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EXACT RELATIVISTIC SOLUTION OF DISORDERED RADIATION WITH PLANE SYMMETRY*

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ABSTRACT

An exact solution of Einstein equations corresponding to an equilibrium distribution of disordered electromagnetic radiation with plane symmetry is obtained. This equilibrium is due solely to the gravitational and pressure effects inherent to the radiation. The distribution of radiation is found to be maximum and finite at the plane of symmetry, and to decrease monotonically in directions normal to this plane.
The solution tends asymptotically to the static plane symmetric vacuum solution obtained by Levi-Civita. Timelike and null geodesics are discussed.

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1. INTRODUCTION

One of the most fascinating physical systems described by the Einstein-Maxwell theory is that of an electromagnetic radiation evolving only under the influence of its own gravitation. Situations might occur in which the gravitational attraction associated to the energy density of the radiation were strong enough to compensate the corresponding pressure; this radiation in equilibrium would then not need any recipient or walls to be confined.

Tolman (1934a) seems to have first provided the sufficient mathematical apparatus for studying that possibility; he explained the circumstances under which a radiation may be treated as a special case of a perfect fluid. And Klein (1948) first applied Tolman's results to a cosmological situation, he studied a spherically symmetric distribution of disordered electromagnetic radiation in equilibrium. He was able to find only an approximate solution, which he presented as a set of series expansins in terms of his dimensionless radial variable $\kappa p_o r^2$.

In the present paper we obtained the exact solution of an unbounded plane symmetric distribution of disordered radiation in equilibrium. Similarly to Klein's sphere our slab distribution shows a larger condensation in the innermost regions, and dilutes monotonically to a vanishing distribution outwards; in the asymptotic regions our solution goes to Levi-Civita's (1918) plane symmetric vacuum solution. Some time-like and null geodesics are discussed.

2. GENERAL EQUATIONS

We start with the static and plane symmetric line element

$$ds^2 = e^{2\alpha}(dx^0)^2 - e^{2\beta}\, dx^2 - e^{\beta-\alpha}(dy^2+dz^2) \quad , \qquad (1)$$

where α and β are functions of x alone; the corresponding non-zero Christoffel symbols of the second kind are

$$\left\{ {0 \atop 01} \right\} = \alpha_1 \quad , \quad \left\{ {1 \atop 00} \right\} = \alpha_1\, e^{2\alpha-2\beta} \quad , \quad \left\{ {1 \atop 11} \right\} = \beta_1 \quad ,$$

$$\left\{ {1 \atop 22} \right\} = \left\{ {1 \atop 33} \right\} = \frac{1}{2}\,(\alpha_1-\beta_1)e^{-\alpha-\beta} \quad , \quad \left\{ {2 \atop 12} \right\} = \left\{ {3 \atop 13} \right\} = \frac{1}{2}(\beta_1-\alpha_1) \quad , \qquad (2)$$

where a subscript 1 means d/dx. And the surviving components of the Ricci tensor are

$$R^0_0 = - e^{-2\beta}\alpha_{11} \quad , \qquad (3)$$

$$R^1_1 = - \left[\beta_{11} + \frac{1}{2}(3\alpha_1+\beta_1)(\alpha_1-\beta_1)\right] e^{-2\beta} \quad , \qquad (4)$$

$$R^2_2 = R^3_3 = \frac{1}{2}(\alpha_{11}-\beta_{11})\, e^{-2\beta} \quad . \qquad (5)$$

Perfect fluids are systems with energy momentum

$$T^\mu_\nu = (\rho c^2 + p)u^\mu u_\nu - p \quad \delta^\mu_\nu \quad , \qquad (6)$$

where ρc^2, p and u^μ are the rest energy density, the pressure and the macroscopic velocity field of the fluid; this last quantity must satisfy $u^\nu u_\nu = 1$. We consider a particular kind of perfect fluid, with equation of state $\rho c^2 = 3\,p$ (Tolman 1934a); for such a fluid with plane symmetry and in static condition ($u^1=u^2=u^3=0$) we have

