

PERIODIC SOLUTIONS OF A NONLINEAR PROPAGATION
EQUATION VIA A FIXED POINT ARGUMENT

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ABSTRACT

In this work the initial value problem for the equation

$$u_t + \beta u_x + \gamma f(u)_x - \delta u_{xxt} = g, \quad \forall x \in \mathbb{R}, \quad \forall t \in [0, T),$$

with periodic boundary conditions is interpreted in the sense of periodic distributions and studied via fixed point arguments. Weak solutions exist if $f \in C^0(\mathbb{R})$ and $g \in L^\infty(L^2(0,1))$. Moreover, regularity in f, g and the initial data implies regularity of solutions.

1. INTRODUCTION

Let β and γ be real numbers and δ and T be positive real constants (under proper conditions T may be $+\infty$). We shall consider the following problem:

Given real functions f and u_0 defined on \mathbb{R} and a real function g defined on $\mathbb{R} \times [0, T)$, find a real function u defined on $\mathbb{R} \times [0, T)$ such that

$$(1.1) \quad u_t + \beta u_x + \gamma f(u)_x - \delta u_{xxt} = g, \quad \forall x \in \mathbb{R}, \quad \forall t \in [0, T),$$

$$(1.2) \quad u(x, 0) = u_0(x), \quad \forall x \in \mathbb{R},$$

and

$$(1.3) \quad u(x+1, t) = u(x, t), \quad \forall t \in [0, T).$$

Equation (1.1), is a generalization of

$$(1.1)' \quad u_t + \beta u_x + \gamma u u_x - \delta u_{xxt} = 0,$$

which was proposed by T.B. Benjamin, J.L. Bona and J.J. Mahony in [1], as an alternative for the KdV equation, to model the propagation of long waves in nonlinear dispersive media.

The homogeneous case of problem (1.1')-(1.3) was studied by L.A. Medeiros and G.P. Menzala in [2]. There they showed existence and uniqueness of classical solutions when the third derivative of u_0 is square integrable. Also M.M. Miranda studied this problem in [5] where theorems concerning the existence of weak periodic solutions are presented. Numerical algorithms for computing solutions of this problem were analysed by M.A. Raupp [9].

The generalized problem (1.1)-(1.3) was first studied by B.P. Neves [7],[8]. In [7] the author obtained theorems asserting existence of classical and weak solution but only for $\beta = 0$, $\gamma < 0$, $g = 0$, and f two times differentiable with positive first derivative. These conditions are clearly not satisfied in equation (1.1)', where $f(s) = \frac{s^2}{2}$. In [8] the restriction that $f' \geq 0$ is relaxed for infinitely differentiable f . In [4] L.A. Medeiros and M.M. Miranda proved the existence of weak solutions of (1.1)-(1.2) with the more stringent boundary condition

$$(1.3)' \quad u(0,t) = u(1,t) = 0, \quad \forall t \in [0,T],$$

which are defined on $[0,1] \times [0,T)$. A numerical analysis of problem (1.1)-(1.3), by C.A. de Moura et al., can be found in [6]. The non-periodic problem (1.1)-(1.2) was studied by Fixed Point techniques by L.A. Medeiros and G.P. Menzala [3].

The aim of this work is to study the existence and regularity of solutions of problem (1.1)-(1.3). In doing this, we follow a line of reasoning that parallels the one of Benjamin, Bona and Mahony in [1] relative to the demonstration of existence and uniqueness of solutions for problem (1.1)'-(1.2). They reduced the problem to a fixed point question by use of the Fourier transform and showed that, for T small enough, the resulting operator is a contraction map and, thus, has a fixed point. By recurrence their solution extends to any T , and a "bootstrap" reasoning ensured regularity.

In the following we interpret problem (1.1)-(1.3) in

the sense of periodic distributions theory. Making the non-linear term independent, we decompose equation (1.1) and solve the associate linear equation by standard Galerkin procedures (the Fourier series method is not appropriate for this linear equation). Then we obtain a fixed point by mean of Schauder's Fixed Point Theorem. A "bootstrap" argument is again used to study regularity.

It is shown that weak solutions exist when f is continuous and are unique if, in addition, f is Lipschitz. Moreover, the solution is as regular as u_0 and a bit more regular than f and g .

In section 2 the framework and the weak form of problem (1.1)-(1.3) that will be used are introduced and it is shown that f can be replaced by a slight different but more convenient function. In section 3 existence and regularity of solutions of the associated linear equation are analysed. Finally, in section 4, a fixed point argument is used to lift the results to the nonlinear equation.

2. THE WEAK FORMULATION

Before presenting the weak form of problem (1.1)-(1.3), we need to define some notation and standard results that will be useful later. The following is a list of spaces for further reference:

$P_0 = H^0(0,1) = L^2(0,1)$, the space of square integrable functions on $(0,1)$;

$H^k(0,1)$, the space of function of P_0 with (generalized)

derivatives up to order k belonging to P_0 ;

P_k , the closed subspace of $H^k(0,1)$ whose functions v are such that $D_x^i v(0) = D_x^i v(1)$ for $i=0, \dots, k-1$;

$C^k(I)$, the space of continuous functions whose derivatives up to order k are continuous on I , $I \subset \mathbb{R}$;

$C^\infty(I)$, the space of infinitely differentiable functions on I .

All these spaces are endowed with their usual topologies. For subspaces, the topology of the larger space is induced. We denote by D_x (resp. D_t) the derivative with respect to x (resp. t). If i is a non-negative integer, D_x^i and D_t^i are the i -th powers of D_x and D_t . In the sequel V will stand for a Banach space. Its dual space will be denoted by V' and the duality pairing will be denoted by $\langle \cdot, \cdot \rangle$. The inner-product and norm of $H^k(0,1)$ (and P_k) are denoted respectively by

$$(2.1) \quad (f, g)_k = \sum_{i=0}^k (D_x^i f, D_x^i g)$$

and

$$(2.2) \quad |f|_k^2 = \sum_{i=0}^k (D_x^i f, D_x^i f),$$

except for $k=0$ when $(\cdot, \cdot) = (\cdot, \cdot)_0$.

