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ON A NON LOCAL RELATIVISTIC QUANTUM THEORY OF FIELDS * +

by

G. Wataghin

Istituto di Física dell'Universita di Torino
and

Centro Brasileiro de Pesquisas Físicas

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I. INTRODUCTORY REMARKS

In this note we want to discuss some features (already suggested) of a non local theory of fields which is free from divergent integrals and infinite constants and satisfies the claim of macroscopic causality. The fundamental problem of the mass quantization connected with the self-energy calculations, will be analysed in another place. Also the discussion of new limitations in the interpretation of physical phenomena like those concerning the measurability on small domains of space and time and the validity of free particle picture in such domains will be only briefly mentioned .

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A revision of the whole S-matrix formalism and of the way of using the second quantization will be necessary. We shall illustrate the method on the example of a collision problem and a self-energy problem in the case of a PS - PV coupling between pions and nucleons, but the same formalism can be applied equally well to the electrodynamics and to a non linear coupling of the type:

$$M\bar{\psi}\left(e^{g\gamma_5\varphi} - 1\right)\psi$$

where φ represents a pseudo-scalar boson field.

Let us start with the analysis of the state of a nucleon having a definite average momentum and energy p_ν. From the theory of the anomalous magnetic moment and of the photo-meson production on nucleons one can deduce that it is possible to represent this state of the nucleon (with its meson cloud and with appropriate vacuum polarisation effects) as a linear superposition of states of bare particles. The coefficients of this expansion can than be interpreted as amplitudes of the probability to find the nucleon without mesons, or with one meson, or with two mesons, and so on. The creation and destruction of particles must be described by the second quantization method of Dirac. But, if the created particle has a definite momentum and energy, its state is supposed to be stationary and the quantized field must have negligible interaction with other fields. Thus the created or absorbed particle should be a "bare" particle, whereas the "bare" state is not a stationary state. This argument shows that only an approximate description of states of bare particles is possible with the use of the 2-d quantization, and will induce us to take into account the finite life time of these states by means of another ap proximation.

The introduction of relativistic cut-off operators will allow us to show that the perturbation expansion is rapidly convergent and thus to explain the success of the application of perturbation calculations in the electrodynamics.

II. DEFINITION OF THE CUT-OFF OPERATORS. FIRST APPROXIMATION TO THE CONSTRUCTION OF THE S-MATRIX.

In order to give in a simple way the rules of the new formalism let us consider the collision between particles having known momenta, spins, polarizations etc. and let us represent them as plane wave solutions of the free particle equations ("bare" particles). Let $P_\nu = \sum p_\nu$ be the total energy-momentum 4-vector and let U_ν be the velocity vector of the Center of Mass (CM) system of the ingoing particles:

$$U_\nu = \frac{1}{m} P_\nu \ , \quad \text{where} \quad m^2 = P_\nu P^\nu > 0$$

Taking the scalar product $K_\nu U^\nu$ of any vector K_ν (projection on U^ν), we shall form two invariants

$$I_t = K_\nu U^\nu \qquad\qquad I_s = \sqrt{I_t^2 - K_\nu K^\nu} \qquad (1)$$

which we shall call projections on the time axis and respectively on the space of the CM system, because in the CM system one has:

$$I_t = K_o \ , \quad I_s = |\vec{K}| \ , \quad I_s^2 = K_o^2 - (K_o^2 - K_1^2 - K_2^2 - K_3^2) = K_1^2 + K_2^2 + K_3^2$$

The cut-off operators will be functions of these invariants.

Let us now consider all the terms of the S-matrix which transform the above mentioned set of ingoing particles in a given set of outgoing particles (also plane waves). This group of terms will transform covariantly under the Lorentz transformation. Knowing the interaction hamiltonian and applying the usual formalism, we can write down easily the term of n-th order S_n in the momentum representation of the collision matrix S, corresponding to a given Feynman graph. Then we can formulate the rule of the relativistic cut-off method in the following way: every non degenerated intermediate or final state of a bare particle having the momentum $\vec{K}$, has a statistical weight $= G^2(I_s)$. Therefore every internal boson line (boson propagator) receives a G^2 factor, every directed internal nucleon line (fermion propagator) receives a factor G, every external line receives also a factor G. Such invariant factors obviously conserve the general covariancy of the theory.

