

A natural fuzzyness of de Sitter space-time

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Abstract

A non-commutative structure for de Sitter spacetime is naturally introduced by replacing (“fuzzyfication”) the classical variables of the bulk in terms of the dS analogs of the Pauli-Lubanski operators. The dimensionality of the fuzzy variables is determined by a Compton length and the commutative limit is recovered for distances much larger than the Compton distance. The choice of the Compton length determines different scenarios. In scenario I the Compton length is determined by the limiting Minkowski spacetime. A fuzzy dS in scenario I implies a lower bound (of the order of the Hubble mass) for the observed masses of all massive particles (including massive neutrinos) of spin $s > 0$. In scenario II the Compton length is fixed in the de Sitter spacetime itself and grossly determines the number of finite elements (“pixels” or “granularity”) of a de Sitter spacetime of a given curvature.

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1 Introduction

Within the framework of Quantum Physics in Minkowskian space-time, an elementary particle, say a quark, a lepton, or a gauge boson, is identified through some basic attributes like mass, spin, charge and flavour. The (rest) mass is certainly the most basic attribute for an elementary particle. Now, for a particle of non-zero mass, its relation to space-time geometry on the quantum scale is irremediably limited by its (reduced) Compton wavelength

$$\lambda_{\text{cmp}} = \frac{\hbar}{mc}. \quad (1)$$

It is sometimes claimed that λ_{cmp} represents “the quantum response of mass to local geometry” since it is considered as the cutoff below which quantum field theory, which can describe particle creation and annihilation, becomes important.

Now we know, essentially since Wigner, that mass and spin attributes of an “elementary system” emerge from space-time symmetry. These arguments rest upon the Wigner classification of the Poincaré unitary irreducible representations (UIR) [1, 2]: the UIR’s of the Poincaré group are completely characterized by the eigenvalues of its two Casimir operators, the quadratic $\mathcal{C}_2^0 = P^\mu P_\mu = P^{02} - \mathbf{P}^2$ (Klein-Gordon operator) with eigenvalues $\langle \mathcal{C}_2^0 \rangle = m^2 c^2$ and the quartic $\mathcal{C}_4^0 = W^\mu W_\mu$, $W_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma$ (Pauli-Lubanski operator) with eigenvalues (in the non-zero mass case) $\langle \mathcal{C}_4^0 \rangle = -m^2 c^2 s(s+1) \hbar^2$.

These results lead us to think that the mathematical structure to be retained in the description of mass and spin is the symmetry group, here the Poincaré group \mathcal{P} , of space-time and not the space-time itself. The latter may be described as the coset \mathcal{P}/L , where L is the Lorentz subgroup. On the other hand we know that a UIR of \mathcal{P} is the quantum version (“quantization”) of a co-adjoint orbit [3] of \mathcal{P} , viewed as the classical phase space of the elementary system. The latter is also described as a coset: for an elementary system with non-zero mass and spin the coset is $\mathcal{P}/(\text{time-translations} \times SO(2))$. This coset is by far more fundamental than space-time.

Since a co-adjoint orbit may be viewed as a phase space or set of initial conditions for the motion of an elementary particle, and so is proper to the latter, the existence of a “minimal” length provided by its Compton wavelength leads us to consider the space-time as a “fuzzy manifold” proper to this system. This raises the question to establish a consistent model of a fuzzy Minkowski space-time issued from the Poincaré UIR associated to that elementary system. The answer is not known in the case of a flat geometry. However note that the Pauli-Lubanski vector components W^μ could be of some use in the “fuzzyfication” of the light-cone in Minkowski, just through the replacement $x^\mu \rightarrow W^\mu$ and by dealing with massless UIR’s of the Poincaré group in such a way that the second Casimir \mathcal{C}_4^0 is fixed to zero. The non-commutativity stems from the rules $[W^\mu, W^\nu] = -i \epsilon_{\mu\nu\rho\sigma} P^\rho W^\sigma$ and the covariance is granted thanks to the rules $[W_\mu, P_\nu] = 0$ and $[J_{\mu\nu}, W_\rho] = i (\eta_{\mu\rho} W_\nu - \eta_{\nu\rho} W_\mu)$.

In this note we show that there exists a consistent way for defining such a structure for any “massive system” if we deal instead with a de Sitter space-time.

The organization of the paper is as follows. In Section **2** we recall the basic features of the de Sitter space-time and of its application to the cosmological data suggesting an accelerating universe. In Section **3** we compactly present the main properties of the de

Sitter group UIR's. In section 4 we discuss the contraction limits of the de Sitter UIR's to the Poincaré UIR's. The main results are discussed in Section 5 and 6. In Section 5 a non-commutative structure is naturally introduced in dS spacetime by assuming the bulk variables being replaced by “fuzzy” variables (similarly to the analogous non-commutative structure of the fuzzy spheres) which, in a given limit, recover the commutative case. The “de Sitter fuzzy Ansatz” implies a lower bound (of the order of the observed Hubble mass $1 \approx 1.2 \cdot 10^{-42} GeV$) for the observed masses of the massive particles of spin $s > 0$. In Section 6 the “de Sitter fuzzy Ansatz” is applied to the desitterian physics and its cosmological applications. In the Conclusions we discuss the implication of these results and outline possible developments.

