$ corresponding respectively to positive (5.2), (5.3) and negative (5.4), (5.5) energy solutions, and

$$N^2(e) = \frac{P(e)}{\omega^4}\left[(1+\gamma_-^2)\int_1^\infty \beta^2(x)dx + (1+\gamma_+^2)\ |\Lambda|^2 \int_1^\infty \alpha^2(x)dx\right]$$

where $P(e=1) = 1$, $P(e=-1) = \gamma_+^2$ . Using expressions (5.6) and (5.7) for $\alpha$ and $\beta$ we calculate

$$<\alpha> = \int_1^\infty \alpha^2(x)dx = \frac{2^{2m-\sqrt{2}|\varepsilon|+3}(j-m)!(\sqrt{2}|\varepsilon|-j-m-1)!((m+1/2)!)^2(j+1/2)}{(\sqrt{2}|\varepsilon|-2j-1)(j+1/2)!(\sqrt{2}|\varepsilon|-j-3/2)!} , \quad (5.38)$$

$$<\beta> = \int_1^\infty \beta^2(x)dx = \frac{1}{4(j+1/2)(\sqrt{2}|\varepsilon|-j-1/2)} <\alpha> \quad (5.39)$$

and we have the result

$$N^2(e) = \frac{2<\alpha>}{\omega^4}\ \frac{|\varepsilon|}{(|\varepsilon|^2 - W_k^2)(|\varepsilon|-eW_k)}\left[1 + \left(\frac{-\mu+\sqrt{\mu^2+k_3^2}}{k_3}\right)^2\right] . \quad (5.40)$$

We denote

$$W_k = \sqrt{\mu^2+k_3^2} - e\sqrt{2}/2 . \quad (5.41)$$

The factor $(2\pi)^2 N^2(e)$ in the right-hand-side of (5.37) can be interpreted as inversely proportional to the local number density of states $(j,m,k_3,e)$, that is, the number density of states in the Fourier space associated to the complete basis of solutions (5.2)–(5.5). It is clear from (5.40) that the local number density of states $(j,m,k_3,e)$ depends strongly on the sign of e. We recall that (5.40) is well defined in the limit $k_3 \to 0$ (cf. (5.1)), and the same is true for the limits [42] $m = -1/2$ as well as $j = m = -1/2$.

Since we have used the local number density $j^{(0)}$ to normalize the wave functions, the normalization depends on the orientation of the field of tetrad frames $e^{\alpha}_{(A)}(x)$, with an arbitrariness due to local Lorentz transformations (4.2). The present orientation of the tetrad frame in which (5.37) and (5.40) were calculated is nevertheless a preferred orientation in the sense that (4.11) is based on the matter flow of the model – actually the zeroth vector of the tetrad frame is defined by the four velocity field of matter $e^{\alpha}_{(0)} = \delta^{\alpha}_{0}$, and (5.37) and (5.40) are invariant under Lorentz transformations which preserve this condition, that is, $L^{0}{}_{A} = \delta^{0}_{A}$. The matter flow of the model singles out (5.37) and (5.40).

## VI. THE GENERALIZED FOURIER SPACE OF FERMION AMPLITUDES

The Fourier space associated to the complete basis (5.2)–(5.5) is constructed as follows. The kernel of the transformation is defined by [43]

$$K(j,m,k_3,\varepsilon;x) = K_{(+)}(j,m,k_3,\varepsilon;x) + K_{(-)}(j,m,k_3,\varepsilon;x) \qquad (6.1)$$

where

$$K_{(+)} = \mathrm{diag}\left(\frac{\beta}{\langle\beta\rangle^{1/2}}, \frac{\alpha}{\langle\alpha\rangle^{1/2}}, \frac{\beta}{\langle\beta\rangle^{1/2}}, \frac{\alpha}{\langle\alpha\rangle^{1/2}}\right)\exp\left[-im\phi+ik_3x^3+i|\varepsilon|t\right] \tag{6.2}$$

and

$$K_{(-)} = \mathrm{diag}\left(\frac{\alpha}{\langle\alpha\rangle^{1/2}}, \frac{\beta}{\langle\beta\rangle^{1/2}}, \frac{\alpha}{\langle\alpha\rangle^{1/2}}, \frac{\beta}{\langle\beta\rangle^{1/2}}\right)\exp\left[im\phi+ik_3x^3-i|\varepsilon|t\right] \quad , \tag{6.3}$$

and $K_+$ and $K_-$ are related by $\gamma^2 K^*_{(\pm)} = K_{(\mp)}\gamma^2$. The kernel (6.1) is a generalization of the exponential kernel of the Fourier transformation (with the substitution of $e^{\pm ikx}$ by the matrix $K_{(\pm)}(x)$). The diagonal form of $K_{(\pm)}$ is the simplest choice, and is derived from an inspection of the system of solutions (5.2)--(5.5): for instance, we construct $K_{(+)}(x)$ with the column vectors appearing in the expansion

$$\left\{\begin{pmatrix}\beta\\0\\0\\0\end{pmatrix} - \gamma_+^2\Lambda\begin{pmatrix}0\\\alpha\\0\\0\end{pmatrix} + \gamma_+\begin{pmatrix}0\\0\\\beta\\0\end{pmatrix} - \gamma_-\Lambda\begin{pmatrix}0\\0\\0\\\alpha\end{pmatrix}\right\}\exp\,(im\phi-ik_3x^3-i|\varepsilon|t)$$

of the positive-energy solution (5.3). The normalizing factors $\langle\alpha\rangle^{1/2}$ and $\langle\beta\rangle^{1/2}$ are used to guarantee the unitary character of $K_{(\pm)}$. The Fourier transform of a fermion field $\psi$ has the expression

$$F[\psi] = \psi_F(j,m,k_3,\varepsilon) = \int \sqrt{-g}\; d^4x\; K(j,m,k_3,\varepsilon;x)\psi(x) \tag{6.4}$$

where the integration is taken over the whole manifold.

For (5.42) we have the unitarity property

$$\int \sqrt{-g}\; d^4x\; K(j',m',k_3',\varepsilon';x)K^\dagger(j,m,k_3,\varepsilon;x) = 2(2\pi)^3\,\mathbb{1}\,\delta_{jj'}\delta_{mm'}\delta(k_3-k_3')\delta(|\varepsilon|-|\varepsilon'|). \tag{6.5}$$

The first term $K_{(+)}$ of the kernel (6.1) can be considered as a projector – with respect to the operation (6.4) – onto positive-energy states, since its action on negative-energy states (5.4), (5.5) results zero; analogously the second term $K_{(-)}$ in (6.1) is a projector onto negative-energy states since its action on positive-energy states (5.2), (5.3) gives zero. Because a projector is not a one-to-one map, the inverse Fourier transform is then defined separately for positive- and negative-energy amplitudes with respective kernels $K_{(+)}$ and $K_{(-)}$. We have [42]

$$\psi(x) = F^{-1}\left[\psi_F(j,m,k_3,\varepsilon;\pm)\right] = \sum_{j=-1/2}^{\infty} \sum_{m=-1/2}^{j} \sum_{e} \int_{\varepsilon>0} \frac{dk_3 d\varepsilon}{(2\pi)^3} \cdot$$

$$\delta(\varepsilon^2 - W_k^2 - 4[j+1/2][\sqrt{2}|\varepsilon|-j-1/2])\; K^{\dagger}_{(\pm)}(j,m,k_3,\varepsilon;x)\psi_F(j,m,k_3,\varepsilon;\pm) \tag{6.6}$$

for positive- and negative-energy states, respectively. The following unitarity properties hold

$$\sum_{j,m,e} \int_{\varepsilon>0} \frac{dk_3 d\varepsilon}{(2\pi)^2}\, \delta(\varepsilon^2-W_k^2-4(j+1/2)(\sqrt{2}|\varepsilon|-j-1/2)) \cdot$$

$$\cdot\; K^{\dagger}_{(+)}(jmk_3\varepsilon;x)K_{(+)}(jmk_3\varepsilon;x') = \frac{\delta^4(x-x')}{\sqrt{-g}}\,\mathbb{1} \tag{6.7a}$$

and

$$\sum_{j,m,e} \int_{\varepsilon>0} \frac{dk_3 d\varepsilon}{(2\pi)^3}\, \delta(\varepsilon^2 - W_k^2 - 4(j+1/2)(\sqrt{2}|\varepsilon|-j-1/2)) .$$

$$\cdot\; K^{\dagger}_{(-)}(jmk_3\varepsilon;x)K_{(-)}(jmk_3\varepsilon;x') = \frac{\delta^4(x-x')}{\sqrt{-g}}\,\mathbb{1} \tag{6.7b}$$

which actually imply $FF^{-1} = F^{-1}F = 1$, as expected.