$$T^{\mu}_{\nu} = p(x) \quad \text{diag}\,(3, -1, -1, -1). \tag{7}$$

Then the Einstein equations

$$R^{\mu}_{\nu} = -\kappa\left[T^{\mu}_{\nu} - \frac{1}{2}\,\delta^{\mu}_{\nu}T\right] \quad , \qquad \kappa = 8\pi G/c^4 \tag{8}$$

reduce to the three equations

$$e^{-2\beta}\alpha_{11} = 3\kappa p \quad , \tag{9}$$

$$\left[\beta_{11} + \frac{1}{2}(3\alpha_1+\beta_1)(\alpha_1-\beta_1)\right]e^{-2\beta} = -\kappa p \ , \tag{10}$$

$$(\alpha_{11}-\beta_{11})\ e^{-2\beta} = 2\kappa p \ . \tag{11}$$

And the contracted Bianchi identity gives the relation

$$p_1 + 4p\alpha_1 = 0 \ . \tag{12}$$

3. SOLUTION OF EQUATIONS

From (9) and (11) we easily obtain $3\beta = a - bx + \alpha$, where <u>a</u> and <u>b</u> are constants of integration. We are free to impose, as boundary conditions, that $g_{00} = -g_{xx} = -g_{yy} = -g_{zz} = 1$ on the plane $x = 0$; from (1) one finds then that $\alpha(0) = \beta(0) = 0$. So with the constant $a = 0$ we have

$$3\beta = \alpha - bx \quad , \quad b = \text{const.} \tag{13}$$

From (9), (10) and (13) we obtain the equation

$$12\alpha_{11} + (10\alpha_1 - b)(2\alpha_1 + b) = 0 \quad , \tag{14}$$

whose solution is

$$10\alpha = -\,5bx + 6\log(c+d\ e^{bx}) \quad , \tag{15}$$

with c and d constants of integration. We want to avoid solutions corresponding to surface density of matter concentrated on the plane x = 0; and we also require that our system present mirror symmetry with respect to that plane. We then impose as another boundary condition that the normal derivative of the metric coefficient g_{00} be zero on the plane x = 0. So with $\alpha(0) = \alpha_1(0) = 0$ we obtain from (15)

$$c = 1/6 \quad , \quad d = 5/6 \quad . \tag{16}$$

The pressure p can now be easily obtained from (9) : the result satisfies the relation (12) and is

$$p = (b^2/36\kappa)\; e^{-4\alpha} \; . \tag{17}$$

If we call $b^2/36\kappa = p_0$ our results become

$$ds^2 = f^3\, e^{-\xi}(dx^0)^2 - f\, e^{-\xi}dx^2 - f^{-1}(dy^2+dz^2), \tag{18}$$

$$p = p_0\; f^{-6}\; e^{2\xi} \quad , \tag{19}$$

where

$$f(\xi) = \left[\frac{1}{6}(1+5e^{\xi})\right]^{2/5} \quad , \quad \xi(x)=6(\kappa p_0 x^2)^{1/2} \geq 0. \tag{20}$$

In regions close to the central plane (x=0) we have the approximate values

$$g_{00} = 1+\xi^2/12 \quad , \quad p = p_0(1-\xi^2/6) \quad , \quad \xi << 1 \quad ; \tag{21}$$

these results will be used in connection with some special geodesics in the next Section.

In studying the properties of the system in regions far from the central plane x = 0 one finds more appropriate the dashed coordinates

$$x^{0'} = (5/6)^{3/5}\, x^0 \quad , \quad x' = (5/6)^{1/5}\, x \quad , \quad y' = (5/6)^{-1/5}\, y \quad ,$$
$$z' = (5/6)^{-1/5}\, z \; ; \tag{22}$$

in terms of these dashed coordinates the exact solution becomes

$$ds^2 = h^3\, e^{-\eta}(dx^{0'})^2 - he^{-\eta}dx'^2 - h^{-1}(dy'^2+dz'^2) \quad , \tag{23}$$

$$p = q\, h^{-6}\, e^{2\eta} \quad , \tag{24}$$

where

$$h(\eta) = (e^{\eta}+1/5)^{2/5} \quad , \quad \eta(x') = 5(\kappa q x'^2)^{1/2} \geq 0 \; , \; q = p_0(6/5)^{12/5} \tag{25}$$