2 The de Sitter hypothesis

In a curved background the mass of a test particle can always be considered as the rest mass of the particle as it should locally hold in a tangent minkowskian space-time. However, when we deal with a de Sitter or Anti de Sitter background, which are constant curvature space-times, another way to examine this concept of mass is possible and should also be considered. It is precisely based on symmetry considerations in the above Wigner sense, i.e. based on the existence of the simple de Sitter or Anti de Sitter groups that are both one-parameter deformations of the Poincaré group. We recall that the de Sitter [resp. Anti de Sitter] space-times are the unique maximally symmetric solutions of the vacuum Einstein's equations with positive [resp. negative] cosmological constant Λ . Their respective invariance (in the relativity or kinematical sense) groups are the ten-parameter de Sitter $SO_0(1, 4)$ and Anti de Sitter $SO_0(2, 3)$ groups. Both may be seen as deformations of the proper orthochronous Poincaré group $\mathbb{R}^{1,3} \times SO_0(1, 3)$, the kinematical group of Minkowski. Exactly like for the flat case, dS and AdS space-times can be identified as cosets $SO_0(1, 4)/\text{Lorentz}$ and $SO_0(2, 3)/\text{Lorentz}$ respectively, and coadjoint (\cong adjoint) orbits of the type $SO_0(1, 4)/SO(1, 1) \times SO(2)$ (resp. $SO_0(2, 3)/SO(2) \times SO(2)$) can be viewed as phase space for “massive” elementary systems with spin in dS (resp. AdS). Since the beginning of the eighties the de Sitter space has been considered as a key model in inflationary cosmological scenario where it is assumed that the cosmic dynamics was dominated by a term acting like a cosmological constant. More recently, observations on far high redshift supernovae, on galaxy clusters and on cosmic microwave background radiation (see for instance [4]), suggest an accelerating universe. This can be explained in a satisfactory way with such a term. This constant, denoted by Λ , is linked to the (constant) Ricci curvature 4Λ of these space-times and it allows to introduce the fundamental curvature or inverse length $Hc^{-1} = \sqrt{|\Lambda|/3} \equiv R^{-1}$, (H is the Hubble constant).

To a given (rest minkowskian) mass m and to the existence of a non-zero curvature is naturally associated the typical dimensionless parameter for dS/AdS perturbation of the minkowskian background:

$$\vartheta_m \stackrel{\text{def}}{=} \frac{\hbar\sqrt{|\Lambda|}}{\sqrt{3}mc} = \frac{\hbar H}{mc^2} = \frac{m_H}{m}, \quad (2)$$

where we have also introduced a ‘‘Hubble mass’’ m_H through

$$m_H = \frac{\hbar H}{c^2}. \quad (3)$$

We can also introduce the Planck units, defining the regime in which quantum gravity becomes important, which are determined in terms of \hbar , c and the gravitational constant G , through the positions

$$\begin{aligned} \text{length} & : l_{Pl} = \sqrt{\frac{\hbar G}{c^3}} \approx 1.6 \times 10^{-33} \text{ cm}, \\ \text{mass} & : m_{Pl} = \sqrt{\frac{\hbar c}{G}}, \approx 2.2 \times 10^{-5} g \approx 1.2 \times 10^{19} \text{ GeV}/c^2, \\ \text{time} & : t_{Pl} = \sqrt{\frac{\hbar G}{c^5}} \approx 5.4 \times 10^{-44} \text{ s}, \\ \text{temperature} & : T_{Pl} = \sqrt{\frac{\hbar c^5}{G k_B^2}} \approx 1.4 \times 10^{32} \text{ K}. \end{aligned} \quad (4)$$

The observed value of the Hubble constant is

$$H \equiv H_0 = 2.5 \times 10^{-18} \text{ s}^{-1}. \quad (5)$$

Associated to the Planck mass, we have the dimensionless parameter ϑ_{Pl} through

$$\vartheta_{Pl} = \frac{m_{H_0}}{m_{Pl}} = t_{Pl} H_0 \approx 1.3 \times 10^{-61}, \quad (6)$$

while

$$\Lambda t_{Pl}^2 c^2 = \Lambda l_{Pl}^2 = 9\vartheta_{Pl}^2 \approx 1.6 \times 10^{-121} \quad (7)$$

(namely, the cosmological constant is of the order 10^{-120} when measured in Planck units) and

$$\frac{R}{l_{Pl}} = (H_0 t_{pl})^{-1} \approx 0.8 \times 10^{61}. \quad (8)$$

As a consequence, if l_{Pl} is a minimal discretized length, $(\frac{R}{l_{Pl}})^3 \approx 10^{180}$ measures the number of discrete elements (‘‘atoms’’) in a quantum de Sitter universe.

We give in Table below the values assumed by the quantity ϑ_m when m is taken as some known masses and Λ (or H_0) is given its present day estimated value. We easily understand from this table that the currently estimated value of the cosmological constant has no practical effect on our familiar massive fermion or boson fields. Contrariwise, adopting the de Sitter point of view appears as inescapable when we deal with infinitely small masses, as is done in standard inflation scenario.

Mass m	$\vartheta_m \approx$
Hubble mass $m_\Lambda/\sqrt{3} \approx 0.293 \times 10^{-68}\text{kg}$	1
up. lim. photon mass m_γ	0.29×10^{-16}
up. lim. neutrino mass m_ν	0.165×10^{-32}
electron mass m_e	0.3×10^{-37}
proton mass m_p	0.17×10^{-41}
W^\pm boson mass	0.2×10^{-43}
Planck mass M_{Pl}	0.135×10^{-60}

Table 1: Estimated values of the dimensionless physical quantity $\vartheta_m \stackrel{\text{def}}{=} \frac{\hbar\sqrt{|\Lambda|}}{\sqrt{3}mc} = \frac{\hbar H_0}{mc^2} \approx 0.293 \times 10^{-68} \times m_{\text{kg}}^{-1}$ for some known masses m and the present day estimated value of the Hubble length $c/H_0 \approx 1.2 \times 10^{26}\text{m}$ [5].