The Fourier transform of a positive-energy amplitude (5.2), (5.3) is the four spinor

$$\psi_F(jmk_3\ e;+) = \frac{(2\pi)^3 g(e)}{\omega^4} \begin{pmatrix} \langle\beta\rangle^{1/2} \\ \dfrac{i\gamma_+^2}{k_3+(-|\varepsilon|-\sqrt{2}/2+\mu)\gamma_+} \quad \langle\alpha\rangle^{1/2} \\ \gamma_+ \ \langle\beta\rangle^{1/2} \\ \dfrac{i\gamma_+}{k_3+(-|\varepsilon|-\sqrt{2}/2+\mu)\gamma_+} \quad \langle\alpha\rangle^{1/2} \end{pmatrix} \cdot$$

$$\cdot\ \delta_{mm'}\delta_{jj'}\delta(k_3-k_3')\delta(|\varepsilon|-|\varepsilon'|) \qquad (6.8)$$

where $g(e) = \gamma_-$ for $e = 1$ and $g(e) = 1$ for $e = -1$.

The Fourier space described above is actually a <u>momentum space</u> for fermions and it is obviously non-local [44].

Because of its greater simplicity we first consider here the Fourier momentum space for neutrinos (or fermions with $k_3^2 \gg \mu^2$). Expressing a neutrino positive-energy solution (5.16), (5.17) as (we note that $eW_k = L\pi_3$ for $\mu = 0$)

$$\psi_{(+)}(L,k_3) = \sum_{jme} \int \frac{dk_3 d\varepsilon}{(2\pi)^3}\, \delta(\varepsilon^2 - \pi_3^2 - 4(j+1/2)(\sqrt{2}|\varepsilon|-j-1/2)) \cdot$$

$$\cdot\ K^{\dagger}_{(+)}(jmk_3\varepsilon;x)\psi_F(jmk_3\varepsilon L;+)$$

and noting (5.13), Dirac equation $\gamma^A \nabla_A \psi = 0$ <u>for neutrinos</u> results in the transformed Dirac equation

$$-i\pi_A\gamma^A\psi_F = 0 \qquad (6.9)$$

where $\pi_A$ is given by

$$\pi_A = (|\varepsilon|, 0, -2[(j+1/2)(\sqrt{2}|\varepsilon|-j-1/2)]^{1/2}, k_3 - L\frac{\sqrt{2}}{2}) \ . \qquad (6.10)$$

We have

$$\pi_A\pi^A = 0 \qquad (6.11)$$

as expected for a massless particle, where $\pi^A = \eta^{AB}\pi_B$. The form of the component $\pi_3$ (along the direction of the vorticity vector) shows that the "leptonic charge" L behaves like the coupling constant in the coupling of the spinor structure of neutrino to the vorticity field. We comment here that it is exactly the zeroth component $\pi_0$ which appears as a factor in (5.40), and characterizes the behaviour of (5.40) under local Lorentz rotations. We shall therefore normalize all Fourier transformed solutions with the remaining factor (cf. (5.40))

$$N = \frac{4<\alpha>}{\omega^4}\,\frac{1}{(|\varepsilon|^2-(\pi_3)^2)(|\varepsilon|-L\pi_3)} \ . \qquad (6.12)$$

This corresponds to have, dropping $\delta$-factors,

$$\psi_F^\dagger\psi_F = |\varepsilon| \ . \qquad (6.13)$$

For a negative-energy neutrino solution (5.18),(5.19),

$$\psi_{(-)}(L,k_3) = \sum_{jme}\int\frac{dk_3 d\varepsilon}{(2\pi)^3}\,\delta(\varepsilon^2 - \pi_3^2 - 4(j+1/2)(\sqrt{2}|\varepsilon|-j-1/2)) \ .$$

$$\cdot\ K^\dagger_{(-)}\ (jmk_3\varepsilon L;x)\psi_F(jmk_3\varepsilon L;-)$$

we analogously obtain (6.9) where $\pi_A$ is now given by

$$\pi_A = (-|\varepsilon|, 0, 2[(j+1/2)(\sqrt{2}|\varepsilon|-j-1/2)]^{1/2} , k_3 - L\frac{\sqrt{2}}{2}) \qquad (6.14)$$

with $\pi_A\pi^A = 0$.

The same results (6.10) and (6.14) are obtained if we used instead the infinite dimensional representation basis discussed in Section IV, the only difference being that the quantum numbers j and m have their range $-1/2 \leq j < \infty$, $-1/2 \leq m < \infty$ completely independent. We remark that $\pi_3$ has the same sign in (6.10) and (6.14) due to our definition of (6.3); in fact if in (6.3) we change $k_3 \to -k_3$ and $L \to -L$ (cf. (5.21),(5.22)) we have in (6.14) that $\pi_3 \to -\pi_3$ without altering other components. It follows that the corresponding $\pi_A$ for negative-energy solutions has the opposite sign of $\pi_A$ for positive-energy solutions, a behaviour characteristic of "plane-wave-type" positive- and negative-energy amplitudes related through property (5.22). This fact is important when we consider symmetry transformations between particle and antiparticle amplitudes.

We now calculate the component (along the vorticity field $\vec{\Omega}$ ) $j_F^{(3)}$ of the local four-current (5.35) of neutrinos. For the positive-energy amplitude (5.16), (5.17) normalized according to (6.12), (6.13), we obtain

$$j_F^{(3)} = (2\pi)^6 (-\pi_3)\delta_{mm'}\delta_{jj'}\delta(k_3-k_3')\delta(|\varepsilon|-|\varepsilon'|) \quad . \qquad (6.15)$$

For the case of a massive fermion we use (6.6) to obtain the transformed Dirac's equation

$$(\gamma^A\pi_A - \mu)\psi_F = 0 \qquad (6.16)$$

where $\pi_A$ is now given by

$$\pi_A = (|\varepsilon|, 0, -2[(j+1/2)(\sqrt{2}|\varepsilon|-j-1/2)]^{1/2}, k_3 - \frac{\sqrt{2}}{2}\gamma^5) \qquad (6.17)$$

for positive-energy solutions, and

$$\pi_A = (-|\varepsilon|, 0, 2[(j+1/2)(\sqrt{2}|\varepsilon|-j-1/2)]^{1/2}, k_3 - \frac{\sqrt{2}}{2}\gamma^5) \qquad (6.18)$$

for negative-energy solutions. We must comment here on the sign of $\pi_3$ in (6.17) and (6.18). The same sign of $k_3$ is due to our definition (6.3); if in (6.3) we change $k_3 \to -k_3$ it results that in (6.18) $\pi_3$ (negative energy) $= -k_3 - \frac{\sqrt{2}}{2}\gamma^5$. Thus for negative-energy solutions the components of $\pi_A$ have the opposite sign of the corresponding positive-energy solutions, except for the term $-\frac{\sqrt{2}}{2}\gamma^5$ appearing in $\pi_3$. This is due to the fact that the symmetry transformation $\psi \to i\gamma^2\psi^*$ change the sign of polar momentum but leave invariant the axial-momentum. In case of neutrinos (cf. (5.29), (5.22) and comments below (6.14)) $L \to -L$ because although $\gamma^5$ do not change sign, its eigenvalues change under $\psi \to i\gamma^2\psi^*$.