Although a definite choice of the cut-off function $G\left(I_s(K)\right)$ and of the universal lenght ℓ cannot be made on the basis of general properties of a non local interaction and will depend probably on the solution of the problem of "mass quantization", some experimental results concerning the mass differences between charged and neutral particles and the momentum distribution of pions (in the CM frame of reference) created in a high energy collision suggest a choice of G of the type [1]

$$G(\xi) = \frac{3[\sin\xi - \xi\cos\xi]}{\xi^3} = 3\sqrt{\frac{\pi}{2}}\,\xi^{-3/2}\,J_{3/2}(\xi) \quad \text{or}$$

$$G(\xi) = \sqrt{\frac{2}{\pi}}\,\frac{1}{1+\xi^2} \qquad (2)$$

where $\xi = \ell I_s(K)$

The first case corresponds to a form-factor $F\left(I_s(\eta)\right)$ in relative coordinates $\eta_\nu = x'_\nu - x''_\nu$, which, in the CM system, is: $F = f\left(\frac{|\vec{\eta}|}{\ell}\right)\delta\left(\frac{\eta_0}{\ell}\right)$ where $f\left(\frac{|\vec{\eta}|}{\ell}\right) = \sqrt{\frac{3}{4\pi}}\ell^{-2}$, if $|\vec{\eta}| < \ell$ and $F = 0$, if $|\vec{\eta}| > \ell$ (3)

In the second case : $f\left(\frac{|\vec{\eta}|}{\ell}\right) \sim e^{-|\vec{\eta}|/\ell}$

This sharp limitation of the small domain of $|\vec{\eta}|$ in which F is not vanishing is also required by the claim of macroscopic causality. In both cases the asymtotic behaviour for $\xi \to \infty$ of the statistical weight factor G^2 is respectively : $\sim \frac{\cos^2(\xi)}{\xi^4}$ or $\sim \frac{1}{\left(1 + \xi^2\right)^2}$

III. DEFINITION OF A NON LOCAL INTERACTION HAMILTONIAN AND OF THE MODIFIED PROPAGATORS.

The simple rules of the calculation of the non local S-matrix in the momentum space given above can be deduced starting from the definition of a non local hamiltonian density in space - time $H'(x)$ and from the assumptions concerning the new rules of calculation of the S-matrix in configuration space. Let us consider the n-th order term S_n of the perturbation expansion of the S-matrix in the usual local theory :

$$S_n = \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} d^4x_1 \dots d^4x_n \; P \left\{ H'(x_1) \dots H'(x_n) \right\} \qquad (4)$$

We shall illustrate the method assuming that the local hamiltonian H' represents a PS - PV interaction :

$$H'(x) = i \frac{g}{m} \bar{\psi}(x) \gamma_5 \gamma_\nu \tau_\sigma \psi(x) \partial_\nu \varphi_\sigma(x) \qquad (5)$$

Each field amplitude φ, ψ, $\bar{\psi}$ can be split, as usualy, into positive and negative frequency components and can be expanded into Fourier components containing absorption and creation operators of particles with momenta K' K'' K''' (we shall adopt the formalism in which the negative energy states are substituted by the "charge conjugated solutions" representing the antiparticles in the positive energy states). For instance in the case (5) we shall have 8 terms of H' with given momenta K' K'' K''' of the 3 interacting fields vorresponding to the different sets of creation and annihilation operators of the type $b^*_{K''} b_{K'} a_{K'}$.