We need also to consider the space of periodic test functions

$$\mathcal{D}_p(0,1) = \{\varphi \in C^\infty([0,1]) \mid D_x^i \varphi(0) = D_x^i \varphi(1), \forall i\},$$

and its dual $\mathcal{D}'_p(0,1)$, the space of periodic distributions.

A sequence $\{\varphi_n\}$ converges to φ in $\mathcal{D}_p(0,1)$ if and only

if $D_x^i \varphi_n \rightarrow D_x^i \varphi$ uniformly for all $i \geq 0$ (note that

$D_x^0 \varphi = \varphi$). Its dual $\mathcal{D}'_p(0,1)$ is endowed with the weak* topology. More information on these spaces can be found in [10, Chap.IV].

The spaces above are spaces whose elements are functions of a single variable (which will be the spatial variable x). To take account of the variable t we need some spaces of strongly measurable functions from $[0,T)$ into V , a Banach space. The space $L^1(V)$ of integrable functions normed by

$$(2.3) \quad |u|_{L^1(V)} = \int_0^T |u(t)|_V dt.$$

Its subspace $L^\infty(V)$, of essentially bounded functions, normed by

$$|u|_{L^\infty(V)} = \operatorname{ess\,sup}_{t \in [0,T)} |u(t)|_V.$$

The subspaces of $L^\infty(P_0)$ defined by

$$K(m,k) = \{u \in L^\infty(P_k) \mid D_t^i u \in L^\infty(P_k) \quad i=1, \dots, m\},$$

are endowed with the product norms

$$|u|_{m,k} = \max_{0 \leq i \leq m} |D_t^i u|_{L^\infty(P_k)}$$

for all non-negative integers m and k .

The following properties of these spaces will be used in the sequel. As P_k is a closed subspace of $H^k(0,1)$ it follows that P_{k+1} is compactly imbedded in P_k . Since $\mathcal{D}_p(0,1)$ contains the trigonometric basis and P_k is a subspace of $C^{k-1}[0,1]$, $\mathcal{D}_p(0,1)$ is densely imbedded in all P_k . This last statement is a consequence

of Fourier series convergence theorems [12]. Finally, by the compactness criterion of Lions-Aubin [11, Chap.I], the space $K(1,1)$ is compactly imbedded in $K(0,0) = L^\infty(P_0)$.

Next we point out in what sense the weak solutions of problem (1.1)-(1.3) are to be considered. Since $\mathcal{D}_p(0,1)$ is dense in P_1 and P_0 , we identify P_0 with its dual and consider all the spaces P_k and their duals as subspaces of $\mathcal{D}_p'(0,1)$. Then the weak problem is to find a function $u \in K(1,1)$ such that for almost every $t \in [0,T)$

$$\langle u_t(t), \varphi \rangle - \beta \langle u(t), \varphi_x \rangle - \gamma \langle f(u(t)), \varphi_x \rangle + \quad (2.5)$$

$$\delta \langle u_{xt}(t), \varphi_x \rangle = \langle g(t), \varphi \rangle, \quad \forall \varphi \in \mathcal{D}_p(0,1),$$

$$(2.6) \quad u(0) = u_0.$$

Since solutions of the weak problem should be in $K(1,1)$ (conditions under which such a solution exists will be made precise in section 4), equation (2.6) makes sense. Moreover, due to the density of $\mathcal{D}_p(0,1)$ as a subspace of P_1 , (2.5) can be modified to read as follows

$$\langle u_t(t), v \rangle - \beta \langle u(t), v_x \rangle - \gamma \langle f(u(t)), v_x \rangle + \quad (2.5)'$$

$$\delta \langle u_{xt}(t), v_x \rangle = \langle g(t), v \rangle, \quad \forall v \in P_1, \quad t \in [0,T) \text{ a.e.}$$

From now on, and without any loss of generality, we shall take $\beta = \gamma = \delta = 1$.

The following lemmas show that under reasonable conditions on f and g , f can be replaced by another function f^* , as regular as f , which is bounded and constant outside a bounded interval. This will be necessary for some arguments

of section 4.

Lemma 2.1 - If $g \in L^1(P_0)$ and $f \in C^0(R)$ then any solution of the weak problem is uniformly bounded. That is, there is a constant C depending only on g and u_0 such that

$$(2.7) \quad \operatorname{ess\,sup}_{t \in [0, T]} \sup_{x \in [0, 1]} |u(x, t)| \leq C.$$

Proof. Choose $v = u(t)$ in (2.5)', then it follows that

$$(2.8) \quad \begin{aligned} (u_t(t), u(t)) + (u_{xt}(t), u_x(t)) &= \\ &= (u_x(t), u(t)) + (f(u(t)), u_x(t)) + (g(t), u(t)) \end{aligned}$$

almost everywhere in t . But if $v \in P_1$,

$$(2.9) \quad (v, v_x) = \int_0^1 v v_x \, dx = \int_0^1 \left(\frac{v^2}{2}\right)_x \, dx = v^2 \Big|_0^1 = 0.$$

Generalizing this, if F is any primitive of f , it follows that

$$(2.10) \quad \begin{aligned} (f(v), v_x) &= \int_0^1 f(v) v_x \, dx = \\ &= \int_{v(0)}^{v(x)} D_x F(s) \, ds = F(v(1)) - F(v(0)) = 0, \end{aligned}$$

for all $v \in P_1$, since $v(0) = v(1)$. Thus, for almost every $t \in [0, T)$,

$$(u(t), u_x(t)) = (f(u(t)), u_x(t)) = 0,$$

and so

$$\frac{1}{2} D_t (|u(t)|_0^2 + |u_x(t)|_0^2) = (g(t), u(t)) \leq |g(t)|_0 |u(t)|_0.$$

