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ABSTRACT

A static spherical distribution of incoherent matter which is a source of Yukawa field is considered, in equilibrium under its gravitational attraction and short range repulsion. Numerical solutions of the full Einstein-Yukawa equations are obtained. The stability of the system under various degrees of concentration is discussed, and the impossibility of static configurarion of massless Yukawa charges under its self gravitation is deduced.

1. INTRODUCTION

It is a general belief that gravity is the only interaction present in any physical system; however, its attractive effect has to be balanced by some kind of repulsive interaction in order to prevent collapse. Long range fields have been tried[1,2], like Coulomb and repulsive scalar fields, but the resulting system proved either unstable or insensitive to each other. Quantum effects are of course essential for microscopic objects, or even for macroscopic ones like neutron stars, where the kinetic energy of constituents due to the Pauli exclusion principle plays an important role; even though, it seems worthwhile to investigate from the purely classical viewpoint the role of the interaction between gravitation and short range fields in the formation and stability of elementary systems.

A simple system was recently considered[3]. It is a static sphere of incoherent dust, which is assumed to be, at the same time, a source of gravitation and of a short range repulsive Yukawa field. It is shown that the linearized solutions of the Einstein-scalar equations are free from the gravitational instability. Since an important feature of General Relativity is exactly its nonlinear character, more interesting results can be expected in the limit of strong nonlinear fields. It is the purpose of this paper to study that system in its nonlinear limits.

Analytic solutions of Einstein equations involving short range scalar fields have not been obtained; we then look for numerical solutions, taking advantage of the fact that only two dimensionless parameters are sufficient for characterizing our system; the parameters are related to the central density of matter and the ratio of the two coupling constants. It is shown that the radius of the distribution should be determined by the boundary condition to the scalar field which consists the eigen value problem for the radius for a given set of the parameters. Some results are demonstrated, and discussed. It is also shown that the pure Yukawa field can not have a stable static configu ration under the gravitation.

2. FIELD EQUATIONS

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We start from the Einstein scalar equations[3]

$$R^{\mu}_{\nu} - \frac{1}{2} R\, \delta^{\mu}_{\nu} = -2\,\varepsilon\, c^{-2}\, T^{\mu}_{\nu} \quad , \quad \varepsilon = 4\pi G/c^2 \quad , \tag{1}$$

$$S^{;\mu}_{;\mu} + S/\ell^2 = f\,\varepsilon\,\rho \quad , \quad f^2 = \mathrm{const} > 1 \quad , \tag{2}$$

$$T^{\mu}_{\nu} = c^2 \rho u^{\mu} u_{\nu} - c^2\, \varepsilon^{-1} \left[S^{;\mu} S_{;\nu} + \frac{1}{2}\, \delta^{\mu}_{\nu} \left(S^2/\ell^2 - S^{;\alpha} S_{;\alpha} \right) \right] \quad , \tag{3}$$

where ρ is the matter density with velocity field u^{μ} and S is a repulsive scalar field with range ℓ ; as usual, the semicolon means the covariant derivative. f denotes the ratio of scalar field charge to gravitational charge of the matter.

As we consider a static, spherically symmetric distribution of matter, we may write the line element as

$$ds^2 = e^{2\eta} (dx^0)^2 - e^{2\alpha}\, dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 \quad ; \tag{4}$$

the functions ρ, S, η, α depend only on r.

In the interior region ($r \leq R$) the above equations reduce to

$$\eta' = - fS' \quad , \tag{5}$$

$$\varepsilon \ell^2 \rho = S(S+f)(f^2-1)^{-1} \quad , \tag{6}$$

$$xS' = f - \left[f^2 - 1 + (1 + x^2 S^2) e^{2\alpha} \right]^{1/2} \quad , \tag{7}$$

$$\alpha' = S'(f - xS') + xS(f+S)(f^2-1)^{-1}\, e^{2\alpha} \quad , \tag{8}$$

where $x = r/\ell$ and a prime means d/dx. For definiteness we considered $f > 1$; a change of sign in S and S' is required for $f < -1$, in these equations.