We note from (6.17) and (6.18) that not all components of $\pi_A$ are scalars, namely the component of $\pi_A$ along the vorticity field is the matrix

$$\pi_3 = k_3(1 - \frac{\sqrt{2}}{2k_3}\gamma^5) \quad , \qquad (6.19)$$

a result which is obviously due to our definition (6.2) and (6.3) for the kernels of the transformation. However a detailed examination shows that we cannot define a kernel $K_{(\pm)}$ which commutes [45] with $\gamma^0$ and for which the corresponding $\pi_A$ are numbers. For massive fermions we then have to live with momenta of the type (6.19).

In what follows we consider the momentum space associated to the choice (6.1)-(6.3) for the kernel of the transformation. The first reason for this choice is simplicity, but the main reason is that the constant of motion S (cf. (4.20)) commutes with K. Then $\psi_F$ is also eigenstate of S with eigenvalue $-e\sqrt{\mu^2+k_3^2}$ and for $k_3^2 >> \mu^2$ $\psi_F$ becomes eigenstate of $\gamma^5$ and of the helicity operator $\vec{\Sigma}.\vec{\pi}$. Using this fact we can transform (6.19) to:

(i) if $k_3 \neq 0$

$$\pi_3 = k_3 \ (1 - \frac{\sqrt{2}}{2} \frac{e\sqrt{\mu^2+k_3^2}}{k_3^2}) - \frac{\sqrt{2}}{2} \frac{\mu}{k_3} \gamma^3\gamma^5$$

associated to the basis $\{\psi_F(j,m,k_3,\varepsilon,e)\}$. Using (6.20) equation (6.16) can be rewritten

$$\gamma^A \ \tilde{\pi}_A\psi_F - \mu(1 - \frac{\sqrt{2}}{2k_3} \gamma^5) \ \psi_F = 0$$

where

$$\tilde{\pi}_A = (\pm\varepsilon,0,\mp \ 2[(j+1/2)(\sqrt{2}|\varepsilon|-j-1/2)]^{1/2}, \ k_3(1-\frac{\sqrt{2}}{2} \frac{e\sqrt{\mu^2+k_3^2}}{k_3^2}) \ ) \ .$$

With respect to the momenta $\tilde{\pi}_A$ and for $\mu \neq 0$ we have the conservation of the four currents [46]

$$(\tilde{\pi}'_A-\tilde{\pi}_A) \ j^A_V = \frac{\sqrt{2}}{2} (\frac{1}{k_3} + \frac{1}{k'_3}) \ \overline{\psi}'_F\gamma^5\psi_F \qquad (6.21a)$$

or equivalently

$$(\tilde{\pi}'_A - \tilde{\pi}_A) \left\{ j_v^A + \frac{\sqrt{2}}{4} \left(\frac{1}{k_3} + \frac{1}{k'_3}\right) j_{ax}^A \right\} = \frac{\mu}{4} \left(\frac{1}{k_3^{\,2}} - \frac{1}{k'^{\,2}_3}\right) \overline{\psi}'_F \psi_F \qquad (6.21b)$$

where $j_v^A = \overline{\psi}'_F \gamma^A \psi_F$ and $j_{ax}^A = \overline{\psi}'_F \gamma^5 \gamma^A \psi_F$. For $\mu = 0$ the two currents are conserved separately.

(ii) if $k_3 = 0$

$$\pi_3 = \frac{\sqrt{2}}{2} e \gamma^3 \qquad (6.22)$$

associated to the basis $\{\psi_F(j,m,k_3,\varepsilon,e)\}$. Using (6.22) equation (6.16) can be rewritten

$$\gamma^A \tilde{\pi}_A \psi_F - \left(\mu - \frac{\sqrt{2}}{2} e\right) \psi_F = 0 \qquad (6.23)$$

where $\tilde{\pi}_A = (\pm|\varepsilon|, 0, \mp 2[(j+1/2)(\sqrt{2}|\varepsilon| - j - 1/2)]^{1/2}, 0)$. Expression (6.23) is valid only for $\mu \neq 0$. In this case the mass invariant

$$\tilde{\pi}_A \tilde{\pi}^A = \left(\mu - \frac{\sqrt{2}}{2} e\right)^2$$

and thus the local Lorentz observers see massive fermions with distinct masses for distinct values of e. In other words, massive fermions have an intrinsic degree of freedom associated to the quantum number e, which is raised by the gravitational coupling of the fermion to a vorticity field, and produces in this case a split of mass. We could use this mass split effect in a gedanken experiment to measure the direction and the intensity of the rotation of the universe: particles in motion in a plane orthogonal to the direction of the vorticity field would present a split of mass proportional to the intensity of the vorticity field. By a change of the plane of motion this split effect would be obliterated because in this case we would need a superposition of solutions (with distinct values of $k_3$) to describe the motion.

For the positive-energy Fourier amplitude (6.8) the component (along the local vorticity field $\vec{\Omega}$) $j_F^{(3)}$ of the local four-current $j_F^{(A)} = \bar{\psi}_F \gamma^A \psi_F$ is given by

$$j_F^{(3)} = 2(2\pi)^6 \, W_k \left(\frac{\mu - \sqrt{\mu^2 + k_3^2}}{k_3}\right) . \frac{1}{1 + \left(\frac{-\mu + \sqrt{\mu^2 + k_3^2}}{k_3}\right)^2} \; .$$

$$. \, \delta_{jj'} \delta_{mm'} \delta(k_3 - k_3') \delta(|\varepsilon| - |\varepsilon'|) \qquad (6.24)$$

We have here normalized (6.8) as in (6.13) for neutrinos (cf. also (5.40)). We shall also use this expression to discuss the microscopic asymmetry of fermion currents in the presence of a local vorticity field.

## VII. SYMMETRY TRANSFORMATIONS FOR FERMION AMPLITUDES AND MICROSCOPIC ASYMMETRIES OF FERMIONS

In order to examine fermion-antifermion symmetry of some processes, we must try to define amplitudes for particle and antiparticle states. To this end we obtain transformations which can be interpreted as leading from particle to antiparticle amplitudes and which actually are symmetry transformations for the present fermions in the sense that they preserve the Hilbert space of fermion solutions generated by the basis (5.2)-(5.5). These transformations can be reasonably understood as corresponding locally to known symmetries of particle physics.

The use of tetrads is practically unavoidable to describe the interaction of fermions with gravitation [27,47] and, in this context, the theory has two groups involved: the local Lorentz rotations (4.2) of the tetrads and the isometry group of the manifold. Spinors are defined with respect to the local Lorentz structure, in the sense that they provide a basis space for a spinorial representation of the local Lorentz group. On the other hand these spinors provide a basis space for a scalar representation of the isometry group of the manifold.

In the definition of fermion and antifermion amplitudes both groups are involved; for instance the energy eigenmodes are related to the Killing vector $\partial/\partial t$ of the isometry group while the charge-conjugation operation must take into account the local spinor structure. In this way, we obtain here consistent fermion- antifermion symmetry transformations of the Hilbert space of fermions amplitudes generated by (5.2)—(5.5) and which then necessarily takes into account the two group structures present.

Let us start by considering the neutrino (or massive fermions with $k_3{}^2 \gg \mu^2$) Hilbert space of solutions generated by (5.16)—(5.19). Starting from a negative-energy solution $\psi_{(-)}(L,k_3)$ (cf. (5.18),(5.19)) we define the transformation

$$\psi_{(-)}(L,k_3) \rightarrow C^{-1}\,\bar{\psi}^T_{(-)}(k_3,L) \qquad (7.1)$$

where C is a matrix of the algebra of Dirac matrices, which satisfies

$$C\gamma^A C^{-1} = -\gamma^{A^T} . \qquad (7.2)$$

In the present representation [29], (7.2) is satisfied by

$$C \sim \gamma^2\gamma^0 \qquad (7.3)$$

where $\sim$ denotes equality up to a constant phase factor. An explicit calculation of (7.1) gives

$$\gamma^2\gamma^0\ \bar{\psi}^T_{(-)}(k_3,L) = \psi_{(+)}(-k_3,-L) \quad . \qquad (7.4)$$