Integrating the above equation we obtain

$$|u(t)|_1^2 \leq |u_0|_1^2 + 2 \int_0^t |g(\tau)|_0 |u(\tau)|_0 \, d\tau,$$

and from this it follows that

$$\begin{aligned} \operatorname{ess\,sup}_{t \in [0, T]} |u(t)|_1^2 &\leq |u_0|_1^2 + 2 \int_0^t |g(\tau)|_0 |u(\tau)|_0 \, d\tau \leq \\ &\leq |u_0|_1^2 + 2 \operatorname{ess\,sup}_{t \in [0, T]} |u(t)|_0 \int_0^t |g(\tau)|_0 \, d\tau. \end{aligned}$$

Therefore, for any $\epsilon < 1$,

$$(2.11) \quad (1-\epsilon) \operatorname{ess\,sup} |u(t)|_1^2 \leq |u_0|_1^2 + \frac{1}{\epsilon} |g|_{L^1(P_0)}^2,$$

(2.11) implies (2.7), as $\sup_x |v(x)| \leq c |v|_1$, if $v \in P_1$.

Lemma 2.2. Let $f \in C^0(\mathbb{R})$ be Lipschitz on any bounded interval. Then there exists at most one weak solution satisfying (2.7).

Proof. Let u_1, u_2 be two solutions of the weak problem. Then $w = u_1 - u_2 \in K(1,1)$ satisfies

$$(2.12) \quad \begin{aligned} &(w_t(t), v) - (w(t), v_x) - \\ &- (f(u_1(t)) - f(u_2(t)), v_x) + (w_{xt}, v_x) = 0, \quad \forall v \in P_1, \end{aligned}$$

for almost every t . For $v = w(t)$, equation (2.12) becomes

$$\begin{aligned} &\frac{1}{2} D_t (|w(t)|_0^2 + |w_x(t)|_0^2) = \\ &= (w(t), w_x(t)) + (f(u_1(t)) - f(u_2(t)), w_x(t)), \end{aligned}$$

which by (2.9) implies that

$$\frac{1}{2} D_t |w(t)|_1^2 \leq |f(u_1(t)) - f(u_2(t))|_0 |w_x(t)|_0.$$

Let L be the Lipschitz constant of f on $[-C, C]$, where C is the constant of (2.7). Then,

$$|f(u_1(t)) - f(u_2(t))|_0 \leq L |w(t)|_0$$

and

$$(2.13) \quad D_t |w(t)|_1^2 \leq L (|w(t)|_0^2 + |w_x(t)|_0^2) = L |w(t)|_1^2$$

from which it follows that

$$\operatorname{ess\,sup}_{t \in [0, T]} |w(t)|_1^2 \leq e^{Lt} |w(0)|_1^2 = 0.$$

Therefore, $w = 0$.

Lemma 2.3. Suppose that f is Lipschitz continuous on bounded intervals and that the solution of the weak problem satisfies (2.7). Then there is a bounded function f^* , as regular as f , such that the solution of the weak problem with equation

$$(2.5)'' \quad \langle u_t(t), v \rangle - \langle u(t), v_x \rangle - \langle f^*(u(t)), v_x \rangle + \\ + \langle u_{xt}(t), v_x \rangle = \langle g(t), v \rangle$$

which satisfies (2.6)' is the same as that of (2.5)'.

Proof. Take C as the constant of (2.7). If f is continuous let

$$(2.14) \quad f^*(s) = \begin{cases} f(-c), & \text{if } s < -c, \\ f(s), & \text{if } |s| < c, \\ f(c), & \text{if } s > c. \end{cases}$$

Now let u^* be a solution of the weak problem for equation (2.5)'' . We can suppose that u^* satisfies (2.7) with the same constant as the original solution u (because, if not, we let C in (2.14) be the maximum of the two constants) and then

$$(2.15) \quad f^*(u^*(x, t)) = f(u^*(x, t))$$

almost everywhere in x and t . Consequently u^* also satisfies equation (2.5)', and by Lemma 2.2 $u^* = u$.

Suppose now that f has more regularity, say $f \in C^k(\mathbb{R})$ for $k \geq 1$. In this case we define f^* to be

$$(2.16) \quad f^*(s) = \begin{cases} f(-C-\epsilon), & \text{if } -s > C+\epsilon \\ \theta_1, & \text{if } C < -s < C+\epsilon \\ f(s), & \text{if } |s| < C, \\ \theta_2, & \text{if } C < s < C+\epsilon, \\ f(C+\epsilon), & \text{if } s > C+\epsilon, \end{cases}$$

where θ_1, θ_2 are generalized Hermite interpolants on $[-C-\epsilon, -C]$ and $[C, C+\epsilon]$, respectively, such that

$$(2.17) \quad \begin{aligned} \theta_1(-C-\epsilon) &= f(-C-\epsilon), \\ D_x^i \theta_1(-C-\epsilon) &= 0, \quad i=1, \dots, k, \\ D_x^i \theta_1(-C) &= D_x^i f(-C), \quad i=0, \dots, k, \end{aligned}$$

and

$$(2.18) \quad \begin{aligned} \theta_2(C+\epsilon) &= f(C+\epsilon) \\ D_x^i \theta_2(C+\epsilon) &= 0, \quad i=1, \dots, k, \\ D_x^i \theta_2(C) &= D_x^i f(C), \quad i=0, \dots, k. \end{aligned}$$

Then clearly $f^* \in C^k(\mathbb{R})$ and u^* satisfies (2.15). Therefore, $u^* = u$.