For the exterior region ($r > R$), where $\rho = 0$, the equations are

$$\eta' = - xS' - \alpha' \quad , \tag{9}$$

$$S'' = Se^{2\alpha} - x^{-1} S' \left[1+(1 + x^2S^2)e^{2\alpha}\right] \quad , \tag{10}$$

$$2x\alpha' = 1 - x^2S'^2 - (1 + x^2S^2)e^{2\alpha} \quad . \tag{11}$$

3. SOLUTION OF EQUATIONS

The coupled ordinary differential equations (7) and (8) for $S(x)$ and $\alpha(x)$ can be numerically integrated when the initial conditions are given. A simple analysis shows that $\alpha = \alpha' = S' = 0$ at $x = 0$; we then fix a value for the parameter f, and also an initial value S_0 for $S(0)$ and start the integration from the origin to outwards.

For a given initial condition, the radius R should be determined uniquely. To find the value of R we first proceed the numerical integration of interior eqs. (7) and (8) up to a certain test radius $r = r_t$; for $r > r_t$ we switch to exterior eqs. (10) and (11). We impose the continuity of α, S and S' through $r = r_t$, and we also impose that S vanishes at infinity. We look for the correct value $r_t = R$ by iteration,which safisfies the above condition.

In possession of $S(x)$ and $\alpha(x)$ we can now obtain the exterior solution for $\eta(x)$ from (9); we impose the asymptotic condition $\eta(\infty) = 0$ which determines the integral constant. The interior solution for $\eta(x)$ is obtained from (5) and must be continuous at $r = R$; the continuity of its radial derivative through $r = R$ follows automatically from the continuity of α, S and S', as can be seen from the following general expression, valid for both interior and exterior regions,

$$2x\eta' = (1+x^2S^2)e^{2\alpha} - (1+x^2S'^2) \quad . \tag{12}$$

4. RESULT AND DISCUSSION

We present now the results obtained corresponding to several sets of values of f and S_0.

In case f = 1.1 and $S_0 = 10^{-5}$ (Fig. 1) the solution almost coincides with that of linearized equations, as expected. The matter density $\rho(x)$ is essentially $x^{-1}\sin x$, which shows a maximum finite value around the center and decreases monotonically to the boundary $R \simeq 0.9\ell$ of the spheres; we plotted $\bar{\rho}/10$, where

$$\bar{\rho}(x) = \varepsilon\ell^2 \rho(x) \tag{13}$$

is a dimensionless quantity. The also dimensionless scalar field S(x) starts from the maximum pre-assigned value $S_0 = 10^{-5}$ on the origin and decreases monotonically to zero at infinity; in the exterior region (x > 0.9) it presents the usual Yukawa behaviour $x^{-1}e^{-x}$. The dimensionless gravitational potential $\eta(x)$ shows the well known pattern (we plotted its negative in order to save space), with maximum slope close to the boundary x = 0.9; in this weak field limit ($S^2 \ll \bar{\rho} \ll 1$) we can relate $\eta(x)$ with the Newtonian potential $\Phi(x)$ produced by the matter density $\rho(x)$,

$$\eta(x) = c^{-2}\Phi(x) \qquad ; \tag{14}$$

these potentials η and Φ then approximately present the usual x^{-1} behaviour for x > 0.9. Finally the metric potential $\alpha(x)$ is related to the ratio between radial physical lengths and the corresponding radial coordinate intervals, $d\ell_{phys} = e^{\alpha}dr$; since $\alpha(x)$ is positive from the center till infinity, all physical radial distances are numerically larger than the corresponding radial coordinate intervals. One finds that $\alpha(x)$ has a parabolic (x^2) behaviour near the origin followed by a slight bending rightwards before reaching the boundary $x = 0.9$. On this boundary α is continuous, but its x-derivative has a discontinuity

$$\alpha'_{int} - \alpha'_{ext} = \varepsilon\ell\, R\rho(R)\, e^{2\alpha(R)} \quad , \tag{15}$$

as can be shown from eqs. (6), (7), (8) and (11). For $x > 0.9$ one finds that $\alpha(x)$ closely follows $-\eta(x)$ in this weak field limit, as it should in a Schwarzschild exterior solution. The gravitational mass of the whole system, as defined by

$$m = -\frac{c^2}{G} \lim_{r\to\infty} \left[r\eta(r) \right] \tag{16}$$

is 0.87×10^{-5} in units $c^2\ell/G$.

