NOTAS DE FÍSICA
VOLUME XVI
Nº 7

AN EXACT SOLUTION OF THE Γ EQUATION

by
Mario Novello

CENTRO BRASILEIRO DE PESQUISAS FÍSICAS
Av. Wenceslau Braz, 71
RIO DE JANEIRO
1970

Notas de Física - Volume XVI - Nº 7

AN EXACT SOLUTION OF THE Γ EQUATION *

Mario Novello **
Centro Brasileiro de Pesquisas Físicas
Rio de Janeiro, Brazil

(Received July 14, 1970)

ABSTRACT:

We try to evaluate a solution of the Γ equation [1] with spherical symmetry and statical condition. We arrive at Schwarzschild's solution, as it would be expected. Some comments about the possibility of obtaining other solutions are set up.

1. INTRODUCTION

In a recent paper [1] we have shown that it is possible to consider Einstein's gravitational theory as a consequence of an interaction between "internal" objects. In this paper it is our purpose to obtain an exact solution of the fundamental Γ equation and to relate this solution to Einstein's equations.

2. THE FUNDAMENTAL VARIABLES

Let us resume, briefly, the properties of the fundamental objets $\Gamma_\mu^{AB}(x)$. In a change of coordinates the Γ's behave as a vector, that is,

$$\Gamma'^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \Gamma^{\alpha}(x) \tag{1}$$

* This work has been suported by the "Coordenação do Aperfeiçoamento de Pessoal de Nivel Superior"(CAPES) and the "Centro Brasileiro de Pesquisas Físicas". This paper will be submitted to publication in the Analles de l'Institut Henri Poincaré.

** On leave of absence at "Université de Genève".

Besides this, they have internal indices, such that they may suffer a transforma tion like

$$\Gamma'^{\mu AB}(x) = M^{A}{}_{C}(x)\ \Gamma^{\mu CD}(x)\ M^{T}\ D^{B}(x) \tag{2}$$

where A,B,C,D may have the values 0,1,2,3.

We do not need, for the time being, to specify the properties of $M^{A}{}_{B}(x)$. We only assume that they are non-singular, everywhere.

The Γ's have the property that

$$\{\Gamma^{\mu}(x),\ \Gamma^{\nu}(x)\} = 2\ g^{\mu\nu}(x)\ \mathbb{1} \tag{3}$$

where

$$\{A,\ B\} = AB + BA \tag{4}$$

$g^{\mu\nu}(x)$ is the metric tensor. $\mathbb{1}$ is the identity of the Clifford algebra.

Furthermore there is a $\Gamma^{5\ AB}(x)$ such that

$$\Gamma'^{5\ AB}(x') = M^{A}{}_{C}(x)\ \Gamma^{5\ CD}(x)\ M^{T}\ D^{B}(x)\ . \tag{5}$$

As a consequence of (1) and (2) we see that the usual derivative is not a good geometrical object, and we obliged to introduce a covariant derivative, defined by

$$\Gamma_{\mu\|\nu}(x) = \Gamma_{\mu|\nu} - \{^{\varepsilon}_{\mu\nu}\}\ \Gamma_{\varepsilon} + \left[\tau_{\nu}\ \ \Gamma_{\mu}\right] \tag{6}$$

where

$$\Gamma_{\mu|\nu}(x) = \frac{2\Gamma_{\mu}(x)}{2x^{\nu}}$$

$\{^{\varepsilon}_{\mu\nu}\}$ is the Christoffel symbol. τ_{ν} is a sort of internal affinity.

We introduce here an hypothesis that makes a strong restriction on the permisible internal transformations. We assume that the M(x) that generates an infinitesimal transformation (5) is of the form

$$M^{AB}(x) \simeq \mathbb{1} + \varepsilon\ \xi_{\alpha}(x)\ U^{\alpha AB}(x) + \varepsilon\ \xi_{\alpha||\beta}(x)\ U^{\alpha\beta AB}(x) + 0(\varepsilon^{2}) \tag{7}$$

where

$$U^{AB}_{\alpha\beta} = \frac{1}{8}\left[\Gamma_\alpha, \Gamma_\beta\right]^{AB} .$$

This choice is such as to permit us to consider the internal transformation as a coordinate transformation of the type

$$x^\alpha \longrightarrow \tilde{x}^\alpha = x^\alpha + \varepsilon\, \xi^\alpha(x) .$$

If we look for the conditions necessary to make this identification we arrive at

$$\Gamma^{AB}_{\mu||\nu}(x) = \left[U_\nu(x), \Gamma_\mu(x)\right]^{AB} . \qquad (8)$$

If we look for an object U_ν which is a member of the Clifford algebra but such that it does not introduce any new field, we see that this object has the form

$$U^{AB}_\nu(x) = \Gamma^{AB}_\nu(x) + (\Gamma_\nu(x)\ \Gamma^5(x))^{AB}. \qquad (9)$$

A straightforward calculation shows that the covariant derivative is not commutative and that we may write

$$\Gamma_{\alpha||\beta||\lambda} - \Gamma_{\alpha||\lambda||\beta} = R_{\alpha\varepsilon\beta\lambda}\ \Gamma^\varepsilon + \left[\mathbb{R}_{\beta\lambda}, \Gamma_\alpha\right] \qquad (10)$$

where

$R_{\alpha\varepsilon\beta\lambda}$ is the Riemann tensor.