3 1+3 De Sitter geometry, kinematics and dS UIR's

Geometrically, de Sitter space-time can be described as an one-sheeted hyperboloid \mathcal{M}_H embedded in a five-dimensional Minkowski space (the bulk):

$$\mathcal{M}_H \equiv \{x \in \mathbb{R}^5; x^2 \stackrel{\text{def}}{=} x \cdot x = \eta_{\alpha\beta} x^\alpha x^\beta = -H^{-2} \equiv -R^2\}$$

$$\alpha, \beta = 0, 1, 2, 3, 4, \quad \eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1, -1), \quad x := (x^0, \vec{x}, x^4).$$

A global causal ordering exists on the de Sitter manifold: it is induced from that one in the ambient spacetime \mathbb{R}^5 : given two events $x, y \in M_H$,

$$x \geq y \text{ iff } x - y \in \overline{V^+}$$

where

$$\overline{V^+} = \{x \in \mathbb{R}^5 : x \cdot x \geq 0, \text{sgn } x^0 = +\}$$

is the future cone in \mathbb{R}^5 . One says that $\{y \in M_H : y \geq x\}$ (resp. $\{y \in M_H : y \leq x\}$) \equiv closed causal future (resp. past) cone of point x in X . Two events $x, y \in M_H$ are in ‘‘acausal relation’’ or ‘‘spacelike separated’’ if they belong to the intersection of the complements of above sets, *i.e.* if $(x - y)^2 = -2(H^{-2} + x \cdot y) < 0$.

There are ten Killing vectors generating a Lie algebra isomorphic to $so(1, 4)$:

$$K_{\alpha\beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha.$$

At this point, we should be aware that there is no globally time-like Killing vector in de Sitter, ‘‘time-like’’ (resp. ‘‘space-like’’) referring to the Lorentzian four-dimensional metric induced by that of the bulk. The de Sitter group is $G = SO_0(1, 4)$ or its universal covering, denoted $Sp(2, 2)$, needed for dealing with half-integer spins.

Quantization (geometrical or coherent state or something else) of de Sitter classical phase spaces leads to their quantum counterparts, namely the quantum elementary systems associated in a biunivocal way to the UIR's of the de Sitter group $SO_0(1, 4)$ or $Sp(2, 2)$. The ten Killing vectors are represented as (essentially) self-adjoint operators in

Hilbert space of (spinor-)tensor valued functions on M_H , square integrable with respect to some invariant inner (Klein-Gordon type) product :

$$K_{\alpha\beta} \rightarrow L_{\alpha\beta} = M_{\alpha\beta} + S_{\alpha\beta},$$

where $M_{\alpha\beta} = -i(x_\alpha \partial_\beta - x_\beta \partial_\alpha)$ (orbital part), and $S_{\alpha\beta}$ (spinorial part) acts on indices of functions in a certain permutational way. Note the usual commutation rules,

$$[L_{\alpha\beta}, L_{\gamma\delta}] = -i(\eta_{\alpha\gamma} L_{\beta\delta} - \eta_{\alpha\delta} L_{\beta\gamma} - \eta_{\beta\gamma} L_{\alpha\delta} + \eta_{\beta\delta} L_{\alpha\gamma}). \quad (9)$$

Two Casimir operators exist whose eigenvalues determine the UIR's :

$$\mathcal{C}_2 = -\frac{1}{2} L_{\alpha\beta} L^{\alpha\beta}, \quad (10)$$

$$\mathcal{C}_4 = -W_\alpha W^\alpha, \quad W_\alpha = -\frac{1}{8} \epsilon_{\alpha\beta\gamma\delta\eta} L^{\beta\gamma} L^{\delta\eta}, \quad (11)$$

where (W_α) is the dS counterpart of the Pauli-Lubanski operator. These W_α 's transform like vectors:

$$[L_{\alpha\beta}, W_\gamma] = i(\eta_{\beta\gamma} W_\alpha - \eta_{\alpha\gamma} W_\beta). \quad (12)$$

In particular we have

$$W_a = i[W_0, L_{a0}], \quad a = 1, 2, 3, 4. \quad (13)$$

The W_α 's commute as:

$$\begin{aligned} [W_\alpha, W_\beta] &= -i\epsilon_{\alpha\beta\alpha_1\alpha_2\alpha_3} W^{\alpha_1} L^{\alpha_2\alpha_3} \\ &= -i(L^{\gamma\delta} L_{\gamma\delta} - 3) L_{\alpha\beta} + \frac{i}{2} \{L^{\gamma\delta}, \{L_{\alpha\gamma}, L_{\beta\delta}\}\}. \end{aligned} \quad (14)$$

The algebra defined by the $L_{\alpha\beta}$ and the W_α generators (regarded as primitive generators) is a non-linear finite W -algebra [6]. This algebra respects the grading $[L_{\alpha\beta}] = 2$, $[W_\alpha] = 3$, $[[,]] = -1$.

As it is the case with non-linear W -algebras, (9), (12) and (14) can be linearized (the ‘‘unfolded’’ version) at the price of introducing an infinite number of generators regarded as primitive generators. E.g., the r.h.s. of Equation (14) can be written as $-i\epsilon_{\alpha\beta\gamma_1\gamma_2\gamma_3} Z^{\gamma_1\gamma_2\gamma_3}$, where $Z^{\gamma_1\gamma_2\gamma_3}$ can be identified with $W^{\gamma_1} L^{\gamma_2\gamma_3}$ ($[Z^{\gamma_1\gamma_2\gamma_3}] = 5$). An infinite tower of extra primitive generators have to be introduced to close the algebra linearly.