3. THE ASSOCIATED LINEAR EQUATION

The study of solutions of the weak problem will be pursued in two steps. First, we reduce the non linear equation (2.5)" treating the non linear term as an indepen-

dent one, thus obtaining a linear problem whose solution depends upon a given function w . That is, given w in a proper space, we study solutions of

$$(3.1) \quad \langle u_t(t), v \rangle - \langle u(t), v_x \rangle + \langle u_{xt}(t), v_x \rangle = \\ = \langle f^*(w(t)), v_x \rangle + \langle g(t), v \rangle, \quad \forall v \in P_1, \quad t \in [0, T],$$

$$(3.2) \quad u(0) = u_0,$$

which belong to $K(1,1)$. Secondly, the mapping that associates u , the solution of (3.1)-(3.2), with the function w is shown to have a fixed point.

In this section we consider the linear problem derived from (3.1)-(3.2) which, given functions $g(x,t)$, $h(x,t)$ and $u_0(x)$, is to find a function $u \in K(1,1)$ satisfying (3.2) such that

$$(3.3) \quad \langle u_t(t), v \rangle - \langle u(t), v_x \rangle + \langle u_{xt}(t), v_x \rangle = \\ = \langle g(t), v \rangle + \langle h(t), v_x \rangle, \quad \forall v \in P_1,$$

almost everywhere in t .

Two results are presented below. The first is concerned with existence and uniqueness of solutions of the linear problem. The second relates the regularity of solutions to the regularity of h, g and u_0 .

Theorem 3.1. If $u_0 \in P_1$, h and $g \in L^\infty(P_0) = K(0,)$, and $T < \infty$, then there exists a unique function $u \in K(1,1)$ which satisfies (3.3) and (3.2).

Proof. The argument follows the usual path of the energy method in which compactness theorems are used. So we shall have three steps:

1. Finite dimensional approximation in x ,
2. A priori estimates,
3. Passage to the limit.

Step 1. If i and m are non-negative integers, let

$$w_i(x) = \begin{cases} c_i \cos(\pi i x) & \text{when } i \text{ is even,} \\ c_i \sin(\pi(i-1)x) & \text{when } i \text{ is odd,} \end{cases}$$

$$V_m = \text{span} \{w_0, \dots, w_m\}$$

and

$$V_\infty = \text{span} \{w_0, \dots, w_m, \dots\} = \bigcup_m V_m,$$

where c_i are normalizing constants. Observe that

$$(3.4) \quad V_m \subset V_n \subset V_\infty \subset \mathcal{D}_p(0,1)$$

whenever $m < n$ and that V_∞ is a dense subspace of P_k for all k , since $\{w_i\}_{i \geq 0}$ is a basis of P_k . This last claim, which implies that $\mathcal{D}_p(0,1)$ is dense in P_k , is a consequence of convergence theorems for Fourier Series [12].

We define the finite dimensional approximation problem to be the finding of a function $u_m: [0, T) \rightarrow V_m$ such that

$$(3.5) \quad (u_{mt}(t), w_i) - (u_m(t), w_{ix}) + (u_{mxt}(t), w_{ix}) = \\ = (g(t), w_i) + (h(t), w_{ix}), \quad 0 \leq i \leq m, \quad t \in [0, T),$$

$$(3.6) \quad u_m(0) = u_{om},$$

where u_{om} is such that $u_{om} \in V_m$ and $u_{om} \rightarrow u_0$ in P_1 . More precisely, we take u_{om} to be the truncated Fourier series of u_0 and recall that, as $u_0 \in P_1$, $u_{om} \rightarrow u_0$ in P_1 . Writing

$$u_m(x, t) = \sum_{i=0}^m \alpha_i(t) w_i(x),$$

$$u_{om}(x) = \sum_{i=0}^m \alpha_{oi} w_i(x),$$

and putting

$$\alpha(t) = (\alpha_1(t), \dots, \alpha_m(t))^T,$$

$$\alpha_o = (\alpha_{o1}, \dots, \alpha_{om})^T,$$

it is easy to see that (3.4)-(3.5) is equivalent to the problem

$$(3.5)' \quad (A+B)\alpha' = C\alpha + F(t),$$

$$(3.6)' \quad \alpha(0) = \alpha_o,$$

where the matrices A, B and C have as their elements the quantities

$$A_{ij} = (w_i, w_j),$$

$$B_{ij} = (w_{ix}, w_{jx}),$$

$$C_{ij} = (w_{ix}, w_j),$$

and the vector F has the components

$$F_i(t) = (h(t), w_{ix}) + (g(t), w_i).$$

The existence of a unique measurable local solution of (3.5)-(3.6) in $[0, t_m)$ is assured, since A+B is non-singular for all m. Actually, as a consequence of the next estimates, we can take $t_m = T$.

Step 2. Since (3.5) is equivalent to

$$(3.7) \quad (u_{mt}(t), v) - (u_m(t), v_x) + (u_{mxt}(t), v_x) = \\ = (g(t), v) + (h(t), v_x), \quad \forall v \in V_m,$$

choosing $v = u_m(t)$ leads to

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (|u_m(t)|_0^2 + |u_{mx}(t)|_0^2) = \\ & = (u_m(t), u_{mx}(t)) + (g(t), u_m(t)) + (h(t), u_{mx}(t)) \end{aligned}$$

and

$$(3.8) \quad \frac{1}{2} \frac{d}{dt} |u_m(t)|_1^2 \leq |g(t)|_0 |u_m(t)|_0 + |h(t)|_0 |u_{mx}(t)|_0,$$

since $(u_m(t), u_{mx}(t)) = 0$. Integrating (3.8), we have

$$|u_m(t)|_1^2 \leq |u_{om}|_1^2 + 2 \int_0^t |g(\tau)|_0 |u_m(\tau)|_0 d\tau + 2 \int_0^t |h(\tau)|_0 |u_{mx}(\tau)|_0 d\tau,$$

and so

$$\begin{aligned} \text{ess sup}_{t \in [0, T]} |u_m(t)|_1^2 & \leq |u_{om}|_1^2 + 2 \int_0^T |g(\tau)|_0 |u_m(\tau)|_0 d\tau + \\ & + 2 \int_0^T |h(\tau)|_0 |u_{mx}(\tau)|_0 d\tau. \end{aligned}$$

