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Abelian Toda field theories on the noncommutative plane

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Abstract

Generalizations of $GL(n)$ abelian Toda and $\widetilde{GL}(n)$ abelian affine Toda theories to the noncommutative plane are constructed. Our proposal relies on the nc extension of the Leznov-Saveliev equations. The special cases of $GL(2)$ Liouville and $\widetilde{GL}(2)$ sinh/sine-Gordon are explicitly studied.

Key-words: Toda Field Theories; $WZNW$ model; Non-commutativeness.

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1 Introduction

Noncommutative Field theories (NCFT) has been a very active research area since the appearance of NCFT as low-energy limits of string theory with magnetic fields turn on [1]. Inside the context of NCFT, noncommutative (nc) extensions of two-dimensional Integrable Field Theories have been investigated [2, 3, 4, 5, 6]. As in two-dimensions, a nc deformation of a model requires a noncommutative time-coordinate, the causality and unitarity properties of the theory can be compromised [7, 8]. However, it is hoped that in exactly solvable systems this situation should be improved or even disappear, as discussed in [9]. In order to avoid the non-unitarity in the two-dimensional case, Euclidean models can be considered.

It is well known that the nc deformation of a theory is not unique since it is always possible to construct different nc extensions that will lead to the same commutative limit (see the nc generalizations of sine-Gordon model in [4], as an example). In this sense, preserving the integrability properties of the theory can be a guiding principle in order to construct nc deformations of two-dimensional theories.

Following the previous direction, in [4] a nc extension of the zero-curvature condition was introduced. The nc extensions of integrable theories, constructed from this condition, have an infinite number of conserved charges, which however, not always guaranteed the complete classical integrability of the theory (see [5]). A way to test classical integrability is to look to the S-matrix of the corresponding quantum theory. If the theory is classically integrable, the tree-level scattering amplitude must vanish which corresponds to no particle production. The S-matrix should be also factorized [10]. In [5] was proved that the nc extension of the sine-Gordon model constructed in [3, 4] suffers from acasual behavior and a non-factorized S-matrix, what means that particle production occurs. In this sense we can say that in the nc extension of the sine-Gordon model as presented in [3, 4] even though it has an infinite chain of conserved charges, the integrability is not preserved.

In the ordinary commutative case, it is well known that the integrable Toda theories can be represented as appropriate gauged Wess-Zumino-Novikov-Witten ($WZNW$) models [11, 12, 13]. The algebraic structure of Toda theories is connected with a $G_0 \in G$ embedding of a G-invariant $WZNW$ model. Abelian Toda theories are connected with abelian G_0 subgroups of G. In order to eliminate the degrees of freedom in the tangent space G/G_0, it is possible to either implement constraints upon specific components of the chiral currents $J, \bar{J}$ of the $WZNW$ model, or equivalently to propose a gauge invariant action and eliminate the unwanted degrees of freedom by choice of the gauge. The equations of motion of the resultant Toda model will be then defined in the G_0 subgroup and it results to be an integrable deformation of a conformal invariant theory.

In this paper we use a nc zero-curvature condition expressed as a nc extension of

Leznov-Saveliev equations [4] to construct nc integrable extensions of $SL(n)$ abelian Toda theories and $\widetilde{SL}(n)$ abelian affine Toda theories. In order to define the zero grade subgroup G_0 we have taken into account that the previous groups are not closed under the noncommutative product, so they should be extended to $GL(n)$ and $\widetilde{GL}(n)$ respectively. This consideration imply the introduction of an additional scalar field which will not decouple. We explicitly studied the $GL(2)$ Liouville and $\widetilde{GL}(2)$ sinh-Gordon extensions. As we will see, any element of the zero grade subgroup can be parameterized in alternative ways, what leads to equivalent nc generalizations of the corresponding original models. This is a consequence of the nonabelian character of the zero grade subgroup in the nc case. The models obtained in the sinh/sine-Gordon case reproduce some of the suggestions for nc sine-Gordon presented in [6]. These generalizations [6] seem to retain most of the properties of the original sine-Gordon model. They have an infinite chain of conserved charges and a seemingly factorized and casual S-matrix, as was checked for some dispersion processes. In this sense we expect that the procedure chosen by us to generalize the abelian Toda models conduce to nc extensions that will keep most of the advantages of the original two-dimensional integrable models.

On the other hand, the nc extensions here constructed do not preserve the conformal invariance of the ordinary Toda theories. As it is well known, the introduction of a constant noncommutative parameter spoils the conformal symmetry. This symmetry has not been very well studied in the nc context, but it seems that in order to define a nc extension of the conformal symmetry the deformation parameter should not be kept fixed [14].

The nc extensions of abelian and abelian affine Toda Theories as presented here have not been studied in any other place, as far as we know. In [15] was proposed a $U(N)$ nc extension of Toda theories as a system of coupled first order equations, that could not be reduced to coupled second order equations. In this sense our versions are more treatable as physical theories since they are described for second order differential equations that can also be derived from an action principle. It remains to see the compatibility of both procedures, specially in relation to the exact solutions.

This paper is organized as follows. In the next section we review firstly the Hamiltonian Reduction process that leads to the Leznov-Saveliev equations of motion [16]. Then we show how to obtain the corresponding action defining a gauge symmetry and after fixing the gauge. At the end of this section we describe how the abelian and abelian affine Toda theories are obtained using the previous procedure and defining the appropriate algebraic structure. At the beginning of section 3 a nc extension of the Hamiltonian Reduction process applied to the nc generalization of $WZNW$ [18] model is presented. The nc Leznov-Saveliev second order equations of motion are derived from the equations of motion of the $WZNW_\star$ model imposing appropriate constraints on the chiral currents. Next the nc Liouville and sinh/sine-Gordon models are constructed, adopting different parameterizations, from the nc extension of the Leznov-Saveliev equations. At the end of Section 3 the nc extensions of $GL(n,\mathbb{R})$ abelian and $\widetilde{GL}(n,\mathbb{R})$ abelian affine Toda theories are introduced. Section 4 provides the conclusions and finally the appendix is dedicated to present some algebraic properties.

2 Abelian Toda theories from $WZNW$ model

Toda theories can be regarded as constrained Wess-Zumino-Novikov-Witten ($WZNW$) models [11, 13]. By placing certain constraints on the currents the G-invariant $WZNW$ model reduces to the appropriate Toda theory. The abelian Toda theories are connected with abelian embeddings $G_0 \subset G$. As it is well known the G-invariant $WZNW$ model [19]

$$S_{WZNW} = -\frac{k}{4\pi}\int d^2z Tr(g^{-1}\partial g g^{-1}\bar{\partial} g) + \frac{k}{24\pi}\int d^3x \epsilon^{ijk}(g^{-1}\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g) \tag{2.1}$$

describes a two dimensional conformal invariant field theory on a group manifold. The fields α parameterize the group element

$$g(z,\bar{z}) = e^{\alpha_a(z,\bar{z})T_a}, \tag{2.2}$$

where T_a are the generators of the Lie algebra of G satisfying the Lie bracket $[T_a, T_b] = f_{abc}T_c$. We are using light cone coordinates $z = t + x$, $\bar{z} = t - x$ and $\partial = \frac{1}{2}(\frac{\partial}{\partial t} + \frac{\partial}{\partial x})$, $\bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial t} - \frac{\partial}{\partial x})$. The equations of motion

$$\bar{\partial} J = \partial \bar{J} = 0, \tag{2.3}$$

are written in function of the chiral currents

$$J = g^{-1}\partial g \quad \text{and} \quad \bar{J} = -\bar{\partial} g g^{-1}. \tag{2.4}$$

In order to eliminate the unwanted degrees of freedom that corresponds to the tangent space G/G_0 it is possible to implement constraints upon specific components of the currents $J, \bar{J}$ [11, 12, 13] (Hamiltonian Reduction) or equivalently to propose a gauge invariant action and eliminate the degrees of freedom by choice of gauge [13].