Then in regions far from the plane $x'=0$ we have the approximate (asymptotic) solution

$$ds^2 = e^{\eta/5}(dx^{0'})^2 - e^{-3\eta/5}\, dx'^2 - e^{-2\eta/5}(dy'^2+dz'^2) \quad , \tag{26}$$

$$p = 0 \quad , \quad \eta \gg 1 \quad ;$$

these results will be discussed later, in connection with the exact Levi-Civita plane symmetric static vacuum solution.

Before closing this Section we evaluate the energy content of our system, per unit area on the plane $x' = 0$. We start from the expression of the energy content of a volume element (Tolman 1934b)

$$d^3E = (-g)^{1/2}(2T_0^0 - T)\; dx'\; dy'\; dz' \quad , \quad g = \det g_{\mu\nu} \quad ; \tag{27}$$

for our fluid (7) with line element (23) and pressure (24) we have

$$d^3E/dy'dz' = 6\; q\; h^{-5}\; e^{\eta}\; dx' \quad . \tag{28}$$

Integrating this differential $d\varepsilon'$ of the surface density of energy between two planes $x' = \pm$ const we get

$$\varepsilon'(\eta) = (4q/\kappa)^{1/2}\,(e^{\eta}-1)(e^{\eta}+1/5)^{-1} \quad ; \qquad (29)$$

for $|x'| = \infty = \eta$ we obtain for the surface density of energy a finite value

$$\varepsilon' = (4q/\kappa)^{1/2} \quad . \qquad (30)$$

This result will also be discussed later, in connection with the Levi-Civita solution.

4. TIMELIKE GEODESICS

In the geodesic equations for a test particle

$$du^{\mu}/ds + \left\{ {\mu \atop \nu\rho} \right\} u^{\nu}u^{\rho} = 0 \qquad (31)$$

we use the Christoffel symbols (2) and obtain

$$du^0/ds + 2\alpha_1\, u^0\, u^1 = 0 \quad , \qquad (32)$$

$$du^1/ds+\alpha_1 e^{2\alpha-2\beta}(u^0)^2 + \beta_1(u^1)^2 + \tfrac{1}{2}(\alpha_1-\beta_1)e^{-\alpha-\beta}\left[(u^2)^2+(u^3)^2\right] = 0, \qquad (33)$$

$$du^2/ds + (\beta_1-\alpha_1)u^1u^2 = 0 \quad , \qquad (34)$$

$$du^3/ds + (\beta_1-\alpha_1)u^1u^3 = 0 \quad . \qquad (35)$$

These equations are not independent, since we must have $u^{\nu}u_{\nu} = 1$; we can easily obtain the first integrals

$$u^0 = D^2\, e^{-2\alpha} \quad , \quad u^2 = B\, e^{\alpha-\beta} \quad , \quad u^3 = C\, e^{\alpha-\beta} \quad , \qquad (36)$$

$$(u^1)^2 = \left[D^4 e^{-2\alpha} - 1 - (B^2+C^2)e^{\alpha-\beta}\right]e^{-2\beta} \quad . \qquad (37)$$

The three constants D^2, B, C are related to the three components of a given "initial" velocity of the test particle; curiously the three covariant components of the velocity

$$u_0 = D^2 \quad , \quad u_2 = - B \quad , \quad u_3 = - C \tag{38}$$

remain constant along the motion of each test particle, only the covariant component u_1 varies along the motion. These results are valid for all plane symmetric static systems with line element (1).