We next consider the case $f = 1.1$, as before, but with $S_0 = 3$ (Fig. 2); this is no longer a weak field solution. The density of matter $\rho(x)$ still shows a larger concentration on the origin, and dilutes monotonically to a nonzero value at the boundary $R = 0.32\ell$. We note the diffused property of the distribution near the surface region which is not observed in the weak field limit. Potentials $\eta(x)$ and $S(x)$ have a behaviour similar to that of the previous case. A somewhat different pattern, however, is presented by the metric potential $\alpha(x)$: it still has a parabolic (x^2) behaviour near the center and reaches a maximum in the region of maximum radial derivative of the gravitational potential (η'), but it now decreases in the tail region of matter distribution. There is a discontinuity of slope (15) on the boundary of the sphere; for increasing x the potential $\alpha(x)$ gradually approaches the Newton-Schwarzschild hyperbolic (x^{-1}) behaviour, since the scalar field density $S^2(x)$ tends to zero exponentially.

It is worthwhile to stress the coincidence of regions in which the material system presents maximum gravitation (as given by η') and maximum dilatation of the physical radial distances (as given by α). It is also interesting to remark that the maximum gravitation occurs in the interior ($x \simeq 0.2 < R/\ell$) of the sphere; this is a consequence of the faint concentration of the outermost shells. The gravitational mass of the system, as defined by (16) is $0.11\ c^2\ell/G$.

We finally consider the case $f = 5$ and $S_o = 1$ (Fig. 3); a few preliminary words are necessary to understand the peculiar situation found in this case. It is known[4] that the "effective energy density" that produces a static gravitational field is $2T^o_o - T$, which in our system is proportional to $\bar{\rho} + S^2$. In the previous two cases the major contribution to the attractive gravitational effects came from the matter density $\rho(x)$, but in the present case the main contribution comes from $S^2(x)$. A trivial calculation starting from (6) shows that one always has $\bar{\rho}(x) \leq S^2(x)$ in situations where $1 \leq (f^2-2)f^{-1}S(x)$; in the present case ($f = 5$) we then have predominance of S^2 contribution in regions where $S(x) > 0.22$, that is, from the center of symmetry where $\bar{\rho} = 0.25$, $S_o^2 = 1$ to the radius given by $x = 13.7$ as can be seen in Fig. 3. Another interesting question concerns the metric potential $\alpha(x)$; in the two previous cases we found a positive parabolic behaviour near the origin. Indeed, a few calculations starting from (7) and (8) show that one always has near the center

$$\lim_{x\to o} \left[x^{-2}\alpha(x)\right] = \frac{1}{6}\, S_o(f^2-1)^{-1} \left[2f-(f^2-3)S_o\right] \quad ; \tag{17}$$

we then have negative values for $\alpha(x)$ in the innermost shells when $2 < (f^2-3)f^{-1}S_o$. That is what happens in the present case ($f = 5$, $S_o = 1$); the metric potential $\alpha(x)$ starts from the zero value on the origin and assumes negative values with increasing x, with a minimum in the region where the gravitational potential $\eta(x)$ shows a minimum derivative; for $x > 13.7$ one finds positive values for $\alpha(x)$, with a maximum near the boundary $x = 15.8$ of the sphere, a region where $\eta(x)$ presents a maximum radial derivative; with increasing x in the external region the two functions $\alpha(x)$ and $-\eta(x)$ asymptotically coalesce as originated by a mass $m = 7.63\ c^2\ell/G$.