Transformation (7.1) has the following properties: (i) it is a symmetry transformation of the Hilbert space of neutrino amplitudes, since it takes a negative-energy solution (5.18) to a positive-energy solution (5.16),and vice-versa; (ii) the S matrix (7.2) and (7.3) has the character of a charge-conjugation operator on the amplitudes (5.16)-(5.19) (in case of charged particles it relates solutions with distinct signs of the charge); (iii) neutrino amplitudes related through (6.1) have opposite helicity L and momentum $k_3$ – the local momentum $\vec{\pi}$ (cf.(6.10) and (6.14)) change sign under (7.1). We note that (5.4) is precisely the symmetry (5.22) between positive- and negative--energy neutrino solutions. From the above properties we interpret (7.1) as a charge-conjugation-parity (CP) transformation for neutrino amplitudes, and hence we have the independent positive--energy wave-functions interpreted as

$$\begin{aligned} &\psi_{(+)}(k_3,L) = \text{neutrino amplitude} \\ &\psi_{(+)}(-k_3,-L) = \text{corresponding antineutrino amplitude} \quad . \end{aligned} \qquad (7.5)$$

The positive-energy amplitudes (7.5) are said CP related in the sense that the corresponding negative-energy amplitudes $\psi_{(-)}(k_3,L)[\psi_{(-)}(-k_3,-L)]$ of one is transformed into the other $\psi_{(+)}(-k_3,L)[\psi_{(+)}(k_3,L)]$ under (7.1). From the local CP invariance of neutrino physics (only negative-helicity neutrinos exist) we take L = -1 for neutrinos, which implies L = +1 for antineutrinos (cf. (7.5)). Neutrino and antineutrino amplitudes have their respective momentum $\vec{\pi}$ with opposite sign.

We make identical analysis for the amplitudes of massive fermions. In what follows we assume that all fermions involved in our considerations are produced by weak interactions. From the local law of physics we are then led to take that – for high momenta ($k_3^2 \gg \mu^2$) – electrons are left-polarized at production by weak interactions, and from CP invariance positrons are right-polarized. In other words high-momentum fermions (antifermions) will have $es(k_3) = -1$ ($es(k_3) = +1$) at production by weak interaction and, since $es(k_3)$ is a constant of motion, they will be characterized by

$$\begin{aligned} &es(k_3) = -1 \text{ or } L = -1 \text{ for high-momentum fermion, neutrino} \\ &es(k_3) = +1 \text{ or } L = +1 \text{ for high-momentum antifermion, antineutrino} \end{aligned} \qquad (7.6)$$

in the absence of interactions other than Gödel's gravitation.

We can now discuss the microscopic asymmetry of <u>high--momentum</u> fermion/neutrino currents along the direction determined by the vorticity vector field. From the expression (6.24) (which reduces to (6.15) for $\mu = 0$) for the component of the local Fourier current $\vec{j}_F$ along $\vec{\Omega}$ we take the relevant factor

$$j_F^{(3)} \cong \left(\frac{\mu-\sqrt{\mu^2+k_3^2}}{k_3}\right)\left(\sqrt{\mu^2+k_3^2} - e\sqrt{2}/2\right) \qquad (7.7)$$

and we distinguish the two cases:

(1) $\sqrt{\mu^2+k_3{}^2} > \sqrt{2}/2$: $j_F^{(3)}$ has the opposite sign of $k_3$; for electrons ($es(k_3) = -1$) we have that $\vec{j}_F$ is large along the direction antiparallel to $\vec{\Omega}$ than along the parallel direction; for positrons ($es(k_3) = +1$), $\vec{j}_F$ is large along the direction parallel to $\vec{\Omega}$ .

(2) $\sqrt{\mu^2+k_3{}^2} > \sqrt{2}/2$ : for electrons the component of $\vec{j}_F$ along $\vec{\Omega}$ is always negative ($\vec{j}_F$ has only antiparallel component along $\vec{\Omega}$); for positrons the component of $\vec{j}_F$ along $\vec{\Omega}$ is always positive.

Analogous behaviour holds for neutrino/antineutrino as (6.24) goes continuously to (6.15) in the limit $\mu = 0$. The diagram of Fig. 1 is illustrative. It is easy to see that the restrictions (7.6) violate P while they are CP invariant. As far as weak interactions are concerned, the selection rules (7.6) are legitimate for the high-momentum fermions produced.

As for the local current $j^{(A)}(x) = \overline{\psi}(x)\gamma^A\psi(x)$, we calculate the component $j^{(3)}(x)$ at the origin $x = 1$ (the results are typical due to the homogeneity of the space-time). In the normalization $\langle\psi|\psi\rangle = |\varepsilon|$ (cf. also (5.40)) we obtain for the positive energy solution (5.2) and (5.3)

$$j^{(3)}(x) = \frac{e}{k_3}\left(-\mu+\sqrt{\mu^2+k_3{}^2}\right)\frac{\omega^4}{\langle\alpha\rangle}\,\frac{(|\varepsilon|^2-W_k^2)(|\varepsilon|-eW_k)}{\left[1+\left(\frac{-\mu+\sqrt{\mu^2+k_3{}^2}}{k_3}\right)^2\right]}$$

$$\cdot\left\{\beta^2-\frac{\alpha^2}{(|\varepsilon|-eW_k)^2}\right\} \tag{7.8}$$

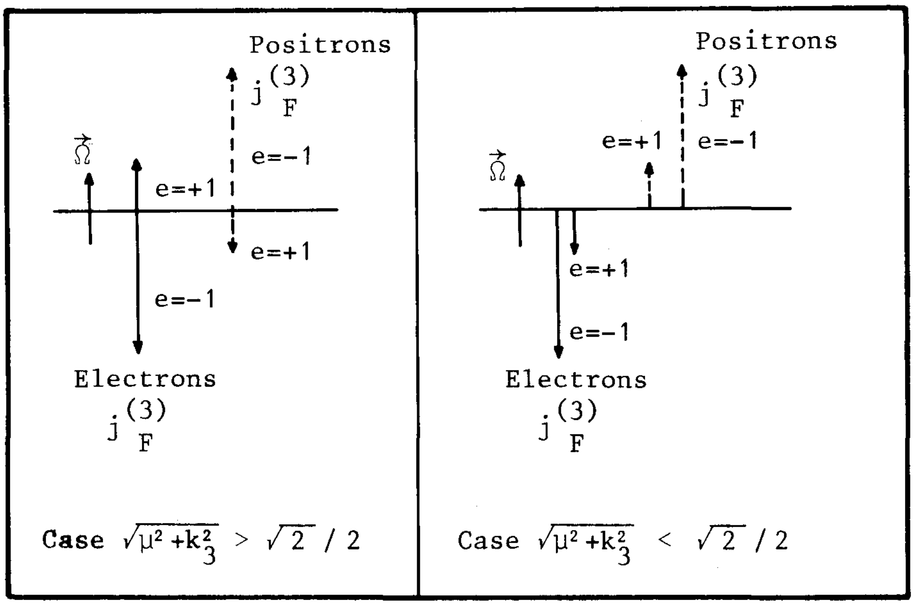


Fig. 1: Diagram of Current Asymmetry.

where $<\alpha>$ is given by (5.38), and $\alpha$ and $\beta$ have their expression in (5.6), (5.7). We note that $j^{(A)}$ depends on the coordinate $x = \cosh 2r$ only. At the origin $x = 1$, we can see that for a given $j \geq 1/2$ only the modes $m = \pm 1/2$ contribute to (6.8), namely for a given $j \geq 1/2$

$$(j^{(3)}(x))_{m=-1/2} = \frac{e}{k_3} R^2 \; (|\varepsilon| - eW_k) \tag{7.9}$$

(cf. Ref. [30]) and

$$(j^{(3)}(x))_{m=+1/2} = \frac{e}{k_3} R^2 \; (-|\varepsilon| - eW_k) \tag{7.10}$$

where $R^2 = 1/2\;(-\mu+\sqrt{\mu^2+k_3{}^2})\;\dfrac{\omega^4}{1+\left(\dfrac{-\mu+\sqrt{\mu^2+k_3{}^2}}{k_3}\right)^2}\;(\sqrt{2}|\varepsilon|-2j-1)$. The total local current along $\vec{\Omega}$ (at the origin x = 1) for a given mode $j \geq 1/2$,

$$j^{(3)}(x=1) = \sum_{m=-1/2,1/2} (j^{(3)}(x=1))_m$$

is then calculated to be

$$j^{(3)}(x=1) = \left(\frac{\mu-\sqrt{\mu^2+k_3^2}}{k_3}\right) \frac{\omega^4}{1+\left(\frac{-\mu+\sqrt{\mu^2+k_3^2}}{k_3}\right)^2} (\sqrt{2}|\varepsilon|-2j-1)W_k \qquad (7.11)$$

The same analysis and diagram for the asymmetry of the Fourier current (6.7) applies to (7.11).