In view of the difficulty in obtaining the subsequent integrals of (36) and (37) with our line element (18) we only consider the motions of test particles with velocities small in comparison with that of light, and in regions not far from the central plane x = 0. In other words, we take the velocity parameters B^2, C^2, D^2-1 and the distance variable ξ all very small. We then obtain from (36) and (37) with the line element (18)

$$dx^0/ds \simeq (1-3\kappa p_0 x^2)D^2 \simeq 1 \quad , \quad dy/ds \simeq B \quad , \quad dz/ds \simeq C \; , \tag{39}$$

$$(dx/ds)^2 \simeq D^4-1-B^2-C^2-3\kappa p_0 x^2 \; . \tag{40}$$

These equations can now be easily integrated; we call x^0 = ct and obtain the approximate (non-relativistic) timelike geodesics

$$dx/dt \simeq cA \sin \omega t \quad ; \quad dy/dt \simeq cB \quad , \quad dz/dt \simeq cC \quad , \tag{41}$$

where

$$A^2 = D^4 - 1 - B^2 - C^2 \quad , \quad \omega = (3\kappa c^2 \, p_0)^{1/2} \; . \tag{42}$$

Inese geodesics represent sinusoidal motions on planes normal to the plane x = 0, and with nodes on this plane x = 0.

5. NULL GEODESICS

Nullgeodesics are also obtained from (31), but now $u^{\mu}u_{\mu} = 0$. A first integral is then

$$dy/dx^0 = B\ e^{3\alpha-\beta} \quad , \qquad dz/dx^0 = C\ e^{3\alpha-\beta} \quad , \qquad (43)$$

$$(dx/dx^0)^2 = e^{2\alpha-2\beta} - (B^2+C^2)e^{5\alpha-3\beta} \quad ; \qquad (44)$$

the two constants B and C are related to a given initial direc tion of the null geodesic.

Let us consider a light ray travelling in the plane z = 0. Making then the constant C = 0 in (43) and (44) we obtain for the trajectory in the (x,y) plane the equation

$$(dy/dx)^2 = B^2\ f^3\ e^{-2\xi}(1-B^2f^4e^{-\xi})^{-1} \quad , \qquad (45)$$

where we used the line element (18). One finds that

$$B^2 = \sin^2\nu \qquad (46)$$

where ν is the angle of incidence of the ray on the plane x=0 , where $\xi = 0$ and f = 1. After crossing this central plane the ray travels outwards until it reaches a maximum distance from the plane x = 0; this distance is given by dx/dy = 0, or

$$e^{\xi/2}\ f^{-2} = \sin\nu \quad . \qquad (47)$$

After having reached this distance the ray proceeds inwards with identical characteristics. For large angles of incidence on the plane x = 0 ($\nu \simeq \pi/2$, $B^2 \simeq 1$) one finds from (47) that the maximum distance reached by the ray is given by

$$\xi_{max} \simeq 3(\pi/2 - \nu)^2 << 1 \quad ; \qquad (48)$$

and for almost normal incidences ($\nu \simeq 0$, $B \simeq 0$) we again obtain from (47)

$$\xi_{max} \simeq -(10/3)\log\nu >> 1 \ . \qquad (49)$$

For an incidence of 45º some computation gives

$$\xi_{max} \simeq 1.5 \quad , \quad \nu = \pi/4 \ . \qquad (50)$$

6. DISCUSSIONS

We obtained the exact unbounded solution given by General Relativity to a class ($\rho c^2 = 3p$) of perfect fluids with plane symmetry, in static condition; two physical examples of such fluids are disordered distributions of electromagnetic radiation (Tolman 1934a), and disordered distributions of neutrinos (Klein 1948). Also distributions of colliding particles with randomly oriented ultrarelativistic velocities can be described, in first approximation, in terms of that class of fluids (Klein 1948).

We found that the gravitational attraction associated to the energy density of these fluids is strong enough to compensate the repulsion produced by the corresponding pressure. It is then possible to have an isotropic electromagnetic radiation bound together in a static equilibrium configuration solely due to its own gravitation.

We have defined (2) our "working x-variable" ξ in a way such that $\xi(x) = \xi(-x)$; this ensured the mirror symmetry of the system across the plane $x = 0$, since the pressure and all metric coefficients have been expressed in terms of ξ. The same remark holds for $\eta(x')$.

The density of our plane symmetric fluid is maximum and finite on the central plane $x = 0$, and decreases monotonically to zero in both directions normal to this plane.