For a better understanding of the behaviour of a system under various degrees of concentration, we plotted in Fig. 4 the gravitational mass m (Schwarzschild mass), the proper mass m_o and the ratio of the binding energy to the total proper mass as functions of the central value of Yukawa field. The value 1.2 is chosen for f. The proper mass (invariant mass) m_o is defined[5] by

$$m_o = 4\pi \int_0^R \rho e^{\alpha} r^2 dr \quad , \qquad (18)$$

and the binding energy is

$$B = (m_o - m)c^2 \quad , \qquad (19)$$

where m is the gravitational mass defined in (16).

All these quantities increase almost linearly in log-log scale up to $\log S_o \simeq 0.5 \times 10^{-2}$ (non-relativistic region), then bend down in the relativistic region. For very high central values of S_o($\log S_o > 5$), the numerical procedure fails due to the computational difficulty.

It is interesting to note that the gravitational mass m has a maximum at $\log S_o \simeq 0.8$. A similar situation is well known in the case of neutron star models when the gravitational mass is plotted against the central density of neutron star[6]. In the latter case the existence of maximum mass is related to the gravitational instability, and solutions with the central density higher than this maximum point are unstable against collapse. In analogy to the above, it seems that the solutions of our system with $\log S_0 > 0.8$ are unstable for the value $f = 1.2$. However we should note that the proper mass does not have a maximum in contrast to the neutron star models. The ratio of the binding energy to the proper mass seems to increase monotonically with S_0, tending to unity. Around the maximum of m, the binding energy reaches about 60% of the total proper mass.

The problem of instability of the Yukawa system becomes clear in the following limit. Let us introduce the "number density" n of the source of Yukawa field by

$$n = \frac{1}{m_0}\rho \quad , \tag{20}$$

where m_0 is the proper mass of the source. The charge of Yukawa field is then

$$q = f\, m_0 \quad . \tag{21}$$

Now consider the limit

$$f \to \infty \quad , \quad m_0 \to 0, \quad q = \text{const} \quad ; \tag{22}$$

this limit corresponds physically to a system of massless sources of Yukawa field under the gravitation created by the field itself. Using a new variable u(x) defined by

$$e^{-2\alpha} = 1 - \frac{u}{x} \tag{23}$$

we get the field equations

$$S' = 0 \quad , \tag{24}$$

$$u' = - x^2 S^2 \quad , \tag{25}$$

$$\eta' = \frac{1}{2} e^{2\alpha} \left(\frac{u}{x^2} + xS^2\right) \quad , \tag{26}$$

$$\varepsilon \ell^2 q n = S \quad , \tag{27}$$

for interior region. For exterior region the equations are the same as before and u satisfies

$$u' = -x^2 (e^{-2\alpha} S'^2 + S^2) \ . \tag{28}$$

The internal equations are easy to integrate; the solution is

$$S = \text{const} = S_o \quad , \tag{29}$$

$$u = - \frac{x^3}{3} S_o^2 \quad , \tag{30}$$

$$\eta = - \alpha + \text{const} \quad , \tag{31}$$

$$n = \frac{S_o}{\varepsilon \ell^2 q} \ . \tag{32}$$

In the exterior region, the derivative of u is negative definite and, for $x = R/\ell$, $u < 0$. Thus u is always negative. This means that the metric never tends to the Schwarzschild behaviour, in which case u is finite and positive.

On the other hand we see from eq. (10) that $S'' > 0$ at $x = R/\ell$, since one always has $S' \leq 0$; this means the scalar field, which is constant in the interior region, begins to increase for $x > R/\ell$.

From the above observations we conclude that there is no static configuration of the system of massless sources of Yukawa field under its self gravitation.

Our Yukawa field has no self interaction. It may be of great interest to study the system of pure Yukawa field with self interaction under the gravitation.

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CAPTIONS FOR THE FIGURES

Fig. 1 - case $f = 1.1$, $S_o = 10^{-5}$.
The scalar field S, dimensionless matter density $\bar{\rho} = \varepsilon \ell^2 \rho$, gravitational potentials α and the negative of η as functions of radial variable $x = r/\ell$.

Fig. 2 - case $f = 1.1$, $S_o = 3$.
The scalar field S, matter density $\bar{\rho}$, gravitational potentials α and $-\eta$ as functions of x.