The subgroup G_0 is defined in terms of a grading operator Q, that decomposes the Lie algebra $\mathcal{G}$ in $\mathbb{Z}$-graded subspaces

$$\mathcal{G} = \bigoplus_i \mathcal{G}_i, \tag{2.5}$$

where

$$[Q, \mathcal{G}_i] = i\mathcal{G}_i, \quad [\mathcal{G}_i, \mathcal{G}_j] \in \mathcal{G}_{i+j}, \tag{2.6}$$

and $\mathcal{G}_0$ is the zero grade subalgebra. Throughout this paper we shall restrict to integer gradings, defined by

$$Q = \sum_{k=1}^{rank\,\mathcal{G}} \frac{\lambda_k \cdot H}{\alpha_k^2}. \tag{2.7}$$

In this case $\mathcal{G}_i (i \neq 0)$ contain the step operators composed of i simple roots (see the appendix for the algebraic nomenclature). It then follows that

$$\mathcal{G}_< = \bigoplus_{i<0} \mathcal{G}_i \quad \text{and} \quad \mathcal{G}_> = \bigoplus_{i>0} \mathcal{G}_i \tag{2.8}$$

are subalgebras generated by negative/positive step operators, while the zero grade $\mathcal{G}_0$ is an abelian subalgebra spanned by the Cartan subalgebra of $\mathcal{G}$, i.e., $\mathcal{G}_0 = U(1)^{rank\,\mathcal{G}}$. A general group element $g \in G$ can be written considering the Gauss decomposition

$$g = NBM, \tag{2.9}$$

where $N = e^{\mathcal{G}_<}$, $B = e^{\mathcal{G}_0}$ and $M = e^{\mathcal{G}_>}$. The Toda models will correspond to the coset $G_- \setminus G/G_+$, for $G_\pm$ generated by positive/negative grade operators.

In the Hamiltonian Reduction procedure [11, 12, 13] constraints are imposed to the chiral currents $J, \bar{J}$ in such a way that the equations of motion (2.3) reduce to equations for the zero grade subgroup only

$$\begin{aligned} &\bar{\partial}(B^{-1}\partial B) + [\epsilon_-, B^{-1}\epsilon_+ B] = 0, \\ &\partial(\bar{\partial} B B^{-1}) - [\epsilon_+, B\epsilon_- B^{-1}] = 0. \end{aligned} \tag{2.10}$$

Here we have introduced the constant generators $\epsilon_\pm$ of grade ± 1 respectively. At this point, it is not difficult to see that the physical fields or Toda fields will parameterize the zero grade subspace and through the Hamiltonian Reduction procedure the degrees of freedom associated to generators in N and M have been eliminated. In this way, (2.10) are the equations of motion of the constraint model and they were firstly obtained by Leznov and Saveliev within the zero-curvature formalism [16].

On the other hand, the Leznov-Saveliev equations (2.10), can also be obtained as the equations of motion of a gauged $WZNW$ model. Demanding (2.1) be invariant under the transformations

$$g \to g' = \alpha_- g \alpha_+, \tag{2.11}$$

where $\alpha_- \in G_-$ and $\alpha_+ \in G_+$, the gauge fields A, $\bar{A}$ transforming as

$$A \to A' = \alpha_- A \alpha_-^{-1} + \alpha_- \partial \alpha_-^{-1}, \qquad \bar{A} \to \bar{A}' = \alpha_+^{-1} \bar{A} \alpha_+ + \bar{\partial} \alpha_+^{-1} \alpha_+ \tag{2.12}$$

must be introduced. The invariant action under the above transformations is constructed introducing also the constant generators $\epsilon_\pm$ of grade ± 1 and it reads

$$S_{G/H} = S_{WZNW}(g) - \frac{k}{2\pi}\int d^2zTr(A(\bar{\partial}gg^{-1} - \epsilon_+) + \bar{A}(g^{-1}\partial g - \epsilon_-) + Ag\bar{A}g^{-1}). \quad (2.13)$$

Since the action (2.13) is $G_\pm$-invariant, it is possible to choose $\alpha_- = N^{-1}$ and $\alpha_+ = M^{-1}$. Taking also into account the graded structure, the functional integration over the gauge fields will render the effective action [13]

$$S_{eff} = S_{WZNW}(B) + \frac{k}{2\pi}\int d^2zTr(\epsilon_+ B\epsilon_- B^{-1}), \quad (2.14)$$

which will have (2.10) as corresponding Euler-Lagrange equations of motion. This is in fact the action of an integrable theory. It is well known that in general the equations of motion of an integrable field theory in two dimensions can be written as a zero-curvature condition

$$\bar{\partial}A - \partial\bar{A} + [A, \bar{A}] = 0\ , \quad (2.15)$$

where A and $\bar{A}$ are two given potentials, which depend on space and time, and on a spectral parameter λ. In the Leznov-Saveliev [16] formulation of Toda models, the equations of motion (2.10) can be deduced from (2.15) taking the potentials A and $\bar{A}$ as

$$A = B\epsilon_- B^{-1} \quad \text{and} \quad \bar{A} = -\epsilon_+ - \bar{\partial}BB^{-1}. \quad (2.16)$$

From this condition the existence of an infinite chain of conserved charges is inferred. Due to that within the zero-curvature formalism the integrability properties model defined by the effective action (2.14) are more explicit.

2.1 $SL(n, \mathbb{R})$ Abelian Toda theories

The simplest class of Toda models is the Abelian Toda which is known to be completely integrable and conformal invariant, that is why they are usually called as Conformal Toda models. These models correspond to an Abelian subgroup $G_0 \subset G$. Let us consider the grading operator (2.7) for $SL(n, \mathbb{R})$,i.e.

$$Q = \sum_{i=1}^{n-1}\frac{2\lambda_i \cdot H}{\alpha_i^2}, \quad (2.17)$$

where H is the corresponding Cartan subalgebra, α_i is the i^{th} simple root and λ_i is the i^{th} fundamental weight that satisfies

$$\frac{2\lambda_i \cdot \alpha_j}{\alpha_i^2} = \delta_{ij}. \quad (2.18)$$

The Abelian subalgebra of grade zero is then $\mathcal{G}_0 = U(1)^{n-1} = \{h_1, h_2, \dots h_{n-1}\}$, where the Cartan generators are defined in the Chevalley basis as $h_i = \frac{2\alpha_i \cdot H}{\alpha_i^2}$. The constant generators of grade ± 1 can be chosen as

$$\epsilon_\pm = \sum_{i=1}^{n-1} \mu_i E_{\pm\alpha_i}, \tag{2.19}$$

where μ_i are constant parameters. A group element B in G_0 is generated by the Cartan subalgebra as $B = e^{\varphi_i h_i}$ and (2.14) yields for this particular case, the action of the Abelian Toda models

$$S_{Toda} = -\frac{k}{4\pi} \int d^2z (k_{ij} \partial\varphi_i \bar{\partial}\varphi_j - 2\sum_{i=1}^{n-1} \mu_i^2 e^{-k_{ij}\varphi_j}), \tag{2.20}$$

where k_{ij} is the Cartan matrix and where we have used the algebraic properties defined on the appendix. The equations of motion of the abelian $SL(n, \mathbb{R})$ Toda model can be easily deduced from (2.20) or (2.10) to be

$$\bar{\partial}\partial\varphi_i = \mu_i^2 e^{-k_{ij}\varphi_j}. \tag{2.21}$$

These are integrable and conformal invariant theories, whose simplest example corresponds to the Lie algebra $SL(2, \mathbb{R})$

$$\bar{\partial}\partial\varphi = \mu^2 e^{-2\varphi}, \tag{2.22}$$

which is Liouville model, a well known theory that appears in applications related to string theory and two-dimensional quantum gravity.