This, by the same reasoning used to arrive at (2.11),

$$\begin{aligned} (3.9) \quad & (1 - \epsilon_1 - \epsilon_2) \text{ess sup}_{t \in [0, T]} |u_m(t)|_1^2 \leq \\ & \leq |u_{om}|_1^2 + \frac{1}{\epsilon_1} |g|_{L^1(P_0)}^2 + \frac{1}{\epsilon_2} |h|_{L^1(P_0)}^2. \end{aligned}$$

Then, for $\epsilon_1 = \epsilon_2$ small enough, it follows that

$$(3.10) \quad \text{ess sup}_{t \in [0, T]} |u_m(t)|_1^2 \leq |u_o|_1^2 + C(|g|_{L^1(P_0)}^2 + |h|_{L^1(P_0)}^2) = K_o^2,$$

from (3.9) and the inequality $|u_{om}|_1 \leq |u_o|_1$.

Estimate (3.10) is enough to guarantee that we can take $t_m = T$ in Step 1 but, to pass to the limit, another estimate is needed and to obtain it we take $v = u_{mt}(t)$ in (3.7). Thus,

$$\begin{aligned}
& |u_{mt}(t)|_0^2 + |u_{mxt}(t)|_0^2 = \\
& = (u_m(t), u_{mxt}(t)) + (g(t), u_{mt}(t)) + (h(t), u_{mxt}(t)).
\end{aligned}$$

Consequently,

$$\begin{aligned}
& |u_{mt}(t)|_1^2 \leq |u_m(t)|_0^2 + \frac{1}{4} |u_{mxt}(t)|_0^2 + \\
& + \frac{1}{2} |g(t)|_0^2 + \frac{1}{2} |u_{mt}(t)|_0^2 + |h(t)|_0^2 + \frac{1}{4} |u_{mxt}(t)|_0^2,
\end{aligned}$$

and

$$(3.11) \quad |u_{mt}(t)|_1^2 \leq 2|u_m(t)|_0^2 + |g(t)|_0^2 + 2|h(t)|_0^2.$$

Therefore,

$$(3.11)' \quad \text{ess sup}_{t \in [0, T]} |u_{mt}(t)|_1^2 = |u_{mt}|_{0,1}^2 \leq 2K_0^2 + |g|_{0,0}^2 + |h|_{0,0}^2.$$

Step 3. Estimates (3.10), (3.11)' mean that the sequence u_m is bounded in $K(1,1)$. Hence, by the weak* compactness of bounded sets in $K(1,1)$, we can extract a subsequence of u_m converging weakly* to a function $u \in K(1,1)$. Denote the subsequence by u_m for convenience.

But $K(1,1)$ can be identified with a closed subspace of $L^\infty(P_1) \times L^\infty(P_1)$. Also $L^\infty(P_1)$ is the dual of $L^1(P'_1)$ and

$$P_1 \subset P_0 \subset P'_1,$$

with dense imbedding. Then $L^1(P_1) \subset L^1(P'_1)$ continuously and the weak* convergence of u_m to u in $K(1,1)$ means that

$$(3.12) \quad \int_0^T (D_x^i D_t^j u_m(t), v(t)) dt \rightarrow \int_0^T (D_x^i D_t^j u(t), v(t)) dt,$$

for $0 \leq i, j \leq 1$ and for all $v \in L^1(P'_1)$. Thus, a fortiori, (3.12) holds for all $v \in L^1(P_1)$.

Therefore u satisfies

$$(3.13) \quad \int_0^T [(u_t(t), v(t)) - (u(t), v_x(t)) + (u_{xt}(t), v_x(t))] dt = \\ = \int_0^T [(g(t), v(t)) + (h(t), v_x(t))] dt, \quad \forall v \in L^1(P_1).$$

In particular, (3.13) holds for $v(t) = \theta(t)w(x) \in L^1(P_1)$, where $w \in P_1$ and $\theta(t) = \mu(A)^{-1} \chi_A(t)$, for A an arbitrary measurable subset of $(0, T)$ with positive measure and χ_A the characteristic function of A . Hence any solution of (3.13) is also a solution of (3.3).

Since (3.11)' implies that u is a continuous map of $[0, T)$ into P_1 , u satisfies (3.2). This accounts for existence.

To show uniqueness, note that, if u_1 and u_2 are two solutions of (3.3)-(3.2), then $w = u_1 - u_2$ satisfies the relation

$$(3.14) \quad (w_t(t), v) - (w(t), v_x) + \\ + (w_{xt}(t), v_x) = 0, \quad \forall v \in P_1, \text{ a.e. } t \in [0, T),$$

and is such that $w(0) = 0$. Take $v = w(t)$. It follows that

$$(3.15) \quad \frac{1}{2} \frac{d}{dt} |w(t)|_1^2 = 0,$$

and since $w(0) = 0$, $w = 0$ in $L^\infty(P_1) \supset K(1, 1)$.

The regularity theorem is as follows.

Theorem 3.2. Suppose that $u_0 \in P_{k+1}$ and that g and h belong to $K(\ell, k)$ for some positive integers ℓ and k . Then u , the solution of (3.3)-(3.2), belongs to

$K(\ell+1, k+1)$.

Proof To prove that $u \in K(\ell+1, k+1)$ it is enough to show that u_m is bounded in this space. To see this, note that the space $K(\ell, k)$ is continuously imbedded in $K(1, 1)$ and can be identified with a closed subspace of a power of $L^\infty(P_1)$. Then repeat the arguments in Step 3 of the proof of Theorem 3.1 to see that any subsequence that converges weakly* in $K(\ell+1, k+1)$ converges to the solution of (3.13).

