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ON THE WEIGHTED POLYNOMIAL APPROXIMATION IN A LOCALLY COMPACT SPACE

by

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ON THE WEIGHTED POLYNOMIAL APPROXIMATION IN A LOCALLY COMPACT SPACE *

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Let E be a locally compact space, C(E) the algebra of all continuous real functions on E, and $C_{\infty}(E)$ the Banach space of all continuous real functions on E vanishing at infinity endowed with its natural norm $f \longrightarrow \|f\| = \sup \{ |f(x)| ; x \in E \}$ for $f \in C_{\infty}(E)$. Let us consider a subalgebra $\mathfrak{a}$ of C(E) containing 1, and a vector subspace $\mathfrak{W}$ of $C_{\infty}(E)$ which is an $\mathfrak{a}$-module, that is $\mathfrak{a} \mathfrak{W} \subset \mathfrak{W}$. The general Bernstein approximation problem on a locally compact space E consists in asking for a description of the closure of such a $\mathfrak{W}$ in $C_{\infty}(E)$. We look for a general treatment similar to the one devised by Stone in the case of the Weierstrass approximation theorem [1]. If we define $x' \sim x''$ provided $x', x'' \in E$ and $f(x') = f(x'')$ for any $f \in \mathfrak{a}$, we obtain an equivalence relation induced on E by $\mathfrak{a}$, to be denoted $E/\mathfrak{a}$. Every equivalence class X of E modulo $\mathfrak{a}$ is closed in E, hence locally compact. We may consider the vector subspace

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$\mathcal{W}|X$ of the Banach space $C_\infty(X)$, formed by the restrictions to X of all members of $\mathcal{W}$. We shall say that $\mathcal{W}$ if of <u>finite type</u> under $\mathcal{A}$ if the following condition is fulfilled: $f \in C_\infty(E)$ belongs to the closure of $\mathcal{W}$ in C_∞ (E) if (and only if) $f|X \in C_\infty(X)$ belongs to the closure of $\mathcal{W}|X$ in $C_\infty(X)$ for every equivalence class X of E modulo $\mathcal{A}$. If $\mathcal{W}$ is of finite type under $\mathcal{A}$ and $\mathcal{A}$ is separating, that is, if $E/\mathcal{A}$ is the identity, then $\mathcal{W}$ is dense in $C_\infty(E)$ if and only if, for every $x \in E$, there is some $w \in \mathcal{W}$ such that $w(x) \neq 0$. In case $\mathcal{W}$, as an $\mathcal{A}$-module, has one generator w, that is $\mathcal{W} = \mathcal{A} w$, then for $\mathcal{W}$ to be dense in $C_\infty(E)$ it is necessary that $\mathcal{A}$ be separating and that w never vanishes; these two conditions being also sufficient in case $\mathcal{W}$ is of finite type under $\mathcal{A}$. When $\mathcal{A} = \mathcal{W}$, we are reduced to the situation in the Weierstrass-Stone theorem. When $\mathcal{A} = R$ is reduced to the constant functions, $\mathcal{W}$ is the most general vector subspace of $C_\infty(E)$, and the general case is to be reduced to this particular case combined with the Hahn-Banach theorem.

<u>THEOREM 1</u>. <u>If all functions in $\mathcal{W}$ have compact supports (in particular, if E is compact), then $\mathcal{W}$ is of finite type under $\mathcal{A}$.</u>

If E is compact and $\mathcal{A} = \mathcal{W}$, this theorem gives us as a particular case the Weierstrass-Stone theorem. On the other hand, Theorem 1 follows from the Weierstrass-Stone theorem, if we want to prove it directly; or it is a consequence of Theorem 2 (more precisely, of Corollary 1 in its exponential form) below. As Theorem 1 points out, a suitable rate of decrease at infinity of the functions in $\mathcal{W}$ with respect to the functions in $\mathcal{A}$ should be sufficient to ensure the finite type case.

To proceed to more precise conditions in such a direction, let us introduce the following notation. Call $\mathcal{P}_n (n = 1, 2 \ldots)$ the algebra of all real polynomials on Euclidean real n-space R^n and let $\mathcal{B}_n (n = 1, 2, \ldots)$ be the set of all continuous, everywhere strictly positive real functions B on R^n that are rapidly decreasing at infinity, that is, $\mathcal{P}_n B \subset C_\infty(R^n)$, and that are fundamental weight functions in the sense of the classical Bernstein approximation problem, that is $\mathcal{P}_n B$ is dense in $C_\infty(R^n)$. Let A be a set of generators for the algebra $\mathcal{A}$, so that every function in $\mathcal{A}$ is a real polynomial in a finite number of functions in A. Let also W be a set of generator for $\mathcal{W}$ as an $\mathcal{A}$-module, so that every function in $\mathcal{W}$ is a finite linear combination of functions in W with coefficients in $\mathcal{A}$.

<u>THEOREM 2</u>. <u>If corresponding to any generators</u> $f_1, \ldots f_n \in A$, $w \in W$, <u>there is some</u> $B \in \mathcal{B}_n$ <u>such that</u>

$$|w(x)| \leq B[f_1(x), \ldots, f_n(x)] \quad \text{for} \quad x \in E,$$

<u>then</u> $\mathcal{W}$ <u>is of finite type under</u> $\mathcal{A}$.

This theorem reduces the search of sufficient conditions for the finite type case in the general Bernstein approximation problem to the search of sufficient conditions for fundamental weight functions in the finite dimensional classical Bernstein approximation problem. The closure problem in the general case is then reduced to the density problem in the finite dimensional case.