2.2 $\widehat{SL}(n, \mathbb{R})$ Abelian Affine Toda theories

A similar construction takes place for $\widehat{SL}(n, \mathbb{R})$, an infinite-dimensional affine Kac-Moody algebra. The model now is described formally by the same action (2.1), but in this case $\mathcal{G}$ is an infinite-dimensional algebra (see the appendix). This model is called two-loop $WZNW$ [13] and is also conformal invariant. The Hamiltonian reduction procedure can be also applied in this case with the difference that N as well as M are now infinite-dimensional. The zero grade subgroup B is however chosen to be finite-dimensional. As result of the reduction process the Leznov-Saveliev equations (2.10) are also obtained for the constrained model as well as the effective action (2.14). Consider now the affine algebra $\widehat{SL}(n, \mathbb{R})$. The grading operator Q in the principal gradation for this algebra is taken as

$$Q = \sum_{i=1}^{n-1} \frac{2\lambda_i \cdot H^{(0)}}{\alpha_i^2} + nd, \tag{2.23}$$

where d is the derivation and its coefficient is chosen such that this gradation ensures that the zero grade subspace $\mathcal{G}_0$ coincides with its counterpart on the corresponding Lie algebra $SL(n,\mathbb{R})$, apart from the generators c, d. The major difference with the finite case is in the constant generators of grade ± 1 that include extra affine generators, say

$$\epsilon_\pm = \sum_{i=1}^{n-1} \mu_i E^{(0)}_{\pm\alpha_i} + m_0 E^{(\pm 1)}_{\mp\psi}, \tag{2.24}$$

here ψ is the highest root of $\mathcal{G} = SL(n,\mathbb{R})$. The Abelian subalgebra of grade zero is now $\mathcal{G}_0 = \{h^{(0)}_1, h^{(0)}_2, \dots h^{(0)}_{n-1}, c, d\}$. If B is parameterized as an exponentiation of these generators the models obtained preserve the conformal invariance and they are called as Conformal Affine Toda models

$$\begin{aligned} \bar\partial\partial\varphi_i &= \mu_i^2 e^{-k_{ij}\varphi_j} - m_0^2 l_i^\psi e^{k_{\psi j}\varphi_j + 2\eta}, \\ \bar\partial\partial\nu &= \frac{2}{\psi^2} m_0^2 e^{k_{\psi j}\varphi_j + 2\eta}. \\ \bar\partial\partial\eta &= 0, \end{aligned} \tag{2.25}$$

where $k_{\psi j} = \frac{2\psi\cdot\alpha_j}{\alpha_j^2}$ is the extended Cartan matrix and $\frac{\psi}{\psi^2} = l_j^\psi \frac{\alpha_j}{\alpha_j^2}$. Taking $\eta = 0$ the conformal invariance is broken, and the first equation in (2.25) reduce to the $\widetilde{SL}(n,\mathbb{R})$ Affine Toda theory [17]. The action that posses (2.25) as equations of motion can be easily obtained from (2.14) and it reads

$$\begin{aligned} S_{CAT} = &- \frac{k}{4\pi} \int d^2 z \left(\frac{2}{\alpha_i^2} k_{ij} \partial\varphi_i \bar\partial\varphi_j + \right. \\ &\left. \bar\partial\nu\partial\eta + \bar\partial\eta\partial\nu - \left(\frac{4}{\alpha_i^2} \sum_{i=1}^{n-1} \mu_i e^{-k_{ij}\varphi_j} + \frac{4}{\psi^2} m_0 e^{k_{\psi_j}\varphi_j + 2\eta} \right) \right). \end{aligned} \tag{2.26}$$

The generalized sinh-Gordon model which corresponds to the affine algebra $\widehat{SL}(2,\mathbb{R})$ is the simplest example of these theories, say

$$\begin{aligned} \bar\partial\partial\eta &= 0, \\ \bar\partial\partial\nu &= \mu^2 e^{2\varphi - \eta}, \\ \partial\bar\partial\varphi &= \mu^2 (e^{-2\varphi} - e^{2\varphi - \eta}), \end{aligned} \tag{2.27}$$

where we have taken the parameters as $\mu_1 = m_0 = \mu$. The corresponding action (2.14) is expressed as

$$S_{gsh-G} = -\frac{k}{4\pi} \int d^2 x \left(2\partial\varphi\bar\partial\varphi + \bar\partial\nu\partial\eta + \bar\partial\eta\partial\nu - 2\mu^2 (e^{-2\varphi} + e^{2\varphi - \eta}) \right). \tag{2.28}$$

Considering the complexification of the real field $\varphi \to i\varphi$ and $\eta = 0$ the sine-Gordon model is obtained [1]

$$S_{s-G} = \frac{k}{2\pi}\int d^2x \left(\partial\varphi\bar{\partial}\varphi + 2\mu^2(\cos\varphi - 1)\right). \tag{2.29}$$

This model, as well as the complexified Affine Toda theories, have the nice property of presenting a real action admitting soliton solutions. Due to that these theories found many applications in phenomena that goes from condensed matter until particle physics.

3 Toda Field Theories on a noncommutative space

In this section we will extend the Hamiltonian Reduction technique to the noncommutative plane. Usually a NCFT [20] is constructed from a given field theory by replacing the product of fields by an associative $\star$-product. When the noncommutative parameter $\theta^{\mu\nu}$ is a constant antisymmetric tensor, the deformed product of functions is expressed trough the Moyal product [21]

$$\phi_1(x)\phi_2(x) \to \phi_1(x)\star\phi_2(x) = e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu^{x_1}\partial_\nu^{x_2}}\phi_1(x_1)\phi_2(x_2)|_{x_1=x_2=x}. \tag{3.30}$$

In the following we will refer to functions of operators in the noncommutative deformation by a $\star$ sub-index, for example $e_\star^\phi = \sum_{n=1}^\infty \frac{1}{n!}\phi_\star^n$ (the n-times star-product of ϕ is understood).

Let us start from the nc generalization of the $WZNW$ model introduced in [18]

$$\begin{aligned} S_{WZNW_\star} &= -\frac{k}{4\pi}\int_\Sigma d^2zTr(g^{-1}\star\partial g\star g^{-1}\star\bar{\partial}g) + \\ &\quad \frac{k}{24\pi}\int_B d^3x\epsilon_{ijk}(g^{-1}\star\partial_i g\star g^{-1}\star\partial_j g\star g^{-1}\star\partial_k g). \end{aligned} \tag{3.31}$$

Here B is a three-dimensional manifold whose boundary $\partial B = \Sigma$. We consider that the coordinates $z, \bar{z}$ or equivalently x, t are noncommutative, but the extended coordinate y on the manifold B remain commutative

$$[z, \bar{z}] = \theta, \qquad [y, z] = [y, \bar{z}] = 0. \tag{3.32}$$

The Euler-Lagrange equations of motion corresponding to $S_{WZNW_\star}$ are the trivial nc extensions of (2.3)

$$\bar{\partial}J = \partial\bar{J} = 0, \tag{3.33}$$

[1] The zero point energy have been subtracted.

where it is proved that the chiral currents J and $\bar{J}$

$$J = g^{-1} \star \partial g, \qquad \bar{J} = -\bar{\partial} g \star g^{-1}. \tag{3.34}$$

are conserved. The fields α_a parameterize the group element $g \in G$ through

$$g = e_\star^{\alpha_a T_a}, \tag{3.35}$$

where T_a are the generators of the corresponding algebra $\mathcal{G}$. Suppose we have defined a grading operator Q in the algebra $\mathcal{G}$ that decomposes it in $\mathbb{Z}$-graded subspaces as in (2.5),

$$[Q, \mathcal{G}_i] = i\mathcal{G}_i, \qquad [\mathcal{G}_i, \mathcal{G}_j] \in \mathcal{G}_{i+j}. \tag{3.36}$$