In order to transform (4) and (5) into non local operators in a covariant way we postulate the following rules : considering a vertex point x and a term of the above Fourier expansion, we substitute in this term the phase factor $\exp(-iK'x)$ belonging to a positive frequency component by $\exp[-iK'(x^+ + \eta')]$ and the phase factor $\exp(iK''x)$ belonging to a negative frequency by $\exp[iK''(x^- + \eta''')]$ and introduce invariant form-factors F operating on the "internal coordinates" η', η'', η'''. These internal coordinates will by definition transform like differences of coordinates obeying the homogeneous Lorentz transformation, and will commute with the momentum operators.

In general case of a graph corresponding to the term (4) one shall have at the time x_o of the vertex x a number of other "boson" and "fermion" lines which do not end or begin in this vertex point. Also each of these lines, let us say the one with a momentum K^{IV}, can be considered as an ingoing particle arriving with the phase-factor $\exp\left[-iK^{IV}(x^+ + \eta^{IV})\right]$ and outgoing with the phase-factor $\exp\left[iK^{IV}(x^- + \eta^{IV})\right]$. Since we shall put in further calculations : $x^+ = x^- = x$, the above phase-factors will not modify the values of the matrix-element (5). In the above example if the term $b^*_{K'''} b_{K''} a_{K'}$ is considered and if there is just one more line (K^{IV}) not passing through the vertex x, the total phase-factor will be :

$$\exp\left[-i\,(K' + K'' + K^{IV})x^+ - i\,(K'\eta' + K''\eta'' + K^{IV}\eta^{IV})\right] \cdot$$
$$\cdot \exp\left[i\,(K''' + K^{IV})x^- + i\,(K'''\eta''' + K^{IV}\eta^{IV})\right] \qquad (6)$$

The form-factors F will be chosen in such a manner as to limit the values of the variables η', η'', η''', η^{IV}, x^+ and x^- to a small domain D_ℓ of space and time containing the vertex-point x. These F's are functions of invariants defined similar to those I_s, I_t introduced in the cut-off operators in the momentum space. Let ∂^+_ν indicate the derivative $\frac{\partial}{\partial x^+_\nu}$ and ∂^-_ν the derivative $\frac{\partial}{\partial x^-_\nu}$. Then the form-factors limiting the domains of internal coordinates η are defined as functions of the invariant operators $I^\pm_s$ and $I^\pm_t$:

$$\left(I_s^+\right)^2 = \frac{(\eta^\nu i\,\partial_\nu^+)^2}{|i\,\partial_\nu^+ i\,\partial^{+\nu}|} - \eta^\nu\eta_\nu = (\eta^\nu U_\nu^+)^2 - \eta^\nu\eta_\nu$$

$$I_t^+ = \frac{\eta^\nu i\,\partial_\nu^+}{|i\,\partial_\nu^+ i\,\partial^{+\nu}|} = \eta^\nu U_\nu^+ \quad \text{where} \quad U_\nu^+ = \frac{K' + K'' + K^{iv}}{|K' + K'' + K^{iv}|} \tag{7}$$

in the case of a positive frequency component, and as function of

$$\left(I_s^-\right)^2 = \frac{(\eta^\nu i\,\partial_\nu^-)^2}{|i\partial_\nu^- i\,\partial^{-\nu}|} - \eta^\nu\eta_\nu = (\eta^\nu U_\nu^-)^2 - \eta^\nu\eta_\nu$$

$$\text{where} \qquad U_\nu^- = \frac{K''' + K^{iv}}{|K''' + K^{iv}|} \tag{7'}$$

in the case of a negative frequency component. Here the expression of the invariants $I_s^\pm$ and $I_t^\pm$ in function of the momenta K', K'', K''' and K^{iv} is specified for the case of the term containing $b^*_{K'''} b_{K''} d_{K'}$ and must be substituted by similar expressions in other cases. The vectors U_ν^+ and U_ν^-, and the invariants $I_s^\pm$, $I_t^\pm$ are obviously analogous to U_ν and I_s, I_t defined in II. Also the function $F(I_s, I_t)$ must be of the type (3) of II in order to have in the reference system in which the $U_\nu^\pm$ components are (1,0,0,0) $\delta(\eta_0)$ dependence of the time and satisfy the limitation