To show the boundness of u_m in $K(\ell+1, k+1)$, observe first that, as g and h belong to $K(\ell, k)$, then $D_t^j g$ and $D_t^j h$ belong to $K(0, 0)$ for $0 \leq j \leq \ell$. Next, since

$$D_t^j(f(t), w_i) = (D_t^j f(t), w_i), \quad \forall i \leq 0,$$

when $D_t^j f \in L^\infty(P_0)$, it follows that (with F as (3.5)')

$$D_t^j F(t) \in L^\infty(\mathbb{R}^m).$$

Due to this fact we can differentiate equation (3.5)' j times ($j \leq \ell$) to arrive at the relation

$$(3.16) \quad D_t^{j+1} \alpha = (A+B)^{-1} C D_t^j \alpha + (A+B)^{-1} D_t^j F(t).$$

Now, by the Caratheodory Theorem, on the existence of solutions of Ordinary Differential Equations,

$$D_t^j \alpha \in L^\infty(\mathbb{R}^m), \quad 0 \leq j \leq \ell+1.$$

Consequently,

$$D_t^j u_m \in L^\infty(V_m), \quad 0 \leq j \leq \ell+1.$$

Moreover, (3.16) is equivalent to

$$(3.17) \quad (D_t^{j+1} u_m(t), v) - (D_t^j u_m(t), v_x) + \\ + (D_t^{j+1} D_x u_m(t), v_x) = (D_t^j g(t), v) + (D_t^j h(t), v_x)$$

for all $v \in V_m$, whenever $0 \leq j \leq \ell$. Thus if we take

$$v = D_t^{j+1} u_m(t) \quad \text{in (3.17),}$$

$$\begin{aligned} & |D_t^{j+1} u_m(t)|_0^2 + |D_x D_t^{j+1} u_m(t)|_0^2 = \\ & = (D_t^j u_m(t), D_x D_t^{j+1} u_m(t)) + (D_t^j g(t), D_t^{j+1} u_m(t)) + (D_t^j h(t), D_x D_t^{j+1} u_m(t)). \end{aligned}$$

Thus,

$$(3.18) \quad |D_t^{j+1} u_m(t)|_1^2 \leq 2 |D_t^j u_m(t)|_0^2 + |D_t^j g(t)|_0^2 + 2 |D_t^j h(t)|_0^2, \quad 0 \leq j \leq \ell$$

(note that for $j=0$ we have (3.11)). Thus, recursively,

$$(3.19) \quad |D_t^{j+1} u_m|_{0,1}^2 \leq C(|u_0|_1^2 + |g|_{L^1(P_0)}^2 + |h|_{L^1(P_0)}^2 + |g|_{j,0}^2 + |h|_{j,0}^2)$$

for $0 \leq j \leq \ell$.

The estimate (3.19) means that u_m is bounded in $K(\ell+1,1)$.

Next we show that the spatial derivatives are bounded.

To do this pick $v = D_x^{2i} u_m(t)$ in equation (3.17) and integrate by parts i times. Then,

$$(3.20) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} (|D_x^i u_m(t)|_0^2 + |D_x^{i+1} u_m(t)|_0^2) = \\ & = (D_x^i u_m(t), D_x^{i+1} u_m(t)) + (D_x^i g(t), D_x^i u_m(t)) + (D_x^i h(t), D_x^{i+1} u_m(t)). \end{aligned}$$

$$\text{But since } D_x^i u_m \in P_1, (D_x^i u_m(t), D_x^{i+1} u_m(t)) = 0,$$

and thus (3.20) implies

$$\begin{aligned} & \frac{d}{dt} (|D_x^i u_m(t)|_0^2 + |D_x^{i+1} u_m(t)|_0^2) \leq \\ & \leq |D_x^i g(t)|_0 |D_x^i u_m(t)|_0 + |D_x^i h(t)|_0 |D_x^{i+1} u_m(t)|_0, \end{aligned}$$

which integrated in time is

$$\begin{aligned} & |D_x^i u_m(t)|_0^2 + |D_x^{i+1} u_m(t)|_0^2 \leq |D_x^i u_{m0}|_0^2 + |D_x^{i+1} u_{m0}|_0^2 + \\ & + \int_0^t |D_x^i g(\tau)|_0 |D_x^i u_m(\tau)|_0 d\tau + \int_0^t |D_x^i h(\tau)|_0 |D_x^{i+1} u_m(\tau)|_0 d\tau. \end{aligned}$$

Since g and $h \in K(0,k)$ and $u_0 \in P_{k+1}$ we have, by the argument used for (3.10), that

$$(3.21) \quad \begin{aligned} & |D_x^i u_m|_{0,0}^2 + |D_x^{i+1} u_m|_{0,0}^2 \leq \\ & \leq |D_x^i u_0|_0^2 + |D_x^{i+1} u_0|_0^2 + C(|g|_{L^1(P_k)}^2 + |h|_{L^1(P_k)}^2), \end{aligned}$$

for $0 \leq i \leq k$. That is, u_m is bounded in $K(0,k+1)$.

One is left now with the estimation of the mixed derivatives. With this purpose, for $0 \leq j \leq l$ and $0 \leq i \leq k$, take $v = D_t^{j+1} D_x^{2i} u_m(t)$ in (3.17) and integrate by parts i times in x to obtain

$$\begin{aligned} & |D_t^{j+1} D_x^i u_m(t)|_0^2 + |D_t^{j+1} D_x^{i+1} u_m(t)|_0^2 = (D_t^j D_x^i u_m(t), D_t^{j+1} D_x^{i+1} u_m(t)) + \\ & (D_t^j D_x^i g(t), D_t^{j+1} D_x^i u_m(t)) + (D_t^j D_x^i h(t), D_t^{j+1} D_x^{i+1} u_m(t)). \end{aligned}$$

Then

$$(3.22) \quad \begin{aligned} & |D_t^{j+1} D_x^i u_m(t)|_0^2 + |D_t^{j+1} D_x^{i+1} u_m(t)|_0^2 \leq \\ & 2|D_t^j D_x^i u_m(t)|_0^2 + |D_t^j D_x^i g(t)|_0^2 + 2|D_t^j D_x^i h(t)|_0^2. \end{aligned}$$