This means that the algebra $\mathcal{G}$ can be represented as the direct sum

$$\mathcal{G} = \bigoplus_i \mathcal{G}_i \tag{3.37}$$

and that the subspaces $\mathcal{G}_0, \mathcal{G}_>, \mathcal{G}_<$ are subalgebras of $\mathcal{G}$, composed of the Cartan and of the positive/negative steps generators respectively. The algebra can then be written using the triangular decomposition

$$\mathcal{G} = \mathcal{G}_< \bigoplus \mathcal{G}_0 \bigoplus \mathcal{G}_>. \tag{3.38}$$

Denote the subgroup elements obtained through the $\star$-exponentiation of the generators of the corresponding subalgebras as

$$N = e_\star^{\alpha_-^{(i)} E_{-\alpha_i}}, \quad B = e_\star^{\alpha_0^{(i)} H_i}, \quad M = e_\star^{\alpha_+^{(i)} E_{\alpha_i}}, \tag{3.39}$$

where we have chosen the nomenclature $\mathcal{G}_< = \{E_{-\alpha_i}\}$, $\mathcal{G}_> = \{E_{\alpha_i}\}$ and $\mathcal{G}_0 = \{H_i\}$ for the generators of the subalgebras. Consider that we can write an element g of the nc group G as

$$g = N \star B \star M. \tag{3.40}$$

The chiral currents $J, \bar{J}$ are written then as

$$J = M^{-1} \star K \star M, \qquad \bar{J} = -N \star \bar{K} \star N^{-1}, \tag{3.41}$$

where

$$\begin{aligned} K &= B^{-1} \star N^{-1} \star \partial N \star B + B^{-1} \star \partial B + \partial M \star M^{-1}, \\ \bar{K} &= N^{-1} \star \bar{\partial} N + \bar{\partial} B \star B^{-1} + B \star \bar{\partial} M \star M^{-1} \star B^{-1}. \end{aligned} \tag{3.42}$$

Introducing (3.41) in (3.33) the equations of motion read

$$\begin{aligned} \bar{\partial} K + [K, \bar{\partial} M \star M^{-1}]_\star &= 0, \\ \partial \bar{K} - [\bar{K}, N^{-1} \star \bar{\partial} N]_\star &= 0. \end{aligned} \tag{3.43}$$

The reduced model is defined by giving the constant elements $\epsilon_{\pm}$ responsible for constraining the currents in a general manner to [2]

$$J_{constr} = j + \epsilon_{-}, \qquad \bar{J}_{constr} = \bar{j} + \epsilon_{+}, \tag{3.44}$$

where $j, \bar{j}$ contain generators of grade zero and positive, and zero and negative respectively, and $\epsilon_{\pm}$ are constant generators of grade ± 1. The effect of the constraints on the chiral currents $J, \bar{J}$ (3.41) translates in the conditions

$$\begin{aligned} B^{-1} \star N^{-1} \star \partial N \star B|_{constr} &= \epsilon_{-}, \\ B \star \bar{\partial} M \star M^{-1} \star B^{-1}|_{constr} &= \epsilon_{+}, \end{aligned} \tag{3.45}$$

because from to the graded structure these are the only terms in (3.42) that contain generators of negative and positive grade respectively. As result of the reduction process, the degrees of freedom in M, N are eliminated and the equations of motion of the constrained model are natural nc extensions of the Leznov-Saveliev equations of motion (2.10)

$$\begin{aligned} &\bar{\partial}(B^{-1} \star \partial B) + [\epsilon_{-}, B^{-1} \star \epsilon_{+} B]_{\star} = 0, \\ &\partial(\bar{\partial} B \star B^{-1}) - [\epsilon_{+}, B\epsilon_{-} \star B^{-1}]_{\star} = 0. \end{aligned} \tag{3.46}$$

Both equations are equivalent. One can see that by $\star$-multiplying B from the left and B^{-1} from the right, the first equation in (3.46), say

$$\begin{aligned} B \star \{\bar{\partial}(B^{-1} \star \partial B) + [\epsilon_{-}, B^{-1} \star \epsilon_{+} B]_{\star} &= 0\} \star B^{-1}, \\ -\bar{\partial} B \star B^{-1} \star \partial B \star B^{-1} + \bar{\partial}\partial B \star B^{-1} - [\epsilon_{+}, B\epsilon_{-} \star B^{-1}]_{\star} &= 0, \end{aligned} \tag{3.47}$$

then using that $\partial(B \star B^{-1}) = 0$ the second equation in (3.46) is obtained. That means that both equations are simultaneously satisfied. In [4] these equations of motion were used to define a nc extension of the sinh/sine-Gordon model. In contrast to the previous suggestion [4], in this paper we propose an alternative definition for B more appropriate to the peculiarities of nc groups. This novel choice will lead to a nc sine-Gordon defined as a system of two coupled second order equations for two scalar fields that reduced to sine-Gordon model and a free scalar field in the commutative limit. The nc extensions of the Leznov-Saveliev equations (3.46) for $GL(2)$ were also obtained in [23] from the nc generalization of the $SL(2)$ affine Toda model coupled to matter (Dirac) fields.

As shown in [4], the equations of motion (3.46) can be expressed as a generalized $\star$-zero-curvature condition

$$\bar{\partial} A - \partial \bar{A} + [A, \bar{A}]_{\star} = 0, \tag{3.48}$$

[2] see [11, 12, 13] for the commutative case

taking the potentials as

$$A = -B \star \epsilon_- \star B^{-1}, \qquad \bar{A} = \epsilon_+ + \bar{\partial} B \star B^{-1}. \tag{3.49}$$

The $\star$-zero-curvature condition (3.48) implies the existence of an infinite amount of conserved charges [4]. For this reason in order to preserve the original integrability properties of the two-dimensional models, (3.46) can be a reasonable starting point to construct nc analogs of Toda models.

3.1 The noncommutative action

In the previous section it was illustrated how ordinary Toda models can be regarded as gauged-$WZNW$ models [11, 12, 13]. On the other side, in the first part of this section we showed that considering a nc extension of the Gauss decomposition, and imposing some constraints on the currents $J, \bar{J}$, the equations of motion of the $WZNW_\star$ reduced to the natural nc extension of the Leznov-Saveliev equations (3.46). One can verify that the equations (3.46) are the Euler-Lagrange equations of motion of the nc extension of (2.14), namely

$$S = S_{WZNW_\star}(B) + \frac{k}{2\pi} \int d^2 z Tr(\epsilon_+ \star B \epsilon_- \star B^{-1}). \tag{3.50}$$

In [22] different gauged $WZNW_\star$ models were constructed. However, the integration over the gauge fields requires special care. For this reason in the present paper we limit to propose the action (3.50) as corresponding to the equations (3.46) and it remains to be proved if the action (3.50) can be obtained from (3.31) gauging the degrees of freedom in M, N and integrating over the corresponding gauge fields.

3.2 NC Liouville

Let us now construct the nc analog of the simplest example of an abelian Toda model, the Liouville theory. This model is obtained through the Hamiltonian procedure connected to $SL(2)$. In the noncommutative case the $SL(2)$ group is not closed. Due to that the nc extension should be related to $GL(2)$. We will parameterize the zero grade group element B as

$$B = e_\star^{\varphi_1 h} \star e_\star^{\varphi_0 I}, \tag{3.51}$$

where h is the Cartan and I is the identity generator. Notice that this subgroup is now nonabelian, i.e, if g_1, g_2 are two elements of the zero grade subgroup G_0

$$g_1 \star g_2 \neq g_2 \star g_1. \tag{3.52}$$

The constant generators of grade ± 1 are taken as

$$\epsilon_{\pm} = \mu E_{\pm\alpha}. \tag{3.53}$$

Introduce (3.51) and (3.53) in the nc generalization of Leznov-Saveliev equations (3.46). Using algebraic properties it is not difficult to see that

$$B \star \epsilon_{-} B^{-1} = e_{\star}^{-2\varphi_1} \epsilon_{-}. \tag{3.54}$$