A special case is the mode $j = m = -1/2$ for which

$$j^{(3)}(x=1)\Big|_{j=m=-1/2} = \frac{e}{k_3} R^2 \sqrt{2}|W_k|(|W_k|-eW_k) \quad . \qquad (7.12)$$

The current asymmetry in this mode is analogous to the two previous cases, for $\sqrt{\mu^2+k_3^2} < \sqrt{2}/2$. For $\sqrt{\mu^2+k_3^2} > \sqrt{2}/2$ we have non-null current components along $\vec{\Omega}$ only for $e = -1$, with electron current antiparallel to $\vec{\Omega}$ and positron current parallel to $\vec{\Omega}$ [48]. This microscopic asymmetry of current – which is a parity violating effect – could be used as a local test for the existence of a rotation of the universe. For instance the decay $\pi^{\pm} \to \mu^{\pm} + \nu(\bar{\nu})$ in the presence of a vorticity field can give rise to parity violating effects – e.g., asymmetry in the muonic current, asymmetry in neutrino current – which could in principle be detected and would be an indication of the presence of rotation.

Finally we draw some interesting conclusions concerning the number density of fermion and antifermion states, CP violation and lepton asymmetry, for the present problem. To this end we note that the number density of states – which we denote by n(e) and is proportional to

$$n(e) = \frac{dk_3}{d|\varepsilon|}\,\frac{1}{N^2(e)} = \frac{|\varepsilon|-\sqrt{2}(2j+1)}{W_k}\,\frac{|\varepsilon|}{N^2(e)} \qquad (7.13)$$

where $N^2(e)$ is given by (5.40) – depends strongly on the sign of e (through $|\varepsilon|$ and $W_k$), <u>for</u> $\sqrt{\mu^2+k_3^2}$ <u>of the order of</u> $\sqrt{2}/2$. Consequently, for a given value of $(j,m,k_3)$ such that $\sqrt{\mu^2+k_3^2}$ is of the order of $\sqrt{2}/2$, we could have a number density of states different for e = -1 and e = +1. This fact can be significative in the presence of CP-violating interactions, as we shall discuss now for the case of creation of fermion-antifermion pairs in the presence of a CP-violating perturbation, when a particle- antiparticle number asymmetry may possibly occur [49].

Since CP transformation does not change the normalization of a wave function, we can split the Hilbert space basis of fermion amplitudes satisfying (7.6) into two distinct sets: fermions and antifermions of type 1 (amplitudes with e = +1) and fermions and antifermions of type 2 (amplitudes with e = -1). Type 1 amplitudes are CP related, and type 2 amplitudes are also CP related. In the diagrams of currents in Fig. 1 the large components of fermion and antifermion currents corresponds to amplitudes of type 2 and are CP related. The small components correspond to CP related amplitudes of type 1, which clearly shows that the asymmetric emission of fermions is CP invariant.

In case of creation of fermion pairs in the present universe, we can distinguish two possibilities:

(i) fermion-antifermion pairs whose amplitudes are CP related, namely $(f_1\bar{f}_1)$ or $(f_2\bar{f}_2)$; for each case the corresponding current diagram is CP invariant, and the number density of fermion states is equal to the number density of antifermion states.

(ii) fermion-antifermion pairs whose amplitudes are not CP related, namely $(f_1\bar{f}_2)$ or $(f_2\bar{f}_1)$. In both cases we note that e has opposite sign for fermion and antifermion amplitudes, which corresponds to a number density of states different for fermions and antifermions . For $(f_1\bar{f}_2)$ or $(f_2\bar{f}_1)$ we have, respectively, the number densities of states (n(e=+1), n(e=-1)) or (n(e=-1), n(e=+1)). Nevertheless if the creation of pairs is due to a CP-invariant perturbation both cases will be equally probable since $(f_1\bar{f}_2) \overset{CP}{\longleftrightarrow} (f_2\bar{f}_1)$ and no net asymmetry in fermion-antifermion number is possible. A net asymmetry (due to different density of states avaiable for fermions and antifermions) will appear if the pair production perturbation violates CP. Indeed if pairs $(f_1\bar{f}_2)$ are produced, the pairs $(f_2\bar{f}_1)$ are then forbidden and a net asymmetry between fermion and antifermion will appear proportional to the ratio

$$\delta_{jmk_3} = \frac{n(e=+1) - n(e=-1)}{n(e=+1) + n(e=-1)} \quad . \qquad (7.14)$$

The ratio (7.14) is significantly non-zero <u>only for</u> $\sqrt{\mu^2+k_3{}^2}$ <u>of the order of</u> $\sqrt{2}/2$.

It could be argued – in the case of creation of charged fermion pairs (for instance electron-positron pairs) – that the non-null ratio (7.14) would violate charge conservation which is a global (space-time) symmetry of the theory (in fact the theory has a global symmetry associated to charge conservation, in contrast to the local CP symmetry referred to above). The answer is that the CP-violating perturbation which creates pairs with $\delta_{jmk_3} \neq 0$ is localized [49] and the local charge conservation is actually violated by the perturbation.

We also remark that the above discussion is independent of the space-time point considered, since we have dealt with scalar quantities only.

## VIII. CONCLUSIONS

One of the main conclusions of our investigation is that the presence of a vorticity field of matter produces, via gravitational coupling, microscopic asymmetries in the physics of spin-1/2 fermions. We have shown our results in the context of the Einstein theory of gravitation, and for technical simplicity we have considered Gödel's universe as the gravitational background because it is the simplest known solution of Einstein field equations which is stationary and in which the matter content of the model has a non-null vorticity. Complete bases of scalar field solutions and Dirac field solutions are obtained, in invariant modes of the total angular momentum which is defined in close connection with the Killing vectors of the space-time. The results follow:

1) we solved the scalar field equation by separation into invariant

modes defined by the global Killing vectors of the space-time, and obtained a complete set of solutions of scalar field amplitudes in the <u>hyperbolic</u> <u>harmonic</u> <u>modes</u> $(\ell, m, \varepsilon, k_3)$. These modes are eigenfunctions of the square of the total angular momentum of the scalar field system – in fact for these modes the scalar field equation is reduced to the eigenvalue equation for the total angular momentum operator of the system. The angular momentum algebra is naturally defined by the Killing vectors of the space-time, up to a complexification of one of the Killing vectors. The field solutions are assumed to be regular over the whole Gödel background due to their test field character, that is, they are assumed to be finite perturbations at any space-time point. Two distinct complete sets of solutions in these hyperbolic harmonic modes are obtained, which constitute bases of representation for the algebra of angular momentum: one infinite-dimensional and the other finite dimensional with dimension $(\ell+1)$, where $\ell$ is a positive integer or zero which caracterizes the latter representation basis. For both cases, the angular-momentum vector space is polarized along the direction determined by the vorticity field $\vec{\Omega}$. The spectrum of energy eigenvalues for both cases are calculated to be $|\varepsilon| = \sqrt{2}\,(2\ell+1) + \sqrt{(2\ell+1)^2 + k_3^2 + \mu^2 a^2 + 1}$ .