It is convenient to reexpress the $L_{\alpha\beta}$, W_α generators of the finite W -algebra in their $SO(4)$ decomposition, ($a, b = 1, 2, 3, 4$), given by T_a , L_{ab} , Z , W_a , where

$$T_\alpha = L_{0\alpha}, \quad Z = W_0.$$

Due to the fact that the W_a 's arise from the commutator $[T_a, Z]$, we can regard the finite non-linear W -algebra with T_a , L_{ab} , Z , W_a primitive generators as an unfolded version of the finite non-linear W -algebra with T_a , L_{ab} , Z primitive generators.

The operator W_0 is the difference of two commuting $su(2)$ -Casimir. To get this, we start from the expression:

$$W_0 = -(L_{12}L_{34} + L_{23}L_{14} + L_{31}L_{24}) = -\mathbf{J} \cdot \mathbf{A}.$$

The operators $\mathbf{J} = (L_{23}, L_{31}, L_{12})^T$ and $\mathbf{A} = (L_{14}, L_{24}, L_{34})^T$ represent a basis for the maximal compact subalgebra $\mathfrak{k} \cong so(4)$:

$$[J_i, J_j] = iJ_k, \quad [J_i, A_j] = iA_k, \quad [A_i, A_j] = iJ_k,$$

with (i, j, k) even permutation of $(1, 2, 3)$. Introducing the two commuting sets of $su(2)$ generators:

$$\mathbf{N}^L := \frac{1}{2}(\mathbf{A} + \mathbf{J}), \quad \mathbf{N}^R := \frac{1}{2}(\mathbf{A} - \mathbf{J}), \quad \left[N_i^L, N_j^R \right] = \pm i N_k^L,$$

we obtain

$$W_0 = -\mathbf{J} \cdot \mathbf{A} = -\mathbf{A} \cdot \mathbf{J} = (\mathbf{N}^L)^2 - (\mathbf{N}^R)^2.$$

In consequence, as an operator on a direct sum of $SU(2)$ UIR's its spectrum is made of the numbers $j_l(j_l + 1) - j_r(j_r + 1)$, $j_l, j_r \in \mathbb{N}/2$. A complete classification [7] of the de Sitter UIR's is precisely based on the following property. Let $Sp(2, 2) \ni g \mapsto \rho(g) \in \text{Aut}(\mathcal{H})$ a UIR of $Sp(2, 2)$ acting in a Hilbert space \mathcal{H} . Then the restriction to the maximal compact subgroup $K \stackrel{\text{def}}{=} SU(2) \times SU(2)$ is completely reducible:

$$\mathcal{H} = \bigoplus_{(j_l, j_r) \in \Gamma_\rho} \mathcal{H}_{j_l, j_r}, \quad \mathcal{H}_{j_l, j_r} \cong \mathbb{C}^{2j_l+1} \times \mathbb{C}^{2j_r+1},$$

where $\Gamma_\rho \subset \mathbb{N}/2$ is the set of pairs (j_l, j_r) such that the UIR $D^{j_l} \otimes D^{j_r}$ of K appears once and only once in the the reduction of the restriction $\rho|_K$. Let $p = \inf_{(j_l, j_r) \in \Gamma_\rho} (j_l + j_r)$ and $q_0 = \min_{\substack{(j_l, j_r) \in \Gamma_\rho \\ j_l + j_r = p}} (j_r - j_l)$, $q_1 = \max_{\substack{(j_l, j_r) \in \Gamma_\rho \\ j_l + j_r = p}} (j_r - j_l)$. Then we have the following exhaustive possibilities:

- (i) $q_1 = p$ and $q_0 = -p$, which correspond to elements of the principal and complementary series, denoted by $\Upsilon_{p, \sigma}$ where $\sigma \in (-2, +\infty)$ (with restrictions according to the values of p in $\mathbb{N}/2$);
- (ii) $q_1 = p$ and $0 < q_0 \equiv q \leq p$, which correspond to elements of the discrete series, denoted by $\Pi_{p, q}^+$;
- (iii) $q_0 = -p$ and $0 < -q_1 \equiv q \leq p$, which correspond to elements of the discrete series, denoted by $\Pi_{p, q}^-$;
- (iv) $q_0 = q_1 = 0 \equiv q$, which correspond to elements lying at the bottom of the discrete series, denoted by $\Pi_{p, 0}$.

Let us now give more details on these three different types of representations.

“Discrete series” $\Pi_{p, q}^\pm$

Parameter q has a spin meaning and the two Casimir are fixed as

$$\begin{aligned} \mathcal{C}_2 &= (-p(p+1) - (q+1)(q-2))\mathbb{I}, \\ \mathcal{C}_4 &= (-p(p+1)q(q-1))\mathbb{I}. \end{aligned}$$

We have to distinguish between

- (i) *the scalar case* $\Pi_{p,0}$, $p = 1, 2, \dots$, which are not square integrable,
- (ii) *the spinorial case* $\Pi_{p,q}^\pm$, $q > 0$, $p = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$, $q = p, p-1, \dots, 1$ or $\frac{1}{2}$. For $q = 1/2$ the representations $\Pi_{p,\frac{1}{2}}^\pm$ are not square-integrable.