$\mathbb{R}^{AB}_{\beta\lambda}$ is the internal curvature.

3. RADIAL SYMMETRY

It is an easy matter to show that Einstein's equations of the gravitational theory are obtained as a consequence of the equation

$$\left[\mathbb{R}_{\alpha\beta}, \Gamma^\alpha\right] = 0 \qquad (11)$$

in the absence of matter.

We will look now for a solution of this equation such that

$$\Gamma_\mu = \Gamma_\mu(r) \tag{12}$$

where r is the radial distance. We will use a spherical system of coordinates and we identify

$$\begin{aligned} x^0 &= t \\ x^1 &= r \\ x^2 &= \theta \\ x^3 &= \phi \end{aligned} \tag{13}$$

Condition (12) implies that the metric tensor has spherical symmetry too. We next assume that

$$\Gamma_o(r)\ \Gamma_o(r) = e^{\nu(r)}\ \mathbb{1} \tag{14}$$

$$\Gamma_1(r)\ \Gamma_1(r) = -e^{\lambda(r)}\ \mathbb{1} \tag{15}$$

$$\Gamma_2(r)\ \Gamma_2(r) = -\ r^2\ \mathbb{1} \tag{16}$$

$$\Gamma_3(r,\theta)\ \Gamma_3(r,\theta) = -\ r^2\ \mathrm{sen}^2\theta\ \mathbb{1} \tag{17}$$

These conditions are equivalent to assume that the infinitesimal length is

$$ds^2 = e^{\nu(r)}(dx^0)^2 - e^{\lambda(r)}(dr)^2 - r^2(d\theta)^2 - r^2\ \mathrm{sen}^2\theta(d\phi)^2 \tag{18}$$

From equations (14), (15), (16) and (17) we obtain

$$\Gamma_{0|r} = \frac{\nu'}{2}\Gamma_o \tag{19}$$

$$\Gamma_{1|r} = \frac{\lambda'}{2}\Gamma_1 \tag{20}$$

$$\Gamma_{2|r} = \frac{1}{r}\Gamma_2 \tag{21}$$

$$\Gamma_{3|r} = \frac{1}{r}\Gamma_3 \tag{22}$$

$$\Gamma_{3|\theta} = \mathrm{cotg}\ \theta\ \Gamma_3 \tag{23}$$

$$\nu' = \frac{d\nu}{dr}.$$

From (6) and (19) we obtain

$$\Gamma_{0||r} = \Gamma_{0|r} - \{^{\varepsilon}_{01}\}\,\Gamma_{\varepsilon} + [\tau_1, \Gamma_0] \tag{24}$$

$$\Gamma_{0||r} = 2\Gamma_1\,\Gamma_0 \tag{25}$$

Then,

$$[\tau_1, \Gamma_0] = 2\Gamma_1\,\Gamma_0 \tag{26}$$

In a same manner we obtain

$$[\tau_1, \Gamma_1] = 2\,e^{\lambda}\,\Gamma^5 \tag{27}$$

$$[\tau_1, \Gamma_2] = 2\,\Gamma_1\,\Gamma_2 \tag{28}$$

$$[\tau_1, \Gamma_3] = 2\,\Gamma_1\,\Gamma_3 \tag{29}$$

Expressions (26) to (29) implies that a non-trivial τ_1 has the form

$$\tau_1 = \Gamma_1 + \Gamma_1\,\Gamma^5 \tag{30}$$

By the same manner we arrive at the following expressions

$$[\tau_0, \Gamma_0] = \frac{\nu'}{2}\,e^{\nu-\lambda}\,\Gamma_1 - 2\,e^{\nu}\,\Gamma^5 \tag{31}$$

$$[\tau_0, \Gamma_1] = 2\,\Gamma_0\,\Gamma_1 + \frac{\nu'}{2}\,\Gamma_0 \tag{32}$$

$$[\tau_0, \Gamma_2] = 2\,\Gamma_0\,\Gamma_2 \tag{33}$$

$$[\tau_0, \Gamma_3] = 2\,\Gamma_0\,\Gamma_3 \tag{34}$$

and we see τ_0 has the form

$$\tau_0 = \Gamma_0 + \Gamma_0\,\Gamma^5 + \frac{\nu'}{4}\,e^{-\lambda}\,\Gamma_1\,\Gamma_0 \tag{35}$$