For computing the equations of motion one can make use of the usual $SL(2)$ representation as Pauli matrices

$$E_{\alpha} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.55}$$

and in this way find the equations that define the nc extension of Liouville model

$$\begin{aligned} \partial(\bar{\partial}(e_{\star}^{\varphi_1} \star e_{\star}^{\varphi_0}) \star e_{\star}^{-\varphi_0} \star e_{\star}^{-\varphi_1}) &= \mu^2 e_{\star}^{-2\varphi_1}, \\ \partial(\bar{\partial}(e_{\star}^{-\varphi_1} \star e_{\star}^{\varphi_0}) \star e_{\star}^{-\varphi_0} \star e_{\star}^{\varphi_1}) &= -\mu^2 e_{\star}^{-2\varphi_1}. \end{aligned} \tag{3.56}$$

One can also combine the previous equations to find the system of coupled second order equations

$$\begin{aligned} \partial(\bar{\partial}(e_{\star}^{\varphi_1} \star e_{\star}^{\varphi_0}) \star e_{\star}^{-\varphi_0} \star e_{\star}^{-\varphi_1} + \bar{\partial}(e_{\star}^{-\varphi_1} \star e_{\star}^{\varphi_0}) \star e_{\star}^{-\varphi_0} \star e_{\star}^{\varphi_1}) &= 0, \\ \partial(\bar{\partial}(e_{\star}^{\varphi_1} \star e_{\star}^{\varphi_0}) \star e_{\star}^{-\varphi_0} \star e_{\star}^{-\varphi_1} - \bar{\partial}(e_{\star}^{-\varphi_1} \star e_{\star}^{\varphi_0}) \star e_{\star}^{-\varphi_0} \star e_{\star}^{\varphi_1}) &= 2\mu^2 e_{\star}^{-2\varphi_1}. \end{aligned} \tag{3.57}$$

The equations of motion written in this way are more suitable for applying the noncommutative limit. When $\theta \to 0$ that system of equations easily decoupled to

$$\partial\bar{\partial}\varphi_0 = 0 \quad \text{and} \quad \partial\bar{\partial}\varphi_1 = \mu^2 e^{-2\varphi_1}, \tag{3.58}$$

where it is evident that the first equation of (3.57) transform to a free field equation for φ_0 and the second leads to the usual Liouville equation (2.22). We can write an action for the nc Liouville model (3.57) using the general expression (3.50) and making use of the nc generalization of the Polyakov-Wiegmann identity

$$\begin{aligned} S_{WZNW_\star}(g_1 \star g_2) &= S_{WZNW_\star}(g_1) + S_{WZNW_\star}(g_2) - \\ &\quad -\frac{1}{2\pi} \int dz d\bar{z} Tr(g_1^{-1} \star \partial g_1 \star \bar{\partial} g_2 \star g_2^{-1}). \end{aligned} \tag{3.59}$$

For instance, introduce (3.51) and (3.53) in (3.50) to obtain the defining action of nc Liouville

$$\begin{aligned} S(\varphi_1, \varphi_0) &= 2S_{PC_\star}(e_{\star}^{\varphi_1}) + 2S_{WZNW_\star}(e_{\star}^{\varphi_0}) + \frac{k}{2\pi} \int d^2 z \mu^2 e_{\star}^{-2\varphi_1} - \\ &\quad -\frac{k}{2\pi} \int d^2 z (\bar{\partial} e_{\star}^{\varphi_0} \star e_{\star}^{-\varphi_0} \star (e_{\star}^{-\varphi_1} \star \partial e_{\star}^{\varphi_1} + e_{\star}^{\varphi_1} \star \partial e_{\star}^{-\varphi_1})), \end{aligned} \tag{3.60}$$

where using the notation of [6] we have defined

$$S_{PC_\star}(g) = -\frac{k}{4\pi}\int_\Sigma d^2 z Tr(g^{-1}\star\partial g\star g^{-1}\star\bar{\partial} g). \tag{3.61}$$

Looking at the action (3.60) it is noticeable the presence of the topological term of $WZNW_\star$, in this case shifted to the φ_0 field. This situation contrast with the ordinary commutative case, where for Liouville and more generally for any abelian subspace this term is identically null.

The nonabelian character of the zero grade subgroup G_0 in the nc case allows an alternative parameterization for the element B, namely

$$B = e_\star^{\varphi_1 h + \varphi_0 I}. \tag{3.62}$$

If one make use of the representation property of the algebra $SL(2,\mathbb{R})$

$$\{h, E_{\pm\alpha}\} = 0, \quad \{\, h,\, h\} \;=\; 2I, \tag{3.63}$$

where by $\{\,,\,\}$ we mean anticommutator, the element B can be also expressed in this parameterization as

$$B = \frac{1}{2}(e_\star^{\varphi_0+\varphi_1} + e_\star^{\varphi_0-\varphi_1})I + \frac{1}{2}(e_\star^{\varphi_0+\varphi_1} - e_\star^{\varphi_0-\varphi_1})h. \tag{3.64}$$

The parameterizations (3.51) and (3.58) for the element B belonging to the zero grade subgroup in the commutative case are identical, what is not the case here. Let us now change variables to

$$\phi_+ = \varphi_0 + \varphi_1 \quad \text{and} \quad \phi_- = \varphi_0 - \varphi_1. \tag{3.65}$$

Then, introduce (3.64) and (3.53) in the nc Leznov-Saveliev equations (3.46). Taking into account that

$$B\star\epsilon_- B^{-1} = e_\star^{\phi_-}\star e_\star^{-\phi_+}\epsilon_-, \tag{3.66}$$

an alternative nc generalization of Liouville model is obtained, i.e.

$$\begin{aligned}\partial(\bar{\partial}(e_\star^{\phi_+})\star e_\star^{-\phi_+}) &= \mu^2 e_\star^{\phi_-}\star e_\star^{-\phi_+},\\ \partial(\bar{\partial}(e_\star^{\phi_-})\star e_\star^{-\phi_-}) &= -\mu^2 e_\star^{\phi_-}\star e_\star^{-\phi_+}.\end{aligned} \tag{3.67}$$

One can now also compute the sum and difference of that equations to find

$$\begin{aligned}\partial(\bar{\partial}(e_\star^{\phi_+})\star e^{-\phi_+} + \bar{\partial}(e_\star^{\phi_-})\star e^{-\phi_-}) &= 0,\\ \partial(\bar{\partial}(e_\star^{\phi_+})\star e^{-\phi_+} - \bar{\partial}(e_\star^{\phi_-})\star e^{-\phi_-}) &= 2\mu^2 e_\star^{\phi_-}\star e_\star^{-\phi_+}.\end{aligned} \tag{3.68}$$

For studying the commutative limit it is better to return to the original variables (3.65). When $\theta \to 0$, the same decoupled model of two fields (3.58) is obtained. The action corresponding to the nc analog of Liouville model obtained in this parameterization (3.68), follows from (3.50), namely

$$S(\phi_+, \phi_-) = S_{WZNW_\star}(e_\star^{\phi_+}) + S_{WZNW_\star}(e_\star^{\phi_-}) + \frac{k}{2\pi}\int d^2z\mu^2 e_\star^{\phi_-} \star e_\star^{-\phi_+}. \quad (3.69)$$

Notice that this action have the left-right local symmetry

$$e_\star^{\phi_+} \to h(z) \star e_\star^{\phi_+(z,\bar{z})} \star \tilde{h}(\bar{z}), \quad e_\star^{\phi_-} \to h(z) \star e_\star^{\phi_-(z,\bar{z})} \star \tilde{h}(\bar{z}), \quad (3.70)$$

where $h(z), \tilde{h}(\bar{z}) \in G_0$. The symmetry properties and the full integrability of the nc Liouville model here proposed will require a more detailed study in future research.

3.3 NC sinh/sine-Gordon

As we already saw the Conformal Affine Toda models are connected with Kac-Moody algebras, and the related affine theories with loop algebras. Specifically, the sinh/sine-Gordon model is connected with the $\widetilde{SL}(2)$ loop algebra. In [12] was discussed how this model can be regarded as a spontaneously broken and reduced version of the corresponding Conformal Affine $\widehat{SL}(2)$ Toda model (2.28). As stated in [12] for process where only the field φ of (2.28) is involved one can simply forget about η and ν and set them to zero in the action.