Fig. 3 - case $f = 5$, $S_o = 1$.
The quantities S, $\bar{\rho}$, α and $-\eta$ as functions of x.

Fig. 4 - The proper mass m_o, the Schwarzschild mass m and the ratio of the binding energy $m_o - m$ to the proper mass (in units $c = 1$) as functions of the central value S_o of Yukawa field, for $f = 1.2$. Log-log scale is used.

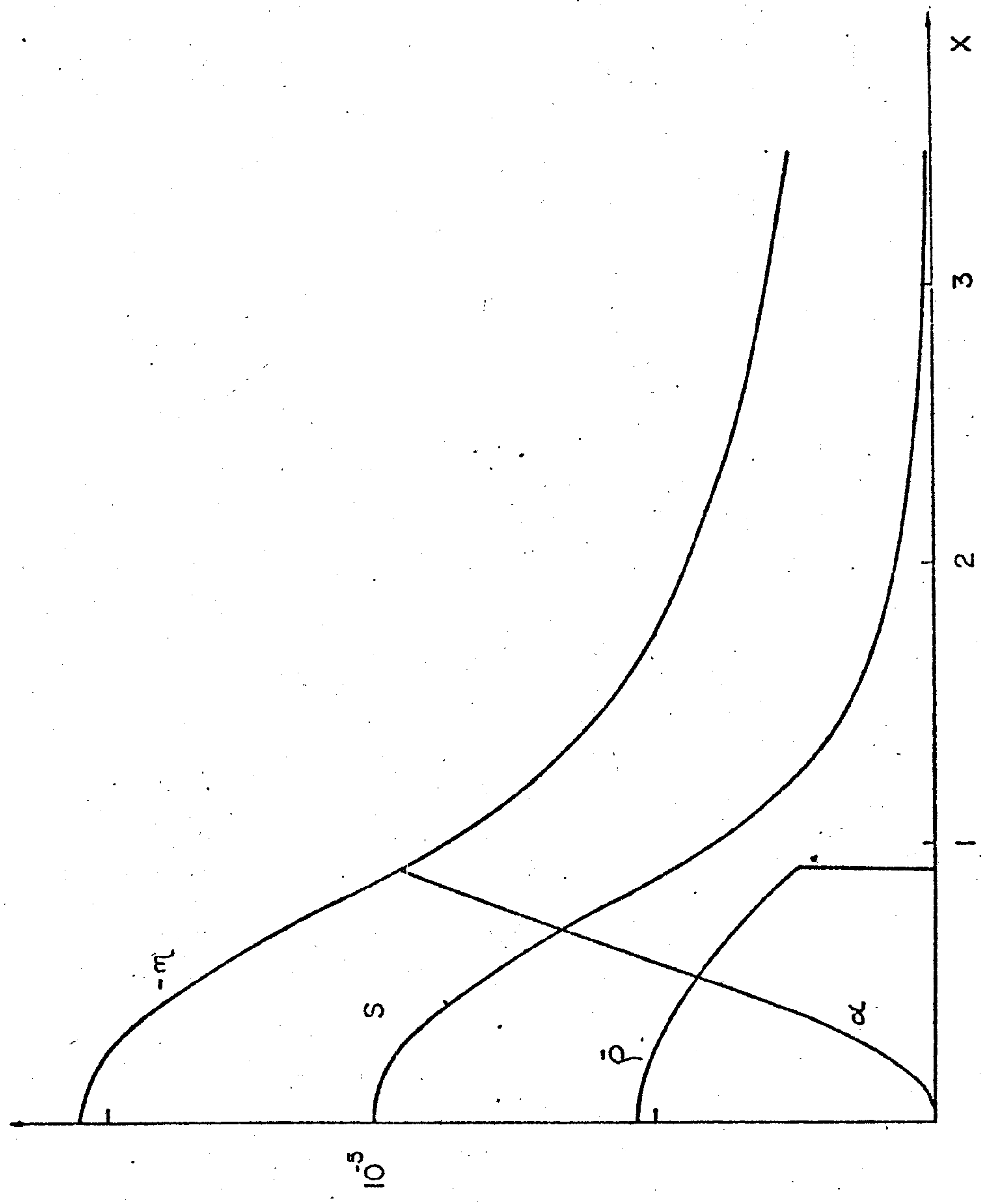

FIG. 1