Analogously, we have

$$[\tau_2, \Gamma_0] = 2\,\Gamma_2\,\Gamma_0 \tag{36}$$

$$[\tau_2, \Gamma_1] = 2\,\Gamma_2\,\Gamma_1 + \frac{1}{r}\Gamma_2 \tag{37}$$

$$[\tau_2, \Gamma_2] = 2\,r^2\,\Gamma^5 - r\,e^{-\lambda}\,\Gamma_1 \tag{38}$$

$$[\tau_2, \Gamma_3] = 2\Gamma_2\,\Gamma_3 \tag{39}$$

and we obtain

$$\tau_2 = \Gamma_2 + \Gamma_2\,\Gamma^5 - \frac{e^{-\lambda}}{2r}\Gamma_2\,\Gamma_1 \tag{40}$$

And, by the same procedure,

$$[\tau_3, \Gamma_0] = 2\,\Gamma_3\,\Gamma_0 \tag{41}$$

$$[\tau_3, \Gamma_1] = 2\,\Gamma_3\,\Gamma_1 + \frac{1}{r}\,\Gamma_3 \tag{42}$$

$$[\tau_3, \Gamma_2] = 2\,\Gamma_3\,\Gamma_2 + \text{cotg}\,\theta\,\Gamma_3 \tag{43}$$

$$[\tau_3, \Gamma_3] = 2\,r^2\,\text{sen}^2\theta\,\Gamma^5 - r\,\text{sen}^2\theta e^{-\lambda}\,\Gamma_1 - \text{sen}\theta\,\cos\theta\,\Gamma_2 \tag{44}$$

and we obtain

$$\tau_3 = \Gamma_3 + \Gamma_3\,\Gamma^5 - \frac{e^{-\lambda}}{2r}\Gamma_3\,\Gamma_1 - \frac{\text{cotg}\theta}{2\,r^2}\Gamma_3\,\Gamma_2\ . \tag{45}$$

With these values of the internal affinities we may evaluate the internal curvature by the expression

$$\mathbb{R}_{\alpha\beta} = \tau_{\alpha|\beta} - \tau_{\beta|\alpha} + [\tau_\beta, \tau_\alpha] \tag{46}$$

If we evaluate this, we obtain

$$\mathbb{R}_{01} = \frac{e^{-\lambda}}{4}\left\{\nu'' - \frac{\nu'\lambda'}{2} + \frac{\nu'^2}{2}\right\}\Gamma_1\,\Gamma_0 \tag{47}$$

$$\mathbb{R}_{02} = \frac{e^{-\lambda}}{4} \cdot \frac{\nu'}{r} \cdot \Gamma_2 \Gamma_0 \tag{48}$$

$$\mathbb{R}_{03} = \frac{e^{-\lambda}}{4} \cdot \frac{\nu'}{r} \cdot \Gamma_3 \Gamma_0 \tag{49}$$

$$\mathbb{R}_{12} = \frac{e^{-\lambda}}{4} \cdot - \frac{\lambda'}{r} \cdot \Gamma_2 \Gamma_1 \tag{50}$$

$$\mathbb{R}_{13} = \frac{e^{-\lambda}}{4} \cdot - \frac{\lambda'}{r} \cdot \Gamma_3 \Gamma_1 \tag{51}$$

$$\mathbb{R}_{23} = \frac{1}{2r^2} \cdot (e^{-\lambda} - 1) \cdot \Gamma_3 \Gamma_2 \tag{52}$$

From these expressions and from the equation (11) we obtain the following equations

$$\left[\mathbb{R}_{0\mu}, \Gamma^{\mu}\right] = 0 \tag{53}$$

gives

$$\nu'' + \frac{\nu'^2}{2} - \nu'\lambda' + \frac{2\nu'}{r} = 0 \tag{54}$$

$$\left[\mathbb{R}_{1\mu}, \Gamma^{\mu}\right] = 0 \tag{55}$$

gives

$$\nu'' + \frac{\nu'^2}{2} - \nu'\lambda' - \frac{2\lambda'}{r} = 0 \tag{56}$$

$$\left[\mathbb{R}_{2\mu}, \Gamma^{\mu}\right] = 0 \tag{57}$$

gives

$$1 - e^{\lambda} = r\lambda' \quad . \tag{58}$$

$$\left[\mathbb{R}_{3\mu}, \Gamma^{\mu}\right] = 0 \tag{59}$$

gives

$$1 - e^{\lambda} = r^{\lambda'} \tag{60}$$

From these equations we see that

$$\nu' + \lambda' = 0 \tag{61}$$

and

$$(r\, e^{-\lambda}) = \text{constant} \tag{62}$$

With the Minkowskian boundary condition we arrive at the Schwarzschild's solution of Einstein's equation. This is not surprising because, as we said, the equation (11) may give origin to Einstein's equation. What we have obtained is another way to arrive at the solutions of these equations, and we expect that others solutions may be obtained by this method.

* * *

ACKNOWLEDGEMENTS

It is a pleasure to thank Professor J. M. Jauch for the hospitality at the "Université de Genève" and for very interesting conversations.

REFERENCES

1. NOVELLO, MARIO - Gravitation as a Consequence of the Self-Interaction of the Γ-Fields. (To be published).
2. ADLER, BAZIN, SCHIFFER - Introduction to General Relativity - (1965), New York - McGraw Hill