The zero grade subspace defined by the gradation operator

$$Q = \frac{\lambda \cdot H^{(0)}}{2} + 2d \quad (3.71)$$

is finite-dimensional and it is extended in relation to the finite-dimensional case by the central charge and the derivation generator, i.e. $\mathcal{G}_0 = \{h^{(0)}, c, d\}$. In order to define the model, one can firstly parameterize B as

$$B = B_0 \star e_\star^{\eta d} \star e_\star^{\nu c}, \quad (3.72)$$

where B_0 is the same zero grade subspace of the related $SL(2)$ Lie algebra and choose the constant generators of grade ± 1 as

$$\epsilon_\pm = \mu(E_{\pm\alpha}^{(0)} + E_{\mp\alpha}^{(\pm 1)}). \quad (3.73)$$

As we are only interested in the generalization of the affine model, we will consider that the fields in the direction of the generators c, d are equal to zero, i.e. $\eta = 0$ and $\nu = 0$.

Introducing (3.73) and (3.51) in (3.46) and working again in the representation (3.55), the nc sinh-Gordon

$$\begin{aligned}
\partial(\bar{\partial}(e_\star^{\varphi_1}\star e_\star^{\varphi_0})\star e_\star^{-\varphi_0}\star e_\star^{-\varphi_1}) &= \mu^2(e_\star^{-2\varphi_1}-e_\star^{2\varphi_1}),\\
\partial(\bar{\partial}(e_\star^{-\varphi_1}\star e_\star^{\varphi_0})\star e_\star^{-\varphi_0}\star e_\star^{\varphi_1}) &= -\mu^2(e_\star^{-2\varphi_1}-e_\star^{2\varphi_1}),
\end{aligned} \tag{3.74}$$

is obtained. Let us now present the sum and difference of the previous equations

$$\begin{aligned}
\partial(\bar{\partial}(e_\star^{\varphi_1}\star e_\star^{\varphi_0})\star e_\star^{-\varphi_0}\star e^{-\varphi_1}+\bar{\partial}(e_\star^{-\varphi_1}\star e_\star^{\varphi_0})\star e_\star^{-\varphi_0}\star e_\star^{\varphi_1}) &= 0, \qquad (3.75)\\
\partial(\bar{\partial}(e_\star^{\varphi_1}\star e_\star^{\varphi_0})\star e_\star^{-\varphi_0}\star e^{-\varphi_1}-\bar{\partial}(e_\star^{-\varphi_1}\star e_\star^{\varphi_0})\star e_\star^{-\varphi_0}\star e_\star^{\varphi_1}) &= 2\mu^2(e_\star^{-2\varphi_1}-e_\star^{2\varphi_1}),
\end{aligned}$$

the complexified version of this system ($\varphi_1 \to i\varphi_1, \quad \varphi_0 \to i\varphi_0$) reproduce one of the suggestions for nc sine-Gordon presented in [6]. Notice that the first equation of this system was also obtained in the nc generalization of the Liouville model (3.57). This equation in the commutative limit $\theta \to 0$ becomes a free field equation for φ_0. The second equation of (3.75) in the same limit produces the usual sinh-Gordon equation (2.27)

$$\partial\bar{\partial}\varphi_1 + 2\mu^2\sinh\varphi_1 = 0. \tag{3.76}$$

The action for (3.74), (3.75) can be obtained from (3.50) with B as in (3.51) and the constant generators $\epsilon_\pm$ as in (3.73), i.e.

$$\begin{aligned}
S(\varphi_1,\varphi_0) &= 2S_{PC_\star}(e_\star^{\varphi_1}) + 2S_{WZNW_\star}(e_\star^{\varphi_0}) + \frac{k}{2\pi}\int d^2z\mu^2(e_\star^{-2\varphi_1}+e_\star^{2\varphi_1}) -\\
&\quad -\frac{k}{2\pi}\int d^2z(\bar{\partial}e_\star^{\varphi_0}\star e_\star^{-\varphi_0}\star(e_\star^{-\varphi_1}\star\partial e_\star^{\varphi_1}+e_\star^{\varphi_1}\star\partial e_\star^{-\varphi_1})).
\end{aligned} \tag{3.77}$$

As was presented in [6], where the scattering amplitudes for some particle dispersion process were computed, no particle production seems to occur in this model what leads to a factorized and casual S-matrix. In this sense, the nc sine-Gordon model as defined in (3.77) apparently retain the integrability properties of the original model.

Consider now the alternative parameterization of B (3.62) to define an equivalent nc extension of the sinh-Gordon model. Using the variables (3.65), the Leznov-Saveliev equations (3.46) will render the system of equations

$$\begin{aligned}
\partial(\bar{\partial}(e_\star^{\phi_+})\star e_\star^{-\phi_+}) &= \mu^2(e_\star^{\phi_-}\star e_\star^{-\phi_+}-e_\star^{\phi_+}\star e_\star^{-\phi_-}),\\
\partial(\bar{\partial}(e_\star^{\phi_-})\star e_\star^{-\phi_-}) &= \mu^2(-e_\star^{\phi_-}\star e_\star^{-\phi_+}+e_\star^{\phi_+}\star e_\star^{-\phi_-}).
\end{aligned} \tag{3.78}$$

Computing the sum and difference of the previous equations

$$\begin{aligned}
\partial(\bar{\partial}(e_\star^{\phi_+})\star e_\star^{-\phi_+}+\bar{\partial}(e_\star^{\phi_-})\star e_\star^{-\phi_-}) &= 0,\\
\partial(\bar{\partial}(e_\star^{\phi_+})\star e_\star^{-\phi_+}-\bar{\partial}(e_\star^{\phi_-})\star e_\star^{-\phi_-}) &= 2\mu^2(e_\star^{\phi_-}\star e_\star^{-\phi_+}-e_\star^{\phi_+}\star e_\star^{-\phi_-}),
\end{aligned} \tag{3.79}$$

an alternative nc analog of sinh-Gordon model is obtained. The complexified version of this system of equations was also obtained in [6]. One more time, the first of these two equations is shared with the nc Liouville generalization (3.68) in this parameterization. Taking the commutative limit it produces a free field equation for φ_0. In [23], the version (3.78) of nc sine-Gordon was derived through a reduction process from the nc Affine Toda model coupled to matter fields.

The action (3.50) corresponding to this nc generalization of the sinh-Gordon model reads

$$S(\phi_+,\phi_-) = S_{WZNW_\star}(e_\star^{\phi_+}) + S_{WZNW_\star}(e_\star^{\phi_-}) \quad + \quad + \quad \frac{k}{2\pi}\int d^2z\mu^2(e_\star^{\phi_-}\star e_\star^{-\phi_+} + e_\star^{\phi_+}\star e_\star^{-\phi_-}). \tag{3.80}$$

Notice that this action also presents the left-right local symmetry (3.70). The nc sinh-Gordon models (3.75) and (3.79) can be obtained by a suitable reduction from nc self-dual Yang-Mills [6]. In this part of the section we have seen that it is possible to construct them starting directly from two-dimensions just taking into account the appropriate algebraic extension.

3.4 NC $GL(n,\mathbb{R})$ abelian Toda theories

In the following we will construct nc analogs of the $SL(n,\mathbb{R})$ abelian Toda theories described in section 2. As was point out at the beginning of this section, in the nc scenario the zero grade subgroup G_0 despite it is spanned by the generators of the Cartan subalgebra, turns out to be nonabelian. That is why abelian makes reference to the property of the original theory.