“Principal series” $U_{s,\nu}$

$$\Upsilon_{p=s,\sigma=\nu^2+\frac{1}{4}} \equiv U_{s,\nu}, \quad q = \frac{1}{2} \pm i\nu, \quad \sigma = q(1-q).$$

$p = s$ has a spin meaning and the two Casimir are fixed as

$$\begin{aligned} \mathcal{C}_2 &= (\sigma + 2 - s(s+1))\mathbb{I} = (9/4 + \nu^2 - s(s+1))\mathbb{I}, \\ \mathcal{C}_4 &= \sigma s(s+1)\mathbb{I} = (1/4 + \nu^2) s(s+1)\mathbb{I}. \end{aligned}$$

We have to distinguish between

- (i) $\nu \in \mathbb{R}$ (i.e., $\sigma \geq 1/4$), $s = 1, 2, \dots$, for the integer spin principal series,
- (ii) $\nu \neq 0$ (i.e., $\sigma > 1/4$), $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$, for the half-integer spin principal series.

In both cases, $U_{s,\nu}$ and $U_{s,-\nu}$ are equivalent. In the case $\nu = 0$, i.e. $q = \frac{1}{2}$, $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$, the representations are direct sums of two UIR’s in the discrete series:

$$U_{s,0} = \Pi_{s,\frac{1}{2}}^+ \oplus \Pi_{s,\frac{1}{2}}^-.$$

“Complementary series” $V_{s,\nu}$

$$\Upsilon_{p=s,\sigma=\frac{1}{4}-\nu^2} \equiv V_{s,\nu}, \quad q = \frac{1}{2} \pm \nu,$$

$p = s$ has a spin meaning and the two Casimir are fixed as

$$\begin{aligned} \mathcal{C}_2 &= (\sigma + 2 - s(s+1))\mathbb{I} = (9/4 - \nu^2 - s(s+1))\mathbb{I}, \\ \mathcal{C}_4 &= \sigma s(s+1)\mathbb{I} = (1/4 - \nu^2) s(s+1)\mathbb{I}. \end{aligned}$$

We have to distinguish between

- (i) *the scalar case* $V_{0,\nu}$, $\nu \in \mathbb{R}$, $0 < |\nu| < \frac{3}{2}$ (i.e., $-2 < \sigma < 1/4$),
- (ii) *the spinorial case* $V_{s,\nu}$, $0 < |\nu| < \frac{1}{2}$ (i.e., $0 < \sigma < 1/4$), $s = 1, 2, 3, \dots$.

In both cases, $V_{s,\nu}$ and $V_{s,-\nu}$ are equivalent.

4 Contraction limits or desitterian Physics from the point of view of a Minkowskian observer

On a geometrical level, $\lim_{H \rightarrow 0} \mathcal{M}_H = \mathcal{M}_0$, the Minkowski spacetime tangent to \mathcal{M}_H at, say, the de Sitter origin point O_H . Then, on an algebraic level, $\lim_{H \rightarrow 0} Sp(2, 2) = \mathcal{P}_+^\uparrow(1, 3) = \mathcal{M}_0 \times SL(2, \mathbb{C})$, the Poincaré group. The ten de Sitter Killing vectors contract to their Poincaré counterparts $K_{\mu\nu}$, Π_μ , $\mu = 0, 1, 2, 3$, after rescaling the four $K_{4\mu} \rightarrow \Pi_\mu = HK_{4\mu}$ (“space-time contraction”). On a UIR level, the question is mathematically more delicate.

The massive case

For the massive case, the principal series representations only are involved (from which the name “de Sitter massive representations”). Introducing ν through $\sigma = \nu^2 + 1/4$ and the Poincaré mass $m = \nu H$, we have the following result [8, 10]:

$$U_{s,\nu} \xrightarrow{H \rightarrow 0, \nu \rightarrow \infty} c_> \mathcal{P}^>(m, s) \oplus c_< \mathcal{P}^<(m, s),$$

where one of the “coefficients” among $c_<, c_>$ can be fixed to 1 whilst the other one vanishes and where $\mathcal{P}^<(m, s)$ denotes the positive (resp. negative) energy Wigner UIR’s of the Poincaré group with mass m and spin s .

The massless case

Here we must distinguish between the scalar massless case, which involves the unique complementary series UIR $\Upsilon_{0,0}$ to be contractively Poincaré significant, and the helicity = s case where are involved all representations $\Pi_{s,s}^\pm$, $s > 0$ lying at the lower limit of the discrete series. The arrows \leftrightarrow below designate unique extension. $\mathcal{P}^<(0, s)$ denotes the Poincaré massless case with helicity s . Conformal invariance leads us to deal also with the discrete series representations (and their lower limits) of the (universal covering of the) conformal group or its double covering $SO_0(2, 4)$ or its fourth covering $SU(2, 2)$ [9]. These UIR’s are denoted by $\mathcal{C}^<(E_0, j_1, j_2)$, where $(j_1, j_2) \in \mathbb{N}/2 \times \mathbb{N}/2$ labels the UIR’s of $SU(2) \times SU(2)$ and E_0 stems for the positive (resp. negative) conformal energy.