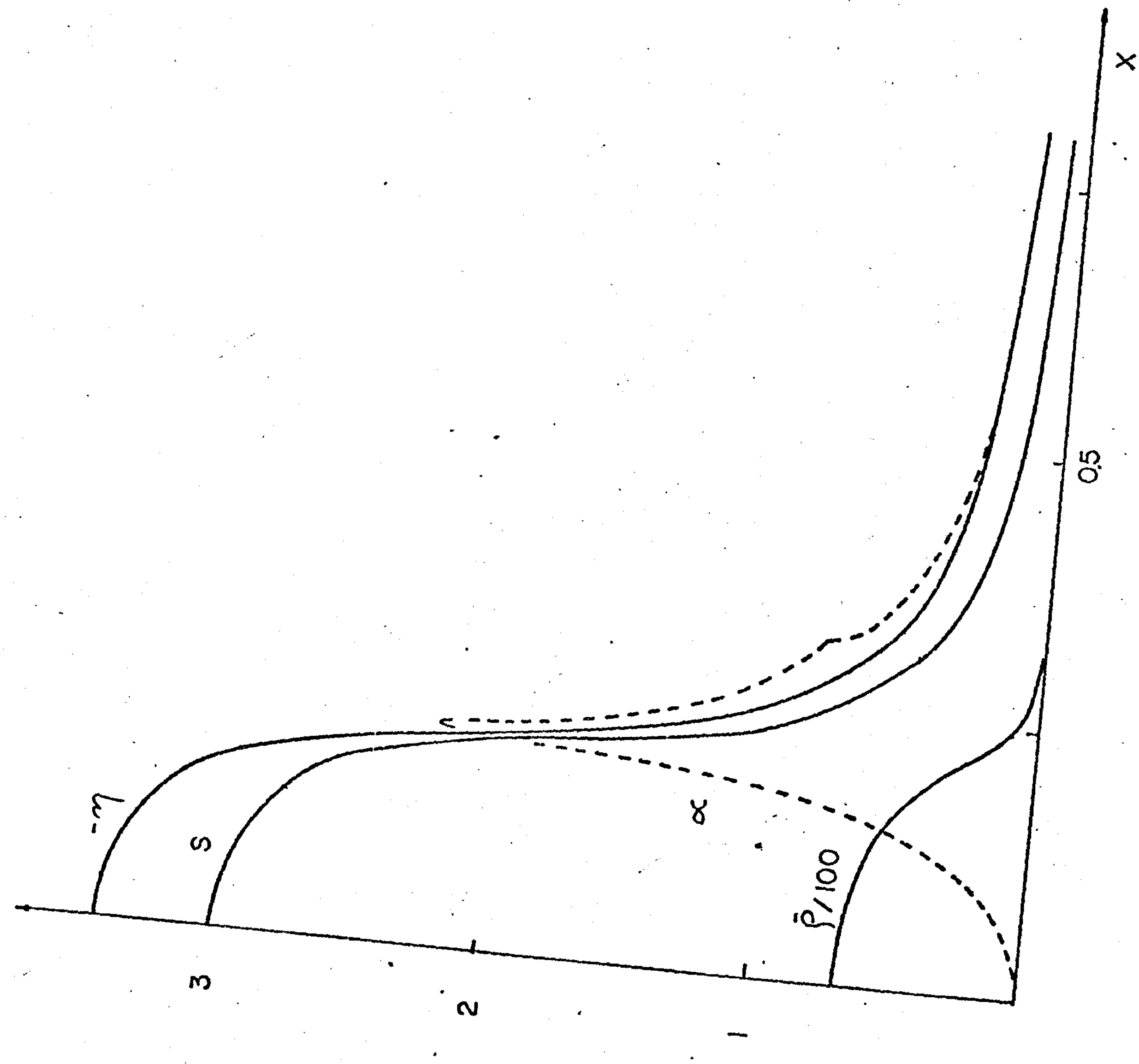

FIG. 2

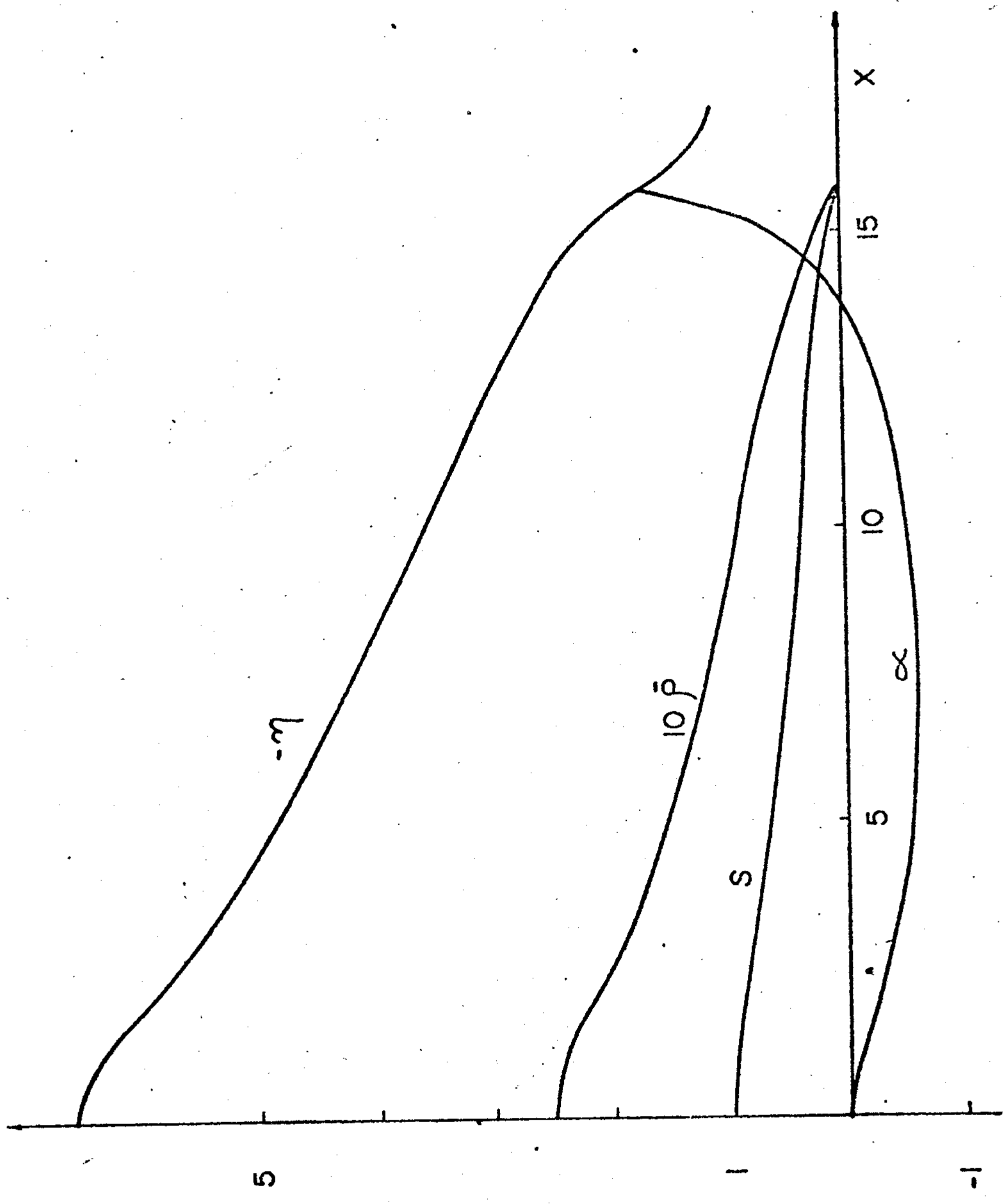

FIG. 3

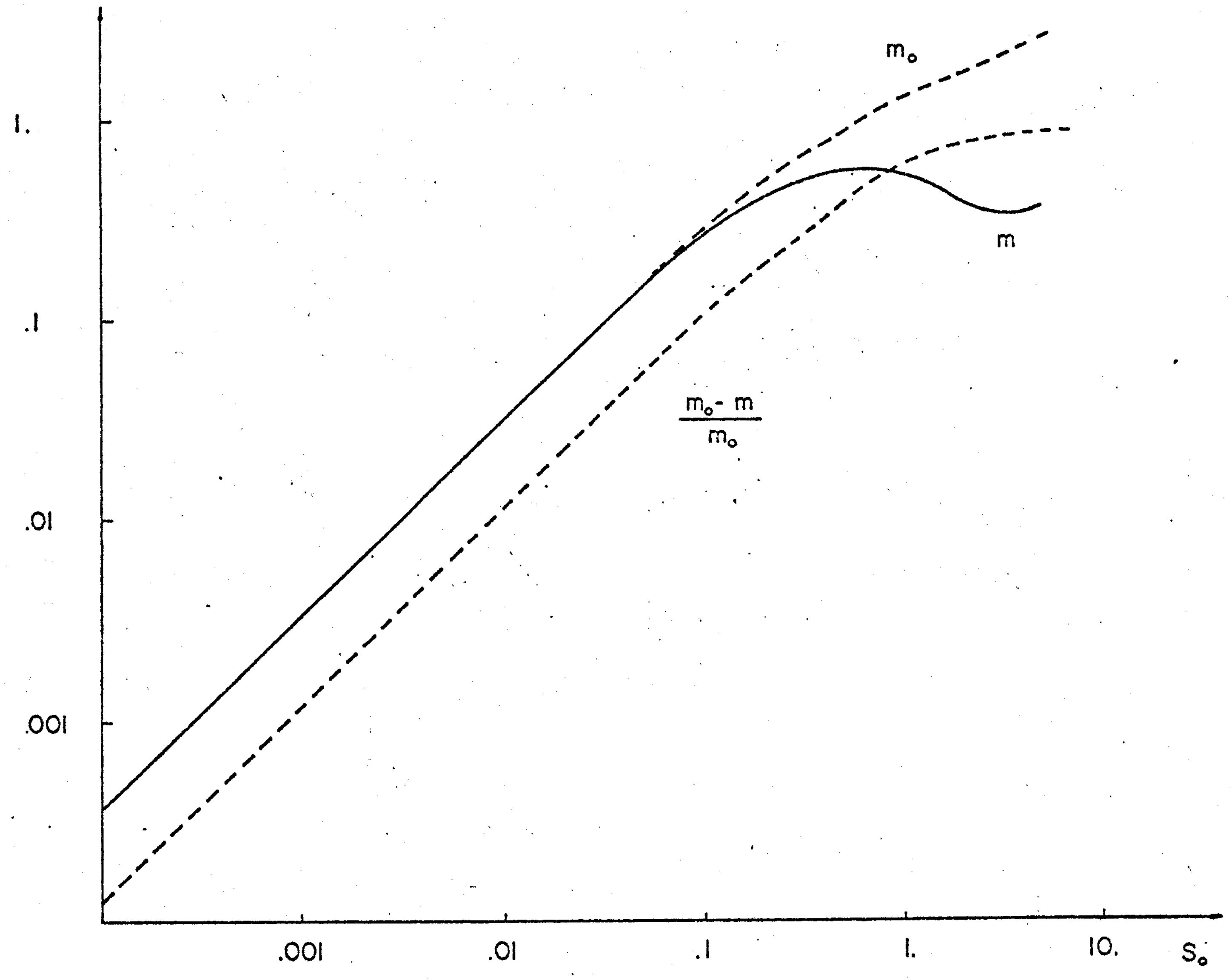

FIG. 4