2) The local dynamics of fermions is obtained from the Dirac equation in Gödel's universe. The Hamiltonian which determines the local dynamics is defined with respect to the global timelike Killing vector $\partial/\partial t$. Contrary to the flat space case, a stationary state of the system cannot have helicity as a good quantum number. There is however a new constant of motion S of the system, whose

existence is crucial for the separation of the equations. For low values of momenta S can be interpreted as the projection of the spin of the system along the direction determined locally by the vorticity vector. For the limit of large momenta S is proportional to $\gamma^5$ and stationary states which are eigenfunctions of S are also eigenstates of the helicity of the system (defined with respect to the local Lorentz frames of the tetrads). S allows to define a conserved projection operator into states which are left- or right-polarized in the large momentum limit. We select a basis of simultaneous eigenstates of S and of the Hamiltonian, in which the separation of Dirac equation is made. The corresponding second-order equations are shown to be the eigenvalue equation for the square of the total angular momentum operator. Two distinct complete stes of <u>normalizable</u> solutions are obtained, which are spinorial generalization of the hyperbolic harmonic modes of the scalar field case. These distinct sets of solutions constitute two representation bases for the algebra of the total angular momentum of the system: one infinite dimensional and the other finite dimensional. The finite dimensional representation basis is characterized by a half-integer $j \geq -1/2$, and for a given j the dimension of the representation is given by $(j + 3/2)$. The angular-momentum vector space appears to be polarized along the direction determined by the local vorticity field $\vec{\Omega}$ . Without loss of generality we analyse only the finite dimensional case. From an examination of the low momentum limit we have that the complete basis of solutions of Dirac equation must be normalized differently for different values of e (quantum number associated to S), in order that the solutions are bounded for all values of the momentum $k_3$. In the limit $k_3 \to 0$, the change in the sign of e corresponds to the mass reversal symmetry of substitution for Dirac equation. Also in this limit $k_3 = 0$ we have not only zero-energy modes, but also states of massless fermions which – due to the gravitational coupling to matter vorticity – are converted to bosons polarized along the direction of the vorticity $\vec{\Omega}$ . In the limit of large momenta ($k_3^2 >> \mu^2$) the solutions assume the form of neutrino ($\mu = 0$) solutions, in which we have the identification $L = e\ \mathrm{sign}(k_3)$ where L is the helicity (or chirality) of the neutrino wave function. We also show that the set of solutions obtained corresponds to a hybrid representation – which mixes the advantages of the usual Foldy-Wouthuysen representation of Dirac solutions (with a good non-relativistic limit) and of the Cini-Toushek representation which is most convenient for large momenta (or massless) particles – with the corresponding correct limits in the non-relativistic case and in the case of large momenta ($k_3^2 >> \mu^2$) or massless ($\mu = 0$) particles.

3) We construct the Fourier space associated to the complete bases and the complete unitarity relations for the kernels of the transformation are obtained. This Fourier space is a <u>momentum space</u> for fermions. In the case of massive fermions, the component of the momentum along $\vec{\Omega}$ is not a number but a matrix (indeed this occurs for any consistent generalized Fourier transformation we can define); for $k_3 = 0$ it appears that massive fermions have an intrinsic degree of freedom associated to the <u>quantum number e</u> which is raised by the gravitational coupling of the fermion to the vorticity field and produces a split of mass.

4) From the symmetry properties of the Hilbert space of fermions

solutions (massive and neutrinos) and its corresponding Fourier space, we are able to define fermion and antifermion amplitudes. We assume that all fermions involved in our considerations are produced by weak interactions. From the local laws of physics we impose that – for high momenta ($k_3^2 >> \mu^2$) massive fermions/ /antifermions, or neutrinos/antineutrinos ($\mu = 0$) – fermions are left-polarized at production by weak interactions, and from CP invariance antifermions are right-polarized. Since $es(k_3)$ – which characterizes the polarization of high momenta massive, or zero mass fermions/antifermions – is a constant of motion, it then follows the selection rule for high momenta or zero mass fermions/antifermions: $es(k_3) = -1$ for fermions and $es(k_3) = +1$ for antifermions, as long as the produced particles have no interaction other than Gödel's gravitation.

5) In our following statements all massive fermions are considered in the high momentum limit ($k_3^2 >> \mu^2$). The Fourier current associated to the fermion amplitudes (electrons, neutrinos) as well as the local current calculated at the origin x = 1 (for a given $j \geq -1/2$, summed over all contributions $-1/2 \leq m \leq j$) are asymmetric along the direction determined by the vorticity field: the component of the fermion current along the direction antiparallel to the vorticity field is larger than the parallel component in some cases the fermion currents are purely antiparallel to $\vec{\Omega}$. Also the Fourier component of antifermion amplitudes (positrons, antineutrinos) as well as the local antifermion current calculated at the origin x = 1 (summed over all contributions $-1/2 \leq m \leq j$) are asymmetric: the component along the direction parallel to the vorticity field is larger than the component along the direction

antiparallel (in some cases the antiparallel component is null). Therefore, at the microscopic level, fermions (electrons, neutrinos) are preferentially emitted antiparallel to the local vorticity field; as well antifermions (positrons, antineutrinos) are preferentially emitted parallel to the local vorticity field. This result is CP invariant.

6) The gravitational coupling of fermions to the matter vorticity field of the cosmological background gives rise to parity violating effects – for example, a split of mass for particles in motion in a plane orthogonal to the vorticity field (cf. Sec. VI), and asymmetries of microscopic currents of fermions (cf. Sec. VII). These parity violating effects could in principle be used in devising experiments to detect the presence of a cosmological rotation of the universe, its direction and intensity, as we have discussed in Secs. VI and VII.

7) In case of production of pairs under CP violation a net number asymmetry appears between fermions and antifermions, which is significantly non-zero for $\sqrt{\mu^2+k_3^2}$ of the order of the vorticity value $\sqrt{2}/2$.

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[3]. I. Damião Soares, Phys. Rev. D23 (1981), 272.

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[5]. K. Gödel, Rev. Mod. Phys. 21 (1949), 447.

[6]. I. Ozsvath and E.L. Schücking, Ann. Phys. 55 (1969), 166.

[7]. This is the multiplication law of hyperbolic quaternions, cf. Ref. [5] and I. Ozsvath, J. Math. Phys. 11 (1970), 2871.

[8]. A left-invariant vector field X on a group G is defined by $vX(a) = X(va)$ for all v, a e G.

[9]. The commutator of two vector fields e,f is defined by $[e,f] = \pounds_e f$, where £ denotes the Lie derivative.

[10]. Correspondingly, by imposing the orthogonality conditions $g(X_a, X_b) = \frac{4}{\omega^2}$ diag(2,-1,-1,-1).

[11]. C.W. Misner, K.S. Thorne, and J.A. Wheeler, Gravitation (Freeman, New York, 1970).

[12]. J.L. Anderson, Principles of Relativity Physics (Academic Press, New York, 1967).

[13]. C. Chevalley, Theory of Lie Groups I (Princeton University Press, Princeton, 1946).

[14]. G. Gibbons, Comm. Math. Phys. 44 (1975), 245; M. Henneaux, GRG 9 (1978), 1031. We must recall at this point that in Section IV we introduce spinors from the point of view of the tetrad formalism, with the tetrad field $e_\alpha^{(A)}(x)$ satisfying obviously $\pounds_{K.V.} e_\alpha^{(A)}(x) = 0$.

[15]. A. Erdély et al., Higher Transcendental Functions (McGraw Hill, New York, 1960), Vol. 1.

[16]. M.Abramowitz and I.A. Stegun (Eds.), Handbook of Mathematical Functions (Dover, New York, 1965).

[17]. The value of $\frac{ab}{2c}$ in (3.20) is explicitly given by

$$\frac{ab}{2c} = \frac{1}{2(m+1)} \{ (m - \frac{\sqrt{2}}{2} \varepsilon + \frac{1}{2})^2 - \frac{n^2}{4} \} \quad .$$

[18]. The choice (3.23) is made in order that $L_\pm$, $L_1$ and $L_2$ be anti-Hermitian, while maintaining $L_3$ Hermitian, with respect to the scalar product to be defined, as we shall see. Weaker conditions as $\lim_{x\to\infty} \phi^*\phi = 0$ would give nothing new.

[19]. We have here discarded some normalizable solutions with m non-integer because they cannot constitute a representation basis for the algebra of angular momentum.

[20]. This analysis follows directly from the asymptotic formula (cf. refs. [15] and [16])

$$\lim_{z\to\infty} F(a,b,c;z) = (-1)^a \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} z^{-a} + (-1)^b \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} z^{-b}$$

[21]. Note the property $F(a,b,c;z) = F(b,a,c;z)$.