Although this theorem has only the nature of a sufficient condition, actually it has the following converse. Let $\mathfrak{F}_n$ $(n=1, 2, \ldots)$ be a set of continuous everywhere strictly positive real functions on R^n. Then Theorem 2 is true for any E, $\mathcal{A}$, $\mathcal{W}$, A, W if we replace $\mathcal{B}_n$ by a given $\mathfrak{F}_n$ if and only if $\mathfrak{F}_n \subset \mathcal{B}_n$ $(n=1, 2, \ldots)$, so that Theorem 2 is the best possible in a sense.

We now turn to a reduction of the general closure problem to the density problem in the one dimensional case. Let $\mathfrak{F}$ be a set of continuous everywhere strictly positive real functions on the real line R such that, if $B_1, \ldots, B_n \in \mathfrak{F}$ and B is defined on R^n by $B(t_1, \ldots, t_n) = \inf \left\{ B_1(t_1), \ldots, B_n(t_n) \right\}$, then $B \in \mathcal{B}_n$ (so that, in particular, $\mathfrak{F} \subset \mathcal{B}_1$).

<u>THEOREM 3.</u> <u>If corresponding to any generators</u> $f \in A$, $w \in W$, <u>there is some</u> $B \in \mathfrak{F}$ <u>such that</u>

$$|w(x)| \leq B[f(x)] \qquad \underline{\text{for}} \quad x \in E,$$

<u>then</u> $\mathcal{W}$ <u>is of finite type under</u> $\mathcal{A}$.

Again, in spite of the fact that this result has only the character of a sufficient condition, it holds true for any E, $\mathcal{A}$, $\mathcal{W}$, A, W and a given $\mathfrak{F}$ if and only if $\mathfrak{F}$ has the property stated immediately before the theorem.

Let $\mathbb{B}_n$ $(n=1, 2, \ldots)$ be the subset of $\mathcal{B}_n$ of all continuous, everywhere strictly positive real functions B on R^n such that $B^a \in \mathcal{B}_n$ for all real exponents $a > 0$.

THEOREM 4. *If corresponding to any generators* $f \in A$, $w \in W$, *there is some* $B \in \mathbb{B}_1$ *such that*

$$|w(x)| \leq B[f(x)] \qquad \text{for} \quad x \in E,$$

then $\mathcal{W}$ *is of finite type under* $\mathcal{A}$.

COROLLARY 1. *If corresponding to any generators* $f \in A$, $w \in W$, *there is an even continuous everywhere strictly positive real function* B *on* R *such that*

$$|w(x)| \leq B[f(x)] \qquad \text{for} \quad x \in E,$$

$$\log 1/B(t) \text{ is a convex function of } \log t \text{ for } t > 0,$$

$$\int^{+\infty} 1/t^2 \cdot \log 1/B(t) \cdot dt = +\infty ,$$

(in particular, if corresponding to any generators $f \in A$, $w \in W$, *there are real constants* $K, k > 0$ such that

$$|w(x)| \leq K \cdot \exp[-k|f(x)|] \quad \text{for} \quad x \in E),$$

then $\mathcal{W}$ *is of finite type under* $\mathcal{A}$.

COROLLARY 2. *If, for every generators* $f \in A$, $w \in W$, *we have*

$$\sum_{n=0}^{+\infty} 1/M_n^{1/n} = +\infty \quad \text{where} \quad M_n = \|f^n w\| \; (n = 0, 1, \ldots),$$

then $\mathcal{W}$ *is of finite type under* $\mathcal{A}$.

For instance, if W is reduced to one function w, that is $\mathcal{W}$ has one generator as an $\mathcal{A}$-module, and the above series diverges for every $f \in A$, then $\mathcal{W}$ is dense in $C_\infty(E)$ if and only if $\mathcal{Q}$ is separating and w never vanishes (Malliavin).

The above results in the case of a separating algebra $\mathcal{Q}$ imply the same results in the general case, provided we deal from the very start with vector valued functions and continuous sums of normed spaces in the sense of von Neumann. This leads also, more generally, to statements of the general Bernstein approximation problem and the preceding theorems and corollaries in terms of commutative algebras of operators in real normed spaces and a suitable notion of generalized local convexity [2]. Extension to commutative self-adjoint algebras to the general complex case is not yet possible at this moment, since even the commutative (not necessarily self-adjoint) complex case on compact spaces (Wermer, etc.) and the self-adjoint (not necessarily commutative) complex case in Hilbert spaces (Kadison, Glimm, etc.) are not yet completely clarified from the viewpoint of the Weierstrass approximation, and a fortiori from the viewpoint of the Bernstein approximation.

FOOTNOTES

1 We refer to M. H. Stone, "The generalized Weierstrass approximation theorem", Mathematics Magazine, vol. 21 (1948); P. Malliavin, "L'approximation polynomiale pondérée sur un espace localement compact, "American Journal of Mathematics, vol. 81 (1959). An excellent report on the classical Bernstein approximation problem is due to S. N. Mergelyan, "Weighted ap-

proximations by polynomials, "Uspehi Matematičeskih Nauk (N.S.), vol. 11 (1956), American Mathematical Society Translations (Series 2), vol. 10 (1958).

2 See L. Nachbin, "Algebras of finite differential order and the operational calculus", Annals of Mathematics, vol. 70 (1959).

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