The above inequality for $j = 0$ is

$$\begin{aligned} & |D_t D_x^i u_m(t)|_0^2 + |D_t D_x^{i+1} u_m(t)|_0^2 \leq \\ & 2|D_x^i u_m(t)|_0^2 + |D_x^i g(t)|_0^2 + 2|D_x^i h(t)|_0^2, \end{aligned}$$

whose right-hand side is bounded for all i as a consequence of (3.21). Therefore, by recurrence on j , (3.22) implies

Theorem 4.1 Let $u_0 \in P_1$ and $g \in K(0,0)$ and let $f \in C^0(\mathbb{R})$ be Lipschitz continuous on bounded intervals. Then if $T < \infty$, there exists function $u \in K(1,1)$ satisfying (2.5)'-(2.6).

Proof. After Lemmas 2.1-2.3, we can work with equation (2.5)". Let $w \in K(0,0)$. Since f^* is bounded, say

$$(4.1) \quad \sup_s |f^*(s)| \leq C^*,$$

then

$$(4.1)' \quad |f^*(w(t))|_0 \leq C^*,$$

and $f^*(w) \in K(0,0)$. Therefore, by Theorem 3.1, for each $w \in K(0,0)$ there is a unique solution u_w of (3.1)-(3.2). Moreover $u_w \in K(1,1) \subset K(0,0)$. Thus, the operator

$$(4.2) \quad \begin{aligned} F: K(0,0) &\rightarrow K(0,0) \\ w &\longrightarrow F(w) = u_w \end{aligned}$$

is well defined.

Next we see that operator F is continuous and maps $K(0,0)$ into a bounded subset of $K(1,1)$.

To show continuity of F let w_1 and w_2 be in $K(0,0)$ and set $u_1 = F(w_1)$ and $u_2 = F(w_2)$. Then, $y = u_1 - u_2$ satisfies

$$(4.3) \quad \begin{aligned} (y_t(t), v) - (y(t), v_x) + (y_{xt}(t), v_x) &= \\ = (f^*(w_1(t)) - f^*(w_2(t)), v_x), & \quad \forall v \in P_1, \quad t \in [0, T) \end{aligned}$$

and $y(0) = 0$. Taking $v = y(t)$ above and using (2.9), equation (4.3) yields

$$\frac{1}{2} \frac{d}{dt} (|y(t)|_0^2 + |y_x(t)|_0^2) \leq |f^*(w_1(t)) - f^*(w_2(t))|_0 |y_x(t)|_0.$$

But since f^* is Lipschitz continuous, with constant L ,

$$(3.23) \quad |D_t^{j+1} D_x^i u_m(t)|_0^2 + |D_t^{j+1} D_x^{i+1} u_m(t)|_0^2 \leq \\ C(|D_x^i u_0|^2 + |D_x^{i+1} u_0|^2 + |g|_{L^1(P_k)}^2 + |h|_{L^1(P_k)}^2 + |D_x^i g|_{j,0}^2 + |D_x^i h|_{j,0}^2),$$

for $0 \leq j \leq \ell$, $0 \leq i \leq k$.

Estimates (3.19), (3.21) and (3.23) show that the sequence u_m of solutions of (3.5) is bounded in $K(\ell+1, k+1)$, and therefore $u \in K(\ell+1, k+1)$.

Remarks

1. We may take $T = \infty$ in Theorems 3.1 and 3.2 by requiring that g and h belong to $K(0,0) \cap L^1(P_0)$, g and h belong to $K(\ell, k) \cap L^1(P_k)$ respectively.
2. The constants appearing in estimates (3.19) and (3.23) depend on ℓ , growing with it. Thus, Theorem 3.2 do not take care of the case $\ell = \infty$. However, this is due to term (u, v_x) in equation (3.3) and, since this term can be embodied into $(f(u), v_x)$, this limitation will have no significance in the nonlinear case.

4. THE NON-LINEAR EQUATION

This section is devoted to showing that, for functions w belonging to a properly chosen space, the operator which associates the solution of (3.1)-(3.2) with w is well defined and has a fixed point. Also, through a "bootstrap" argument, solutions of (2.5) are shown to be as regular as u_0 , f and g .

$$|f^*(w_1(t)) - f^*(w_2(t))|_0 \leq L|w_1(t) - w_2(t)|_0.$$

And

$$\frac{d}{dt} |y(t)|_1^2 \leq L^2 |w_1(t) - w_2(t)|_0^2 + |y(t)|_1^2,$$

which implies, integrating and using Gronwall Lemma, that

$$(4.4) \quad |y(t)|_1^2 \leq L^2 \exp(t) \int_0^t |w_1(\tau) - w_2(\tau)|_0^2 d\tau.$$

Therefore,

$$(4.5) \quad |u_1 - u_2|_{0,0} \leq L^2 \exp(T/2) |w_1 - w_2|_{0,0},$$

so that F is continuous.

To prove F maps $K(0,0)$ into a bounded subset of $K(1,1)$ let $v = u(t) = F(w)(t)$ in (3.1). Then,

$$(u_t(t), u(t)) + (u_{xt}(t), u_x(t)) = (f^*(w(t)), u_x(t)) + (g(t), u(t)),$$

as a consequence of (2.9). This implies the inequality

$$\frac{1}{2} \frac{d}{dt} |u(t)|_1^2 \leq |f^*(w(t))|_0 |u_x(t)|_0 + |g(t)|_0 |u(t)|_0,$$

and so

$$\begin{aligned} |u(t)|_1^2 &\leq |u_0|_1^2 + \int_0^t |f^*(w(\tau))|_0 |u_x(\tau)|_0 d\tau + \\ &+ \int_0^t |g(\tau)|_0 |u(\tau)|_0 d\tau. \end{aligned}$$

Then, by the same arguments as for (2.11), (3.10) we arrive

$$(4.6) \quad |u|_{0,1}^2 \leq 2(|u_0|_1^2 + |g|_{L^1(P_0)}^2 + \int_0^T |f^*(w(t))|_0 dt),$$

and since, by (4.1)',

$$\int_0^T |f^*(w(t))|_0 dt \leq TC^*,$$

inequality (4.6) yields the bound

$$(4.7) \quad |u|_{0,1} \leq 2(|u_0|_1^2 + |g|_{L^1(p_0)}^2 + TC^*) = K_1,$$

where K_1 does not depend on w .