- Scalar massless case :

$$\Upsilon_{0,0} \leftrightarrow \begin{array}{ccc} \mathcal{C}^>(1, 0, 0) & \xrightarrow{H=0} & \mathcal{C}^>(1, 0, 0) \leftrightarrow \mathcal{P}^>(0, 0) \\ \oplus & & \oplus \\ \mathcal{C}^<(-1, 0, 0) & & \mathcal{C}^<(-1, 0, 0) \leftrightarrow \mathcal{P}^<(0, 0), \end{array}$$

- Spinorial massless case :

$$\begin{array}{ccc} \Pi_{s,s}^- \leftrightarrow & \begin{array}{ccc} \mathcal{C}^>(s+1, s, 0) & \xrightarrow{H=0} & \mathcal{C}^>(s+1, s, 0) \leftrightarrow \mathcal{P}^>(0, s) \\ \oplus & & \oplus \\ \mathcal{C}^<(-s-1, s, 0) & & \mathcal{C}^<(-s-1, s, 0) \leftrightarrow \mathcal{P}^<(0, s), \end{array} \\ \Pi_{s,s}^+ \leftrightarrow & \begin{array}{ccc} \mathcal{C}^>(s+1, 0, s) & \xrightarrow{H=0} & \mathcal{C}^>(s+1, 0, s) \leftrightarrow \mathcal{P}^>(0, -s) \\ \oplus & & \oplus \\ \mathcal{C}^<(-s-1, 0, s) & & \mathcal{C}^<(-s-1, 0, s) \leftrightarrow \mathcal{P}^<(0, -s), \end{array} \end{array}$$

We can see from the above that there is energy ambiguity in de Sitter relativity, exemplified by the possible breaking of dS irreducibility into a direct sum of two Poincaré UIR’s with positive and negative energy respectively. This phenomenon is linked to the existence in the de Sitter group of the discrete symmetry that sends any point $(x^0, \mathcal{P}) \in M_H$ into its mirror image $(x^0, -\mathcal{P}) \in M_H$ with respect to the x^0 -axis. Under such a symmetry the four generators L_{0a} , $a = 1, 2, 3, 4$, (and particularly L_{04} which contracts to energy operator!) transform into their respective opposite $-L_{0a}$, whereas the six L_{ab} ’s remain unchanged. All representations that are not listed in the above contraction limits have either non-physical Poincaré contraction limit or have no contraction limit at all.

In order to get a unifying description of the dS-Poincaré contraction relations, the following “mass” formula has been proposed by Garidi [11] in terms of the dS UIR parameters p and q :

$$m_{\text{Gar}}^2 = \langle \mathcal{C}_2 \rangle_{\text{dS}} - \langle \mathcal{C}_{2p=q} \rangle_{\text{dS}} = [(p - q)(p + q - 1)]m_H^2, \quad m_H = \hbar H/c^2. \quad (15)$$

The minimal value assumed by the eigenvalues of the first Casimir in the set of UIR in the discrete series is precisely reached at $p = q$, which corresponds to the “conformal” massless case. The Garidi mass has the advantages to encompass all mass formulas introduced within a de-sitterian context, often in a purely mimetic way in regard with their minkowskian counterparts.

Now, given a minkowskian mass m and a “universal” length $R = \sqrt{3/\Lambda} = cH^{-1}$, nothing prevents us to consider the dS UIR parameter ν (principal series), specific of a “physics” in constant-curvature space-time, as meromorphic functions of the dimensionless physical (in the minkowskian sense!) quantity ϑ_m , as was introduced in Equation (2) in terms of various other quantities introduced in this context, Note that ϑ_m is also the ratio of the Compton length of the minkowskian object of mass m considered at the limit with the universal length R yielded by dS geometry. Thus, we may consider the following Laurent expansions of the dS UIR parameter ν (principal series) in a certain neighborhood of $\vartheta_m = 0$:

$$\nu = \nu(\vartheta_m) = \frac{1}{\vartheta_m} + e_0 + e_1\vartheta_m + \cdots e_n(\vartheta_m)^n + \cdots, \quad (16)$$

with $\vartheta_m \in (0, \vartheta_{\text{max}})$ (convergence interval). The coefficients e_n are pure numbers to be determined. We should be aware that nothing is changed in the contraction formulas from the point of view of a minkowskian observer, except that we allow to consider positive as well as negative values of ν in a (positive) neighborhood of $\vartheta_m = 0$: multiply (16) by ϑ_m and go to the limit $\vartheta_m \rightarrow 0$. We recover asymptotically the relation

$$m = |\nu|\hbar H/c^2 = |\nu|\frac{\hbar}{c}\sqrt{\frac{|\Lambda|}{3}}. \quad (17)$$

As a matter of fact, the Garidi mass is a good example of such an expansion since it can be rewritten as the following expansion in the parameter $\vartheta_m \in (0, 1/|s - 1/2|]$:

$$\nu = \sqrt{\frac{1}{\vartheta_m^2} - (s - 1/2)^2} = \frac{1}{\vartheta_m} - (s - 1/2)^2 \left(\frac{\vartheta_m}{2} + O(\vartheta_m^2) \right), \quad (18)$$

Note the particular symmetric place occupied by the spin 1/2 case with regard to the scalar case $s = 0$ and the boson case $s = 1$.

5 Fuzzy de Sitter space-time with Compton wavelength

Examining the equation (11) that fixes the value of the quartic Casimir in terms of the operators W_α it is tempting, if no natural, to introduce a non-commutative structure for

the dS spacetime by replacing the classical variables of the bulk x^α with the suitably normalized W^α operators through the “fuzzy” variables \widehat{x}^α

$$x^\alpha \rightarrow \widehat{x}^\alpha = rW^\alpha, \quad (19)$$

where r ($[r] = l$) has been introduced for dimensional reasons. In principle a different r has to be introduced for any given irreducible representation characterized by p, q (therefore $r \equiv r_{p,q}$).