[22]. Then all positive-energy solutions are orthogonal to any negative energy solution by $\phi$-integration only, because $m \geq 0$ always.

[23]. In the above discussion we have discarded normalizable solutions which could not constitute a basis of representation for the algebra of angular-momentum (for instance $\ell$ = positive half-integer) because successive application of $L_-$ would lead to non-normalizable functions with $m < 0$. We mention that some of these solutions have interesting features as zero energy and eigenvalue of $L_3$ equal to a half-integer.

[24]. N. Ia. Vilenkin, Fonctions Spéciales et Théorie de la Representacion des Groupes, Dunod (Paris, 1969).
N. Naimark, A. Stern, Théorie des Representations des Groupe Editions MIR (Moscow, 1979).

[25]. We have used the following formula for the hypergeometric function appearing in type I solutions,

$$F(m-\ell,m+\ell-\sqrt{2}\,\varepsilon+1,m+1;z) = \frac{\frac{d^{\ell-m}}{dz^{\ell-m}}\{z^{\ell}(1-z)^{\ell-\sqrt{2}\,\varepsilon}\}}{(\frac{\ell!}{m!})\, z^{m}\,(1-z)^{m-\sqrt{2}\,\varepsilon}}$$

(cf. ref. [15]).

[26]. In the right-hand side of (3.34) the energy delta $\delta(|\varepsilon|-|\varepsilon'|)$ is redundant (cf. expression (3.30)). The orthogonality corresponding to $\delta_{\ell\ell'}$ for the same $\varepsilon$ and m can be shown as follows. We start from the solutions m = 0, which we express a

$\phi_{\varepsilon\ell}(x) = (x+1)^{-\frac{\sqrt{2}}{2}|\varepsilon|}\, U_{\ell,\varepsilon}(x)$ where $U_{\ell,\varepsilon}(x) =$

$= \frac{(x+1)^{\sqrt{2}|\varepsilon|}}{2^{\ell}\ell!}\frac{d^{\ell}}{dx^{\ell}}\{(x-1)^{\ell}(x+1)^{-\sqrt{2}|\varepsilon|+\ell}\}$. We can show that

$$\frac{d}{dx}\left[(1-x)(1+x)^{-\sqrt{2}|\varepsilon|+1}\{U_{\ell'\varepsilon}\frac{d}{dx}U_{\ell\varepsilon} - U_{\ell\varepsilon}\frac{d}{dx}U_{\ell'\varepsilon}\}\right] =$$

$= (\ell'-\ell)(\ell+\ell'-\sqrt{2}|\varepsilon|+1).(1+x)^{-\sqrt{2}|\varepsilon|}\, U_{\ell\varepsilon}U_{\ell'\varepsilon}$ . The integral in x, taken between 1 and $\infty$, of the LHS of the above expression can be shown to vanish. We remark that for $x \to \infty$ the expression inside brackets in the LHS behaves as $x^{\ell+\ell'-\sqrt{2}|\varepsilon|+1}$ and $\ell+\ell'-\sqrt{2}|\varepsilon|+1 < 0$ always. We then have for the RHS

$$(\ell-\ell')\int_1^{\infty}(1+x)^{-\sqrt{2}|\varepsilon|}U_{\ell\varepsilon}U_{\ell'\varepsilon}(x)\,dx = 0$$

which implies either $\ell = \ell'$ or $\int_1^{\infty}(1-x)^{-\sqrt{2}|\varepsilon|}U_{\ell\varepsilon}U_{\ell'\varepsilon}dx = 0$ for $\ell \neq \ell'$. The orthogonality of $\phi_{\varepsilon\ell m}$ and $\phi_{\varepsilon\ell' m}$ can be obtained from $\phi_{\varepsilon\ell m} \sim (L_+)^m\,\phi_{\varepsilon\ell 0}$, and using the hermiticity properties of angular momentum operators with respect to the scalar product. It is then obvious that in the RHS of (5.43) the energy delta $\delta(|\varepsilon|-|\varepsilon'|)$ is redundant

[27]. D.R. Brill and J.A. Wheeler, Rev. Mod. Phys. 29 (1957), 465.

[28]. Capital Latin indices are tetrad indices and run from 0 to 3; they are raised and lowered with the Minkowski metric $\eta^{AB},\eta_{AB}$ = diag(+1,-1,-1,-1). Greek indices run from 0 to 3 and are raised and lowered with $g^{\alpha\beta},g_{\alpha\beta}$; throughout the paper we use units such that $\hbar = c = 1$.

[29]. $\gamma^A$ are the constant Dirac matrices; we use a representation such that $\gamma^{A\dagger} = \gamma^0\gamma^A\gamma^0$, with $(\gamma^0)^2 = -(\gamma^k)^2 = \mathbb{1}$, $k = 1,2,3$, and $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$. Explicitly $\gamma^0 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, $\gamma^k = \begin{pmatrix}0 & \sigma^k\\ -\sigma^k & 0\end{pmatrix}$. We use Pauli matrices in the representation $\sigma^1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $\sigma^2 = \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}$, $\sigma^3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$.

[30]. With the regularity conditions at $r = 0$ and $r = \infty$ to be imposed, and with respect to the scalar product to be defined in the next section, the momentum $\vec{\pi}$ is Hermitian.

[31]. Actually the fermion wave function $(-g)^{1/4}\psi$ is a pseudo scalar with respect to general coordinate transformations.

[32]. Because successive application of $J_{(-)}$ would lead to non-normalizable functions, namely, $m+m'/2 < -1/2$.

[33]. We use the result $\lim_{c\to 0} cF(a,b,c;\lambda) = ab\lambda F(a+1,b+1,2;\lambda)$.

[34]. J. Tiomno, N. Cim. 1 (1955), 226.

[35]. Actually we have $A\,S(k_3=0)A^\dagger = -\Sigma^3$.

[36]. We remark that in the limit $\mu = 0$ solutions for different e differ by the factor $L = \pm 1$.

[37]. L.L. Foldy and S.A. Wouthuysen, Phys. Rev. 78 (1950), 29.

[38]. M. Cini and B. Toushek, Nuovo Cimento 7 (1958), 422.

[39]. J. Tiomno, Physica 53 (1971), 58.

[40]. Besides the coordinate independence of the definition (5.36) for the normalization integral, we have analogous convenience as discussed in the case of the normalization (3.35) for the scalar field solutions. When a global space-like hypersurface is avaiable in the geometry of the space-time, the definition (5.36) can also be shown equivalent to the usual definition of the conserved scalar product for fermions solutions which are eigenstates of the Hamiltonian of the system.

[41]. In the right-hand-side of expression (5.37) the Kroenecker delta $\delta_{ee'}$ is obviously redundant (cf. expression (5.12)).

[42]. We note the relation

$$|\varepsilon|^2 - W_k^{\,2} = 4(j+1/2)(\sqrt{2}|\varepsilon|-j-1/2) \ .$$

[43]. In the remaining of this section we take for simplicity $\omega = 1$.

[44]. The local Lorentz group (4.2), (4.3) — with respect to which the spinor structure is defined — induces on the Fourier space the group of transformations

$$\tilde{\psi}_F(jmk_3\,\varepsilon\,e\ ;\pm) = \sum_{j'm'e'} \int \frac{dk_3{}'d\varepsilon'}{(2\pi)^3}\,\delta(\varepsilon^2 W_k^{\,2} - 4(j+1/2)(\sqrt{2}|\varepsilon|-j-1/2)$$

$$.S(jmk_3\varepsilon e;j'm'k_3'\varepsilon'e')\psi_F(j'm'k_3'\varepsilon'e';\pm)$$

where

$$S(jmk_3\varepsilon e;j'm'k_3'\varepsilon'e') = \int \sqrt{-g}\ d^4x\ K_{(\pm)}(jmk_3\varepsilon e;x)S(x)\ .$$

$$.\ K^{\dagger}_{(\pm)}(j'm'k_3'\varepsilon'e';x)$$

The infinitesimal version of the above transformations are easier to handle.

[45]. We impose the condition $[K_{\pm},\gamma^0] = 0$ in order to have $\pi_0 = \pm|\varepsilon|$ always.