Also, taking $v = u_t(t)$ in (3.1), we have

$$\begin{aligned} & |u_t(t)|_0^2 + |u_{xt}(t)|_0^2 = \\ & = (u(t), u_t(t)) + (f^*(w(t)), u_{xt}(t)) + (g(t), u_t(t)). \end{aligned}$$

Pursuing now the same steps that were used to obtain (3.11)', the above equality implies that

$$|u_t(t)|_1^2 \leq 2|u(t)|_0^2 + |f^*(w(t))|_0^2 + 2|g(t)|_0^2$$

so that, using (4.1)',

$$(4.8) \quad |u_t|_{0,1}^2 \leq 2K_1 + (C^*)^2 + 2|g|_{0,0}^2 = K_2,$$

where K_2 does not depend on w .

Estimates (4.7)-(4.8) show that $\mathbb{F}(w)$ belongs to a bounded subset of $K(1,1)$ for all $w \in K(0,0)$. Hence \mathbb{F} is a continuous operator which maps the closed convex set $K(0,0)$

into itself in such a way that its range is a pre-compact subset of $K(0,0)$. Therefore Schauder's Fixed Point Theorem ensures the existence of a fixed point for \mathbb{F} . That is, there is an element $u \in K(0,0)$ such that $\mathbb{F}(u) = u$. This is obviously a solution of (2.5)'. Moreover u belongs to the range of \mathbb{F} and thus $u \in K(1,1)$.

Remark 3. Theorem 4.1 gives solutions of (2.5)'' for $T < \infty$.

If the hypothesis are changed to require that

$$(4.9) \quad g \in L^1(P_0) \cap L^\infty(P_0),$$

we may take $T = \infty$. This can be done in the following way.

Let $T_1 < \infty$ be such that $nT_1 = T$. Theorem 4.1 says that there is a function $u_1: [0, T_1) \rightarrow P_1$ solution of (2.5)' and

Lemma 2.1 implies that

$$|u(T_1)|_1^2 \leq |u_0|_1^2 + C \int_0^{T_1} |g(t)|_0 dt.$$

Since the arguments of Theorem 4.1 are independent of the origin of time there is a function $u_2: [T_1, T_2) \rightarrow P_1$ solution of (2.5)' such that $u_2(T_1) = u_1(T_1)$. Therefore

$$u(t) = \begin{cases} u_1(t), & \text{if } t \in [0, T_1) \\ u_2(t), & \text{if } t \in [T_1, T_2) \end{cases}$$

is a solution of (2.5)' on $[0, T_2)$ satisfying

$$u(0) = u_0$$

and

$$\begin{aligned} |u(T_2)|_1^2 &= |u_2(T_2)|_1^2 \leq |u_1(T_1)|_1^2 + \int_{T_1}^{T_2} |g(t)| dt \leq \\ &\leq |u_0|_1^2 + \int_0^{T_2} |g(t)| dt. \end{aligned}$$

Repeating the process, we can define u on $[0, T)$. Clearly this procedure can be repeated indefinitely, and u may be extended to $[0, \infty)$.

The next theorem tell us about the regularity of solutions of (2.5)". First note that equations (1.1) and (2.5) are invariant when f (or f^*) are replaced by $f - f(0)$ ($f^* - f^*(0)$). Thus we can suppose, without loss of generality, that $f(0) = 0$. In this case, using a result of Sobolev [5], it is easy to

see that $f(w) \in K(m,k)$ if $w \in K(m,k)$ and $f \in C^{\ell}(\mathbb{R})$, where $\ell = \max(m,k)$. Then we have:

Theorem 4.2. In the case that $f \in C^{\ell}(\mathbb{R})$, $g \in K(m,k)$, $u_0 \in P_{k+1}$ and $T < \infty$, the solution u of (2.5)" (or(2.5)) is such that

$$(4.10) \quad u \in K(m+1,k+1).$$

Proof. Since $K(m,k) \subset K(0,0)$ and $P_{k+1} \subset P_1$ there exists a function $u \in K(1,1)$ which is a solution of

$$(4.11) \quad \begin{aligned} (u_t(t), v) - (u(t), v_x) + (u_{xt}(t), v_x) &= \\ &= (f(u(t)), v_x) + (g(t), v), \quad \forall v \in P_1. \end{aligned}$$

If $m = k = 0$ there is nothing to do. If $m, k > 0$, let $h = f(u(t))$. Since $u \in K(1,1)$ so does h . Then, by Theorem 3.2, $u \in K(2,2)$. This reasoning can be repeated until $h \in K(m,k)$, showing that $u \in K(m+1,k+1)$.

Remark 4. If $g \in K(m,k) \cap L^1(P_k)$ the result of Theorem 4.2 remains true for $T = \infty$ because Theorem 3.2 can be used when $T = \infty$ for $f(w) \in L^1(P_k)$ and, if $f(0) = 0$ and $f \in C^k(\mathbb{R})$, $w \in L^1(P_k)$ implies that $f(w) \in L^1(P_k)$.

Remark 5. With a slight modification in the argument of Theorem 3.2, one can weaken the hypothesis on g , on both regularity theorems, by requiring that $g \in K(m,k-1)$.

Concluding, we observe that the weak solutions for $m > 0$ and $k > 1$ are indeed classical solutions.

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