Considering the extension of the $SL(n,\mathbb{R})$ algebra to $GL(n,\mathbb{R})$, what is necessary in order to obtain a nc closed group, the gradation operator (2.17) defines the subalgebra of grade zero $\mathcal{G}_0 = U(1)^n = \{I, h_i,\ \ i = 1\dots n-1\}$. The zero grade group element B is then expressed through the $\star$-exponentiation of the generators of the zero grade subalgebra $\mathcal{G}_0$, i.e. the $SL(n,\mathbb{R})$ Cartan subalgebra plus the identity

$$B = e_\star^{\Sigma_{i=1}^{n-1}\varphi_i h_i + \varphi_0 I}. \tag{3.81}$$

The constant generators of grade ±1 are chosen as

$$\epsilon_\pm = \Sigma_{i=1}^{n-1}\mu_i E_{\pm\alpha_i}, \tag{3.82}$$

where $E_{\pm\alpha_i}$ are the steps generators associated to the positive/negative simple roots of the algebra. In the ordinary commutative case this consideration will lead to a combined model of a free field φ_0 plus (2.20), but in the presence of the noncommutative parameter the field associated to the identity generator does not decouple.

Let us consider the $n \times n$ matrix representation

$$(h_i)_{\mu\nu} = \delta_{\mu\nu}(\delta_{i,\mu} - \delta_{i+1,\mu}), \quad (E_{\alpha_i})_{\mu\nu} = \delta_{\mu,i}\delta_{\nu,i+1}, \quad (E_{-\alpha_i})_{\mu\nu} = \delta_{\nu,i}\delta_{\mu,i+1}. \tag{3.83}$$

It is not difficult to see that in this case the zero grade group element is represented by the $n \times n$ diagonal matrix

$$B = \begin{pmatrix} e_\star^{\varphi_1+\varphi_0} & 0 & 0 & 0 & \dots & 0 \\ 0 & e_\star^{-\varphi_1+\varphi_2+\varphi_0} & 0 & 0 & \dots & 0 \\ 0 & 0 & e_\star^{-\varphi_2+\varphi_3+\varphi_0} & 0 & \dots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & e_\star^{\varphi_{n-1}-\varphi_{n-2}+\varphi_0} & 0 \\ 0 & 0 & 0 & \dots & 0 & e_\star^{-\varphi_{n-1}+\varphi_0} \end{pmatrix}. \tag{3.84}$$

Change variables to

$$\begin{aligned} \varphi_1 + \varphi_0 &= \phi_1, \\ -\varphi_k + \varphi_{k+1} + \varphi_0 &= \phi_{k+1}, \quad \text{for k = 1 to n} - 1. \end{aligned} \tag{3.85}$$

In these new fields the components of the gauge connections (3.49) are written as

$$\bar{A}_{ij} = \bar{\partial}(e_\star^{\phi_i}) \star e_\star^{-\phi_i}\delta_{ij} + \mu_i\delta_{i+1,j} \quad \text{and} \quad A_{ij} = -\mu_i e_\star^{\phi_{i+1}} \star e_\star^{-\phi_i}\delta_{i,j+1}, \tag{3.86}$$

with $i, j = 1 \dots n$. One can now introduce the gauge potentials (3.86) in the $\star$-zero-curvature condition (3.48) to obtain the equations of motion that define the nc extension of abelian Toda models in this parametrization

$$\partial(\bar{\partial}(e_\star^{\phi_k}) \star e_\star^{-\phi_k}) \;=\; \mu_k^2 e_\star^{\phi_{k+1}} \star e_\star^{-\phi_k} - \mu_{k-1}^2 e_\star^{\phi_k} \star e_\star^{-\phi_{k-1}}, \tag{3.87}$$

a system of n-coupled equations ($k = 1 \dots n$) and where $\mu_0 = \mu_n = 0$. To compute the commutative limit we sum all the equations in (3.87). As next step, introduce the original fields $\varphi_0, \varphi_1, \dots \varphi_{n-1}$ (3.85) and apply the limit $\theta \to 0$, which transform $e_\star^\phi \to e^\phi$. At the end we find, as expected, that the field φ_0 decouples, i.e.

$$\partial\bar{\partial}\varphi_0 = 0 \tag{3.88}$$

and the remaining $n-1$ equations become the usual abelian Toda equations (2.21)

$$\partial\bar{\partial}\varphi_i = \mu_i^2 e^{-k_{ij}\varphi_j}. \tag{3.89}$$

The action whose Euler-Lagrange equations of motion leads to (3.87), can be obtained from (3.50) with (3.84) and making use of the nc Polyakov-Wiegmann identity (3.59). At the end it reads

$$S(\phi_1,\ldots,\phi_n) = \sum_{k=1}^{n} S_{WZNW_\star}(e_\star^{\phi_k}) + \frac{k}{2\pi}\int d^2z \sum_{k=1}^{n-1}\mu_k^2(e_\star^{\phi_{k+1}}\star e_\star^{-\phi_k}). \tag{3.90}$$

This model reduces to the nc Liouville extension (3.69) for $n=2$. So, we have constructed nc analogs of abelian Toda theories that have an infinite number of conserved charges. It is an open question to see how this infinite chain of conserved charges will influence the integrability properties of the theories.

3.5 NC $\widetilde{GL}(n,\mathbb{R})$ affine Toda

Among Toda theories the affine models are quite special due, essentially, to the presence of soliton type solutions. Looking at possible applications in particle physics and in condensed matter, in this section we will extend these theories to the nc scenario. Let us now move to the case where the algebra is the $\widetilde{GL}(n,\mathbb{R})$ infinite-dimensional loop algebra. One can repeat the procedure described in this paper to construct the corresponding nc generalizations of Toda theories. The zero grade group element B is parameterized as

$$B = e_\star^{\Sigma_{i=1}^{n-1}\varphi_i h_i^{(0)}+\varphi_0 I} \tag{3.91}$$

and the constant generators of grade ±1 are chosen as

$$\epsilon_\pm = \Sigma_{i=1}^{n-1}\mu_i E_{\pm\alpha_i}^{(0)} + m_0 E_{\mp\psi}^{(\pm1)}. \tag{3.92}$$

Working in the $n\times n$ representation

$$(h_i)_{\mu\nu}^{(0)} = \delta_{\mu\nu}(\delta_{i,\mu}-\delta_{i+1,\mu}), \quad (E_{\alpha_i}^{(\pm1)})_{\mu\nu} = \lambda^{\pm1}\delta_{\mu,i}\delta_{\nu,i+1}, \quad (E_{-\alpha_i}^{(\pm1)})_{\mu\nu} = \lambda^{\pm1}\delta_{\nu,i}\delta_{\mu,i+1}, \tag{3.93}$$

where λ is the spectral parameter, and using the variables (3.85) the components of the gauge potentials (3.49) read

$$\begin{aligned}\bar{A}_{ij} &= \bar\partial(e_\star^{\phi_i})\star e_\star^{-\phi_i}\delta_{ij} + \mu_i\delta_{i+1,j} + \lambda m_0\delta_{i,n}\delta_{j,1},\\ A_{ij} &= -\mu_i e_\star^{\phi_{i+1}}\star e_\star^{-\phi_i}\delta_{i,j+1} - \frac{m_0}{\lambda}e_\star^{\phi_1}\star e_\star^{-\phi_n}\delta_{i,1}\delta_{j,n}.\end{aligned} \tag{3.94}$$

Introducing the potentials (3.94) in the $\star$-zero-curvature condition (3.48), the n-coupled equations of motion

$$\begin{aligned}\partial(\bar\partial(e_\star^{\phi_k})\star e_\star^{-\phi_k}) &= \\ &\mu_{k-1}^2 e_\star^{\phi_k}\star e_\star^{-\phi_{k-1}} - \mu_k^2 e_\star^{\phi_{k+1}}\star e_\star^{-\phi_k} - m_0^2(\delta_{n,k}-\delta_{1,k})e_\star^{\phi_1}\star e_\star^{-\phi_n},\end{aligned} \tag{3.95}$$

are obtained. In the previous expression $\mu_0 = \mu_n = 0$ and $k = 1 \ldots n$. It is not difficult to see that in the commutative limit with the appropriate algebraic parameters the system of equations (2.25) (remember $\eta = \nu = 0$) is recovered, plus an additional equation for a free scalar field. The defining action from which the nc $\widetilde{GL}(n)$ affine equations (3.95) can be derived reads

$$S(\phi_1, \ldots, \phi_n) = \sum_{k=1}^{n} S_{WZNW_\star}(e_\star^{\phi_k}) + \frac{k}{2\pi}\int d^2z \left(\sum_{k=1}^{n-1} \mu_k^2 e_\star^{\phi_{k+1}} \star e_\star^{-\phi_k} + m_0^2 e_\star^{\phi_1} \star e_\star^{-\phi_n}\right), \tag{3.96}$$

and it include the sinh-Gordon model (3.81) for $n = 2$. The study of the possible soliton type solutions of these models deserve special attention in future research.