$$F = \text{const.} \neq 0 \text{ , if } |\vec{\eta}| \leq \ell$$

The next step in the calculation is the integration over all "internal coordinates η', η'', η''', and this gives us the cut-off operators $G\left(I_s(K)\right)$ for each momentum K', K'', K''', and a factor $= 1$ for K^{iv}(because $\int F(\eta^{iv}) d^4\eta^{iv} = 1$). Further step concerns the condition $x^+ = x^- = x$. We can associate $\delta(x^+ - x)$ and $\delta(x^- - x)$ factors and integrate over x^+ and x^-. (The introduction of new form-factors of the type (3) instead of these δ does not give any new result, due to our special choice of these form-factors). After this integration remains the phase-factor $\exp\left[-i(K' + K'' - K''')x\right]$. Successive integration over x will give us the usual $\delta^4(K' + K'' - K''')$ factor. It is easy to see that one obtains in this way : $U^+_\nu = U^-_\nu = U_\nu$ in all vertex-points, and therefore one has the important result that all the cut-off and form-factors are refered to the same 4-vector U_ν in all vertex-points and this 4-vector U_ν is defined as the velocity-vector of the CM system of the incident (and also of the outgoing) particles.

The remaining part of the calculation of the S_n can be performed applying the usual commutation relations between the creation and annihilation operators $a_{K'}$, $a^*_{K''}$, $b_{K'''}$ etc. in the momentum space (obviously the commutators, anti-commutators and propagators in space-time are modified by the cut-off). One obtains instead of the usual causal boson propagator $\Delta_c(x - y)$ the following non-local propagator :

$$\Delta_c(x - y) = \frac{1}{(2\pi)^4}\int_c \frac{G\left(\ell\, I_s(K)\right) e^{-iK_\nu(x^\nu - y^\nu)}}{K^2 - m^2}\, d^4 K \tag{8}$$

and similar expression for S_c. Since Δ_c is an invariant , we can calculate it in the CM system where $I_s(K) = |\vec{K}|$. Then the first integration over K_o in (8) can be performed with the usual prescription about the contour. In this manner the causality condition of Stueckelberg and Feynman (which requires that for $x^o - y^o > 0$ only the pole $K_o = \sqrt{\vec{K}^2 + m^2}$ contributes to (8) and for $x^o - y^o < 0$ only the pole $K_o = -\sqrt{\vec{K}^2 + m^2}$).

A simple analysis shows that making the choice (2) and (3) of the cut-off, and considering particles having a rest-mass $m \sim \ell^{-1}$, one finds that the domains of $x - y$, in which these propagators Δ_c and S_c do not vanish, of the order of ℓ^3 in the space and are of the order of ℓ in the time, if measured in the CM system. In other reference systems these domains appear naturally Lorentz contracted in space and expanded in time. In the case of particles of vanishing rest-mass, as photons or neutrinos, the domain is a small zone on both sides of the light cone. In this last case an elementary calculation gives:

$$D_c(x) = \frac{1}{(2\pi)^3} \int \frac{d^3K}{2|\vec{K}|} G(\ell|\vec{K}|)\, e^{-iK_\nu x^\nu} =$$

$$= \frac{1}{(2\pi)^2} \frac{1}{2r} G(\ell K)\left[e^{iK(r-t)} - e^{iK(r+t)}\right] =$$

$$= \frac{1}{2(2\pi)^2} \frac{1}{r} \left\{ f\left(\frac{r-t}{\ell}\right) - f\left(\frac{r+t}{\ell}\right)\right\} \qquad (9)$$

where $r = |\vec{x}|$; $K = |\vec{K}|$; $t = x_0 > 0$ and f is the form-factor (3).

Since in the non local theory proposed here the particle which emits or absorbs photons can at best be localized with an uncertainty $\sim \ell$ corresponding to the form-factor (3), (9) is not in contradiction with the claim that no signal propagates with velocity $> c$.