The non-commutativity reads as follows

$$[\widehat{x}^\alpha, \widehat{x}^\beta] = -ir\epsilon_{\alpha\beta\gamma_1\gamma_2\gamma_3}\widehat{x}^{\gamma_1}L^{\gamma_2\gamma_3} \quad (20)$$

and goes to zero in the limit $r \rightarrow 0$.

On the other hand, the dS inverse curvature R , arising from the classical constraint

$$-x_\alpha x^\alpha = R^2, \quad (21)$$

is replaced in the “fuzzy” case by the equation

$$-\widehat{x}_\alpha \widehat{x}^\alpha = -r^2 p(p+1)q(q-1), \quad (22)$$

inducing the identification

$$R = r\sqrt{-p(p+1)q(q-1)}. \quad (23)$$

What is r ? A natural interpretation consists in assuming it to be a Compton length l_{cmp} of the associated particle. The most immediate approach then consists in looking at the Minkowski limit fixing the Minkowskian observational mass m_{obs} as the limit of the Garidi mass. In this case we have to work with the principal series which allows taking the limit $\nu \rightarrow \infty$. We get that (15) can be rewritten as

$$m_{\text{Gar}} = \frac{\hbar}{cR} \sqrt{\left(s - \frac{1}{2}\right)^2 + \nu^2}. \quad (24)$$

In agreement with (16), we assume for ν a dependence on R such as

$$\nu(R) = c_{-1}R + c_0 + \frac{c_1}{R} + \frac{c_2}{R^2} + \dots, \quad (25)$$

and we can safely take the $R \rightarrow \infty$ limit in the r.h.s. and obtain

$$m_{\text{obs}} = \frac{\hbar}{c} c_{-1}. \quad (26)$$

We are now in the position to use as r in (23) the Compton length associated to the observational mass m_{obs} . Since we are working with the principal series, (23) can be in this case expressed as

$$R_0 = \frac{\hbar}{m_{\text{obs}}c} \sqrt{s(s+1)\left(\frac{1}{4} + \nu^2\right)}, \quad (27)$$

where $R_0 = c/H_0$. The physical interpretation of the (observational) parameter ν is the fact that it connects the observational curvature of the dS spacetime with the observational mass of the elementary system (for the given spin $s > 0$). We obtain

$$\nu^2 = \left(\frac{R_0^2 m_{\text{obs}}^2 c^2}{\hbar^2} - \frac{s(s+1)}{4} \right) \frac{1}{s(s+1)}. \quad (28)$$

Since ν^2 must be positive we end up with a constraint on the observational masses of the massive elementary systems with spin $s > 0$. The constraint is given by the relation

$$m_{\text{obs}} \geq \frac{\hbar}{2c} \sqrt{s(s+1)} \frac{1}{R_0} = \frac{m_{H_0}}{2} \sqrt{s(s+1)}. \quad (29)$$

One can say that for all known massive particles this lowest bound is of the order of the observed Hubble mass $\approx 1.6 \cdot 10^{-42} \text{ GeV}$. Combining the upper limit of Table 1 and this lowest limit, we obtain the allowed mass range for massive neutrinos:

$$\approx 1.2 \cdot 10^{-42} \text{ GeV} \leq m_n \leq \approx 9.7 \cdot 10^{-11} \text{ GeV}.$$

It is worth pointing out that the ‘‘de Sitter fuzzy Ansatz’’ implies a lower bound for the observed masses of all massive particles of positive spin s . The lower bound does not depend on the electric charge (positive, negative or vanishing) of the particles. It depends smoothly on their spin s . For large spin s the lower bound grows linearly with s .

We refer to the construction of the fuzzy de Sitter spacetime discussed in this Section as ‘‘Scenario I’’. In this scenario the Compton length has been defined in the limiting Minkowski spacetime. In the next Section we will discuss another scenario (referred to as ‘‘Scenario II’’), such that the Compton length is defined in the de Sitter spacetime itself.

6 Fuzzy de Sitter and Garidi mass

Another interpretation of r consists in assuming it to be a Compton length l_C of the associated particle with its desitterian Garidi mass. The Compton length is in this case defined in the de Sitter spacetime itself. We refer to this scenario as ‘‘Scenario II’’. For the irreducible representation characterized by p, q the Compton length is

$$l_C = \frac{\hbar}{m_{\text{Gar}} c}. \quad (30)$$

Scenario II offers different possibilities of constraining the relevant p, q UIR’s which satisfy some given properties. We explicitly discuss two such cases. In the first case we have

$$l_C = \frac{\hbar}{cm_H} \sqrt{\frac{1}{(p-q)(p+q-1)}} = l_{Pl} \frac{1}{\vartheta_{Pl}} \sqrt{\frac{1}{(p-q)(p+q-1)}} \quad (31)$$

and

$$\frac{R}{l_{Pl}} = \frac{1}{\vartheta_{Pl}} \sqrt{\frac{-p(p+1)q(q-1)}{(p-q)(p+q-1)}}. \quad (32)$$

For the principal series we can set

$$\begin{aligned} p &= s, \\ q &= \frac{1}{2} + i\nu, \end{aligned} \quad (33)$$

where s is the spin and $\nu \in \mathbb{R}$. For this series we obtain that (32) reads as

$$\frac{R}{l_{Pl}} = \frac{1}{\vartheta_{Pl}} \sqrt{\frac{s(s+1) + \frac{s(s+1)}{4\nu^2}}{1 + \frac{4s^2 - 4s + 1}{4\nu^2}}} \approx \frac{1}{\vartheta_{Pl}} \sqrt{s(s+1)}. \quad (34)$$

The last equality holds in the limit $\nu \rightarrow \infty$.