Before this section ends, let us remark that in the construction of the nc generalizations of abelian and abelian affine Toda theories we have chosen only one type of parameterization for the element B of the zero grade subgroup. Whereas it is possible to use the alternative parameterization

$$B = (\prod_{i=1}^{n-1} e_\star^{\varphi_i h_i^{(0)}}) \star e_\star^{\varphi_0 I}, \tag{3.97}$$

which will lead to generalizations of (3.56) and (3.74) for $n > 2$.

4 Conclusions

Starting from a nc extension of the Leznov-Saveliev equations proposed in [4], we have constructed nc generalizations of abelian and abelian affine Toda theories associated respectively to the algebras $GL(n, \mathbb{R})$ and $\widetilde{GL}(n, \mathbb{R})$. As particular examples for $n = 2$ the nc Liouville and nc sinh/sine-Gordon have been discussed. We have seen how the zero grade subgroup in the nc scenario looses the abelian character. Due to that one can choose alternative parameterization schemes that will lead to equivalent nc extensions of the same model. As was explained in the last section for the sinh-Gordon theory, the construction scheme proposed here gives the possibility of extending the models directly in two dimensions without apparently loosing the integrability properties of the original theory. The crucial point is that the deformation must be done in a consistent way, respecting the algebraic properties of the theory.

The nc generalization of Toda theories as a system of coupled second order differential equations with the nice property of having an action principle from which can be derived, as far as we know, have not been presented in other work and constitute the main contribution of this paper.

There are many directions to follow in future research work. Among them 1-) to obtain the action (3.50) from $WZNW_\star$ gauging the extra fields and after integrating in the gauge potentials, 2-) to study the symmetries and solutions of the models proposed, which for the affine case could possibly include solitons, 3-) finally to study the full integrability of the models presented deserves special attention.

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Appendix A

Let us introduce here some of the algebraic structures used in this paper. The Lie algebra $SL(n,\mathbb{R})$ in the Chevalley basis is defined through the commutation relations:

$$
\begin{aligned}
[h_i, h_j] &= 0, \\
[h_i, E_{\alpha_j}] &= \sum_{b=1}^{n-1} m_b^{\alpha_j} k_{bi} E_{\alpha_j}, \\
[E_{\alpha_i}, E_{\alpha_j}] &= \begin{cases} \sum_{b=1}^{n-1} l_b^{\alpha_i} h_b, & \text{if} \quad \alpha_i + \alpha_j = 0, \\ \varepsilon(\alpha_i,\alpha_j) E_{\alpha_i+\alpha_j}, & \text{if} \quad \alpha_i+\alpha_j \quad \text{is a root}, \\ 0 & \text{otherwise}, \end{cases}
\end{aligned}
\tag{4.98}
$$

where $\varepsilon(\alpha_i,\alpha_j)$ are constant such that $\varepsilon(\alpha_i,\alpha_j) = -\varepsilon(\alpha_j,\alpha_i)$, k_{ij} is the Cartan matrix

$$
k_{ij} = \frac{\alpha_i \cdot \alpha_j}{\alpha_j^2}, \quad i = 1 \ldots n-1,
\tag{4.99}
$$

with α_i, α_j in this case simple roots, and for any root we have that

$$
\frac{\alpha_i}{\alpha_i^2} = \sum_{b=1}^{n-1} l_b^{\alpha_i} \frac{\alpha_b}{\alpha_b^2} \quad \text{and} \quad \alpha_i = \sum_{b=1}^{n-1} m_b^{\alpha_i} \alpha_b.
\tag{4.100}
$$

The bilinear form

$$
\begin{aligned}
Tr(h_i h_j) &= k_{ij}, \\
Tr(E_{\alpha_i} E_{\alpha_j}) &= \frac{2}{|\alpha_i|^2} \delta_{\alpha_i+\alpha_j,0} \quad \text{for any root}, \\
Tr(E_{\alpha_i} h_j) &= 0,
\end{aligned}
\tag{4.101}
$$

is also introduced. The loop algebra $\widetilde{SL}(n,\mathbb{R})$ is the Lie algebra of traceless matrices with entries which are Laurent polynomials in λ

$$\widetilde{SL}(n,\mathbb{R}) = C(\lambda,\lambda^{-1}) \otimes SL(n,\mathbb{R}). \tag{4.102}$$

The structure of the Lie algebra is introduced by the relation

$$[\lambda^n \otimes T_i, \lambda^m \otimes T_j] = \lambda^{n+m} \otimes f_{ijk} T_k, \tag{4.103}$$

where $m, n \in \mathbb{Z}$ and the elements of the form $1 \otimes T_i$ are identified with the algebra $SL(n,\mathbb{R})$ which is a subalgebra of $\widetilde{SL}(n,\mathbb{R})$. In this sense we can write $\lambda^n \otimes T_i$ as $\lambda^n T_i$.

The affine algebra $\widehat{SL}(n,\mathbb{R})$ is obtained from $\widetilde{SL}(n,\mathbb{R})$ by adjoining the derivation $d = \lambda \frac{d}{d\lambda}$ and the one dimensional center operator c. In this way, the commutation relations for the Kac-Moody algebra are defined by

$$[h_i^{(m)}, h_j^{(n)}] = mck_{ij}\delta_{m+n,0}, \tag{4.104}$$

$$[h_i^{(m)}, E_{\alpha_j}^{(n)}] = \sum_{b=1}^{n-1} m_b^{\alpha_j} k_{bi} E_{\pm\alpha_j}^{(m+n)}, \tag{4.105}$$

$$[E_{\alpha_i}^{(m)}, E_{-\alpha_i}^{(n)}] = \sum_{b=1}^{n-1} l_b^{\alpha_i} h_i^{(m+n)} + \frac{2}{\alpha_i^2} mc\delta_{m+n,0}, \tag{4.106}$$

$$[E_{\alpha_i}^{(m)}, E_{\alpha_j}^{(n)}] = \varepsilon(\alpha_i,\alpha_j) E_{\alpha_i+\alpha_j}^{(m+n)} \quad \text{if} \quad \alpha_i + \alpha_j \quad \text{is} \quad \text{a} \quad \text{root}, \tag{4.107}$$

$$[d, E_{\alpha_i}^{(n)}] = nE_{\alpha_i}^{(n)}, \quad [d, h_i^{(n)}] = nh_i^{(n)}, \quad [c,d] = [c, E_{\alpha_i}^{(n)}] = [c, h_i^{(n)}] = 0, \tag{4.108}$$

where $|\alpha_i|^2 = 2$ for the simple roots and the symmetric bilinear form

$$Tr(h_i^{(m)} h_j^{(n)}) = k_{ij}\delta_{m+n,0}, \tag{4.109}$$

$$Tr(E_{\alpha_i}^{(m)} E_{\alpha_j}^{(n)}) = \frac{2}{|\alpha_i^2|}\delta_{\alpha_i+\alpha_j,0}\delta_{m+n,0}, \tag{4.110}$$

$$Tr(cd) = 1 \tag{4.111}$$

is also introduced.

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