[46]. Expressions (5.48) show that the gravitational coupling of fermions to matter vorticity violates parity: in fact the conservation equations (5.48) are not invariant under <u>active</u> parity transformations of the system, for instance an active reflexion through planes which contain the $x^3$-axis. We note that the parity transformations considered here are local transformations defined with respect to the local Lorentz frames of the tetrads.

[47]. P.A.M. Dirac, in Recente Developments in General Relativity (Pergamon Press, New York, 1962) pp. 191-200.

[48]. The preferential emission of spin-1/2 fermion (antifermion)

along the direction antiparallel (parallel) to the local vorticity field has a macroscopic analog in the case of neutrino evaporation by a rotating black hole. Cf. A. Vilenkin, Phys. Rev. Lett. 41 (1978), 1575; Phys. Rev. D20 (1979), 1807 and D.A. Leahy and W.G. Unruh Phys. Rev. D19 (1979), 3509.

[49]. Provided also that the characteristic length of the pair-creation perturbation be much smaller than the characteristic dimensions of Gödel's universe. In other words, the pair-creation perturbation must be localized in a small space-time volume of Gödel's universe.

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# GRAVITATIONAL COUPLING OF SCALAR AND FERMION FIELDS TO MATTER VORTICITY: MICROSCOPIC ASYMMETRIES

I. Damião Soares and J. Tiomno

## ERRATA

pg 29 - equation (4.26) should read

$$J^2 = (x^2-1)\frac{d^2}{dx^2} + \left[2x + m' - m\right]\frac{d}{dx} + \frac{1}{(x+1)}\left[\varepsilon^2 + \frac{\sqrt{2}}{2}\varepsilon\,(m'+m) + \right.$$

$$\left. + \frac{m'+m+\sqrt{2}\,\varepsilon}{2}\,\Sigma^3\right] - \frac{1}{(x^2-1)}\,A\begin{pmatrix}(m-1/2)\;(m'-1/2)\mathbb{1} & 0\\ 0 & (m+1/2)\;(m'+1/2)\mathbb{1}\end{pmatrix}A$$

pg 31 - equation (4.43) should read

$$J_+\,\psi(m,m') = -\frac{ab}{m+m'+3}\,\psi(m+1,m'+1) \qquad (4.43)$$

Pages 13, 14, 33 and 34 should be substituted by the following ones.

$$\lim_{x\to 1} \phi^*\phi = \text{finite} \quad , \qquad (3.22)$$

$$\lim_{x\to\infty} \sqrt{-g}\, \phi^*\phi = 0 \quad . \qquad (3.23)$$

From the explicit expression (3.17) for $\phi_{\varepsilon,m}(x)$ condition (3.22) implies that

$$m \geq 0 \quad . \qquad (3.24)$$

So starting from a given regular solution $\phi_{\varepsilon,m}$ with m=positive integer or zero, we can generate a sequence of regular solutions [19] by successive application of $L_-$, which by (3.21) necessarily terminates at m = 0.

On the right the sequence could in principle extend to infinite values of m by successive application of $L_+$. However from (3.23) two distinct possibilities arise. By using the asymptotic expression of the hypergeometric function [20], condition (3.23) is satisfied if either [21]

(I) $\quad a$ = negative integer or zero $\qquad$ (3.25)

or

(II) $\quad c-b = m+1-b$ = negative integer or zero $\qquad$ (3.26)

with

$$a = m + \frac{\sqrt{2}}{2}\varepsilon + \frac{1}{2} + \frac{n}{2} \qquad (3.27)$$

$$b = m + \frac{\sqrt{2}}{2}\varepsilon + \frac{1}{2} - \frac{n}{2} \qquad (3.28)$$

for both cases (I) and (II). Two distinct sets of solutions arise:

Type I solutions

We express any negative integer or zero by $m-\ell$, where $\ell$ = integer $\geq$ m, that is,

$$0 \leq m \leq \ell \quad . \qquad (3.29)$$

From (3.25) and (3.27) we then have $\ell + \frac{\sqrt{2}}{2}\,\varepsilon + \frac{1}{2} + \frac{n}{2} = 0$ which implies

$$\varepsilon = -\left[\sqrt{2}\,(2\ell+1) + \sqrt{(2\ell+1)^2 + k_3^{\,2} + \mu^2 a^2 + 1}\right] \quad . \qquad (3.30)$$

The corresponding positive-energy solutions are obtained by taking the complex-conjugate solution $\phi^*$. We note that for type I, the negative-energy solutions are associated to the negative exponential $e^{-im\phi}$ [22]. The eigenvalue of $L_3$ for this case are given by $\mp\,(m - \frac{\sqrt{2}}{2}\,|\varepsilon|)$ for positive/negative energy. Also from (3.20) we have that

$$L_+\,\phi_{\varepsilon,\ell} = 0$$

as expected.

Type II solutions

We denote here any negative integer or zero by $-\ell$, where

$$\ell = \text{integer} \geq 0 \quad . \qquad (3.31)$$

From (3.26) and (3.28) we have $\ell - \frac{\sqrt{2}}{2}\,\varepsilon + \frac{1}{2} + \frac{n}{2} = 0$ which implies

$$\varepsilon = \sqrt{2}\,(2\ell+1) + \sqrt{(2\ell+1)^2 + k_3^{\,2} + \mu^2 a^2 + 1} \quad . \qquad (3.32)$$

any of its components along a direction orthogonal to $\vec{\Omega}$ is not. The allowed "rotations" in this space maintain the direction $\vec{\Omega}$ invariant.

To proceed we shall then make use of regularity and boundary conditions on the wave functions, and obtain two distinct sets of solutions, one infinite dimensional and the other finite dimensional representation basis of the algebra of angular momentum. On the set of solutions (4.30) we now impose boundary and regularity conditions, namely that Dirac fields (which are test fields and do not contribute to the curvature of the cosmological background) are finite perturbations at any space-time point. We impose similarly to (3.22), (3.23), that

$$\lim_{x\to 1} \psi^\dagger\psi = \text{finite} \tag{4.47}$$

$$\lim_{x\to\infty} \sqrt{-g}\; \psi^\dagger\psi = 0 \quad . \tag{4.48}$$

The quantity $\psi^\dagger\psi$ is the component of the Dirac current $\bar{\psi}\gamma^\mu(x)\psi$ along the four-velocity of the matter content of the model. By using (4.31) and (4.32), the regularity condition (4.47) implies

$$m \geq 1/2 \quad . \tag{4.49}$$

So starting from a given regular solution $\psi(m,m')$ and by successively applying $J_{(-)}$ we necessarily arrive at a solution which does not satisfy (4.49) unless $J_-\psi = 0$ for some value $(m,m')$. From (4.42) we have that the sequence finishes on the left for $m+m'/2 = -1/2$, and we must then have

$$\frac{m+m'}{2} > -1/2 \tag{4.50}$$

that is, $m+m'/2$ takes half-integer values greater or equal to $-1/2$. The lower bound $\frac{m+m'}{2} = -1/2$ is not in contradiction with the regularity condition (4.47) because we have [33] $\lim_{\frac{m+m'}{2} \to -1/2} \alpha$ = finite for all x.

On the right the sequence could in principle extend to infinite values of $m+m'/2$ by successive application of $J_+$.

Condition (4.48) will nevertheless impose an upper bound on the values of $m+m'/2$.

From (4.48) two distinct possibilities arise [20,21]. Either (cf. the scalar field case)

(I) a = negative integer or zero (4.51)

or

(II) c-b = negative integer or zero (4.52)

with

$$a = \frac{m+m'}{2} + \frac{\sqrt{2}}{2}\varepsilon + \frac{1}{2} + \frac{n}{2} \qquad (4.53)$$

$$b = \frac{m+m'}{2} + \frac{\sqrt{2}}{2}\varepsilon + \frac{1}{2} - \frac{n}{2} \qquad (4.54)$$

for both cases (I) and (II) and we obtain the two distinct sets of solutions:

Type I solutions

We denote any negative integer or zero by $m+m'/2 - j$, with $j$ = half-integer $\geq m+m'/2$, that is,

$$-1/2 \leq \frac{m+m'}{2} \leq j \quad . \qquad (4.55)$$