Similar calculations show that also in the case of $m \neq 0$ the propagator (8) has no singularities on the light cone.

If the mass of the particle is $\sim \rho^{-1}$, then the propagator (8) vanishes (with a law depending on the form-factor) for $r = |\vec{x} - \vec{y}| > \rho$ and also in the case of $t = |x_0 - y_0| > \rho$. The case of $m \ll \rho^{-1}$ requires a special consideration and has a particular importance in electrodynamics. Here we shall restrict ourselfs to the consideration of mesons and nucleons.

IV. CONVERGENCY OF THE PERTURBATION SOLUTION.

In the introduction we made a remark concerning the bare state of a particle being not a stable state. Also the states of a nucleon with n mesons in the cloud or of a pion which can be substituted by a pair (nucleon + anti-nucleon) are not stable having a finite life time, determined by the total probability of all transitions from these states. The introduction of the cut-off factors makes the total life-time of each of these states finite.

The process described by a given Feynman diagram of n-th order can be thought of as a chronological sequence of n transitions having each a definite probability. From the discussion of the propagators modified by the cut-off follows that the average life-time of intermediate states is $\sim \rho$, if measured in the CM system. In order to take it into account approximately we can assume that the n successive processes are statistically independent and have all the same probability (total width) $\sim \rho^{-1}$. Then, as is well known, the Poisson formula gives us the approximate probability of the process of n-th order. Since the usual perturbation theory does not take into account the finite life-times of the intermediate states, we shall take them into account associating to each term of the perturbation expansion the Poisson factor :

$$\sim \frac{1}{n!} e^{-\ell\mathcal{T}} (z\ell^{-1}\mathcal{T})^n \approx \frac{z^n}{n!} e^{-1} \qquad (10)$$

$$\text{when} \quad \mathcal{T} \approx \ell \quad \text{and} \quad 0 < |z| < 1$$

This approximate method indicates that the convergency of the perturbation expansion can depend on the finite life-time of the intermediate states.

Peierls and Mac Manus suggested the use of form-factors which are functions only of the 4-dimensional intervals $(x' - x'')^2$. C. Bloch showed that such factors do not always give convergent results and Stueckelberg and Wanders showed that these factors give rise to a contradiction with the claim of macroscopic causality . Our method is free from both objections. Naturally we cannot have microscopic causality condition satisfied in small domains D_ℓ defined by the form-factors, but the macroscopic causality is satisfied due to the properties of the propagators illustrated above. This question and the applications of the above method will be subject of another paper.

V. RELATIVISTIC INVARIANCE. CONCLUDING REMARKS

The formalism suggested above differs from the formalism of the relativistic local theory of fields by the introduction in the hamiltonian density and in the S-matrix of invariants operators. These operators do not change the covariant properties of any single term of the S-matrix and thus also do not destroy the general covariancy of the theory. The macroscopic causality is fulfilled as was shown above and as follows from the remark that the local interaction is by us substituted by an interaction which is limited to a finite domain restricted as well in space as in time. In the CM system this domain can be, e.g. a sphere of radius ℓ in space and a time-interval $\Delta t \sim \ell$ in time. The hermitian nature of the hamiltonian and thus the unitarity of the S-matrix is also conserved. Indeed terms of the hamiltonian containing operators $b^*_{K''} b_{K''} a_{K'}$ and $b_{K''} b^*_{K''} a^*_{K'}$ receive the same cut-off factors and remain conjugated.

The method of relativistic cut-off allows a natural description of the multiple meson production if one introduces non linear coupling e.g. of the type mintioned in I. The theory of this multiple production and of the mass quantization will be subject of a next publication.

The significance of the internal coordinates and their connection with discontinous parameters as spin, iso-spin and strangeness parameters will be also discussed with reference to the mass quantization problem.

(1) Units - we use : $\hbar = 1$, $c = 1$; and metric tensor: $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$