In our system the scalar curvature R^{μ}_{μ} vanishes every--where, however $R^{\mu}_{\nu}R^{\nu}_{\mu} = 12\ \kappa^2 p^2$ as can easily be obtained from (7) and (8); this quantity also is finite in the central plane and decreases monotonically to zero outwards.

In the central zone ($\xi<<1$) the density of the fluid is nearly uniform, as can be seen from (21); as a consequence we obtained the sinusoidal motion (41) for slowly moving test particles in that zone.

It is known from non-relativistic mechanics that an infinite slab of homogeneous fluid of density of mass μ pro - duces internal motions of test particles which are sinusoidal with frequency $\omega^2 = \kappa c^4 \mu/2$. If we compare this result with ours $\omega^2 = \kappa c^4 \rho$ obtained in (42) we find that our fluid with $T^0_0 = c^2\rho = 3p$ produces a gravitational field which at first approximation ressembles that produced by a homogeneous fluid with active mass density $\mu = 2\rho$. The same conclusion could be drawn from (27): since the trace T is zero one finds from that equation that the time component $T^0_0 = c^2\rho$ contributes twice to the energy of our fluid (Tolman 1934c).

It is also known from non-relativistic gravistatics that an infinite homogeneous slab with surface density of mass σ

produces an external acceleration field which is uniform and directed inwards, of strength $2\pi G\sigma$(or $\kappa c^4\sigma/4$); to this acceleration field it corresponds a Newtonian potential

$$\phi(x') = (\kappa c^4\sigma/4)|x'| \quad . \tag{51}$$

Newtonian potentials ϕ are often related to the metric coefficient g_{00} of relativistic descriptions according to $g_{00} = \exp(2\phi/c^2)$. Indeed, the exact Levi-Civita (1918) static vacuum solution with plane symmetry can be written as

$$ds^2 = e^{2\phi/c^2}(dx^{0'})^2 - e^{-6\phi/c^2}\, dx'^2 - e^{-4\phi/c^2}(dy'^2 + dz'^2), \tag{52}$$

with ϕ given in (51). One finds that this line element coincides with our asymptotic line element (26). Our surface density of energy $\varepsilon' = 2(q/\kappa)^{1/2}$ obtained in (30) coincides with the Levi-Civita's surface density of energy $c^2\sigma = 2(q/\kappa)^{1/2}$, obtained by comparing (25) and (26) with (51) and (52).

Again in non-relativistic gravistatics of usual perfect fluids one finds that the condition for local equilibrium is grad $p = -\mu'$ grad ϕ where p is the presure, $\mu' \gg p/c^2$ is the mass density and ϕ is the Newtonian potential. If we now compare this equation with the result $p_1 = -(4\rho/3)c^2\alpha_1$ stated in (12) we find that in a first approximation ($\phi = c^2\alpha$) our fluid behaves as a passive mass density $\mu' = 4\rho/3$. In usual incompressible fluids the contribution p/c^2 of the pressure p to the passive mass density $\mu' = \rho + p/c^2$ is negligible ($p/c^2 \ll \rho$), but in our fluid this contribution is considerable, and amounts to $p/c^2 = \rho/3$.

From (29) and (30) one finds that half of the total energy of our fluid slab is contained between the two planes given by $(e^{\eta}-1)(e^{\eta}+1/5)^{-1} = 1/2$, or $\eta = \xi = 0.8$. We may take then this value $2\xi = 1.6$ as a measure of the thickness of our slab. Referring then to (20) one finds that this thickness is given in light years by $|2x| = 186\ p_0^{-1/2}$, with the central pressure p_0 in atmospheres; the thickness of the slab is thus seen to decrease with increasing central pressure. Some representative values of radiation pressures are 0.002 atm. and 10^7 atm., corresponding to the situation on sun's surface and at the beginning of a thermonuclear reaction, respectively (Band 1955). The values of the thickness $|2x|$ corresponding to these values for the central pressure p_0 are 4,000 light years and 20 light days, respectively.

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