On the other hand, it is possible to choose $\nu \in \mathbb{R}$ such that the r.h.s. of (34) does not depend on s . For $s = 1$ we get $\sqrt{\frac{s(s+1) + \frac{s(s+1)}{4\nu^2}}{1 + \frac{4s^2 - 4s + 1}{4\nu^2}}} = \sqrt{2}$ no matter which is the value of ν . In order to reproduce this value for any positive (half-)integer spin s , ν must be given by

$$\nu = \sqrt{\frac{7s^2 - 9s + 2}{4s^2 + 4s - 8}}. \quad (35)$$

The above equation always admits real solutions (for $s = \frac{1}{2}$, $\nu = \sqrt{\frac{3}{28}}$, for $s = \frac{3}{2}$, $\nu = \sqrt{\frac{17}{28}}$, for $s = 2$, $\nu = \frac{3}{4}$, for $s = \frac{5}{2}$, $\nu = \sqrt{\frac{103}{28}}$, ... with, in the limit $s \rightarrow \infty$, $\nu \rightarrow \sqrt{\frac{7}{4}}$).

In the massless case we have $m_{Gar} = 0$, which implies either $p = q$ or $p = 1 - q$. In both cases

$$R = r\sqrt{p^2(1-p^2)}, \quad (36)$$

which is positive only for $p = \frac{1}{2}$ (we are dealing with the $\Pi^{\pm}_{\frac{1}{2}, \frac{1}{2}}$ irrep). Therefore

$$R = r\sqrt{\frac{3}{16}}. \quad (37)$$

In order to obtain the ‘‘universal formula’’

$$\frac{R}{l_{Pl}} = \frac{1}{\vartheta_{Pl}} \sqrt{2} \quad (38)$$

we have to fix r , the Compton length l_C of the massless spin $\frac{1}{2}$ system, to be given by

$$r \equiv l_C = \frac{l_{Pl}}{\vartheta_{Pl}} \sqrt{\frac{32}{3}}. \quad (39)$$

Summarizing the results obtained so far, in the first case of Scenario II we can constrain the p, q UIR's such that the ratio $\frac{R}{l_{Pl}}$ between the de Sitter curvature and the Planck length becomes universal (i.e., the allowed p, q produce formula (38)).

In a second case for Scenario II, the Compton length for the Hubble mass is associated with the curvature R through

$$m_H = \frac{\hbar}{cR}. \quad (40)$$

By assuming (40) we can reexpress the Garidi mass m_{Gar} in terms of p , q and the curvature R (besides the constants c and \hbar). We obtain

$$m_{\text{Gar}} = \frac{\hbar}{cR} \sqrt{(p-q)(p+q-1)}. \quad (41)$$

Since

$$R = r \sqrt{-p(p+1)q(q-1)}, \quad (42)$$

with r the Compton length associated to the Garidi mass ($r = \frac{\hbar}{m_{\text{Gar}}c}$), we end up with the equality

$$R = R \sqrt{\frac{-p(p+1)q(q-1)}{(p-q)(p+q-1)}}. \quad (43)$$

In this case the constraint

$$\frac{-p(p+1)q(q-1)}{(p-q)(p+q-1)} = 1 \quad (44)$$

has to be satisfied, giving a restriction on the allowed (p, q) UIR's.

For the principal series ($p = s$, $q = \frac{1}{2} + i\nu$) the restriction corresponds to the equation

$$s(s+1)\left(\frac{1}{4} + \nu^2\right) = \left(s - \frac{1}{2}\right)^2 + \nu^2, \quad (45)$$

which implies

$$\nu^2 = \frac{1}{2} \sqrt{\frac{3s^2 - 5s + 1}{s^2 + s - 1}}. \quad (46)$$

The above equation finds solutions for any positive spin $s > 0$.

7 Conclusions

In this work we described a natural fuzzyness of the de Sitter spacetime. In analogy to what happens in the case of the fuzzy spheres, a non-commutative structure for dS can be introduced by replacing the classical variables of the bulk with the dS analogs of the Pauli-Lubanski operators, suitably normalized by a dimensional parameter which plays the role of the Compton length and measures the granularity of the non-commutative dS space-time. We adopt the point of view of the elementary system described by UIR's of the de Sitter group (parametrized by the pair (p, q)) and determine the mathematical

and physical consequences of the “de Sitter fuzzy Ansatz”. Two different scenarios have been investigated according to the possibility that the Compton length of the elementary system is described in the Minkowski limit of the de Sitter spacetime (Scenario I; in this case the suitable UIR’s of the de Sitter spacetime belong to the principal series and the UIR’s of the Poincaré group are recovered in the limit $\nu \rightarrow \infty$) or in the de Sitter spacetime itself (Scenario II). The “fuzzy dS Ansatz” in Scenario I has some observational implications. The observed masses of all massive particles with spin $s > 0$ admits a lower bound of the order of the Hubble mass. For large s the lower bound grows linearly with s . The lower bound is only function of the spin (not of the charge of the particles) and applies in particular to the case of the massive neutrinos. In scenario II the Compton length is fixed in the de Sitter spacetime itself and grossly determines the number of finite elements (“pixels” or “granularity”) of a de Sitter spacetime of a given curvature. In Scenario II the allowed UIR’s (parametrized by p and q) are determined in accordance with the properties that have to be required for the fuzzy dS spacetime. We discussed in particular two cases. A first case such that the ratio $\frac{R}{l_{Pl}}$ between the dS curvature and the Planck length is universal. A second case such that the Compton length for the Hubble mass is given by the dS curvature.

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