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STOKESIAN FLUIDS AND COSMOLOGY

M. Novello
Centro Brasileiro de Pesquisas Físicas

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* This paper is dedicated to the memory of Bronislav Kuchowicz.

I - GENERAL FLUIDS AND COSMOLOGY

Let me start by emphasizing that we will discuss here models of fluid behavior which are apparently beyond any experimental evidence on terrestrial materials. I will consider fluids which have anisotropic stress and which responses to an external stimulus in a non-linear way. Such behavior can be either of a stokesian type (the response depends only on the deformation) or of a non-stokesian form (it can depends, for instance, on the local rotation of the fluid). Instead of looking for special materials which could have such properties we will show how it is possible to detect similar behavior in usual ordinary forms of matter under certain specific situations. In this vein, Einstein's equations will be used in the present paper as a tool to construct explicit examples of such circumstances. It is indeed very stimulating that one can easily find models of Universes which can be interpreted as having such unusual materials as the main source of its curvature. Astonishing enough in these examples the "strange materials" are nothing but classical long range fields coupled to ordinary matter in a curved space.

From the cosmological point of view the analysis of matter which are not limited to the perfect fluid behavior should be of importance in order to describe strong deviations from the usual models in such drastic periods which certainly occured in early epochs.

In the last decade the interest on viscosity effects on Cosmology has grown up very rapidily. The main reasons for this are the following:

(i) Tentatives of explanation of the highly value of the entropy per baryon present in the Universe;

(ii) The speculative remark that viscosity could be an efficient mechanism to annihilate primordial arbitrary anisotropy;

(iii) The observation that in order to consider the effect of new particles created by the non stationary gravitational field in a self-consistent way, one should enlarge the usual perfect fluid cosmologies by allowing viscous phenomena, which should represent the macroscopic behavior of the new matter.

Hitherto, the analysis of the viscosity effects have been limited to simple linear models, e.g. the Cauchy rheological equation which relates the anisotropic stress $\Pi^i{}_j$ to the shear $\sigma^i{}_j$ through a viscous coefficient [(1)]:

$$(1) \qquad \Pi^i{}_j = \xi \, \sigma^i{}_j$$

(from here on latin indices runs 1,2,3; greek indices runs o,1,2,3. We choose a system of coordinates (co-moving) such that $V^\alpha = \delta^\alpha_0$)

Expression (1) is the simplest form of the so-called Stokesian fluidity condition which says that the **response** of a fluid to a deformation depends only on the matrix of deformation.

We represent the dilatation (deformation) tensor by $\Theta^i{}_j$ which is nothing but the shear $\sigma^i{}_j$ added to the expansion factor term $\frac{1}{3}\Theta\,\delta^i{}_j$, where $\Theta = \Theta^i{}_i$. We can represent the Stokes fluidity condition by saying that the anisotropic pressure $\Pi^i{}_j$ is a functional of the dilatation $\Theta^i{}_j$:

$$(2) \qquad \Pi^i{}_j = F^i{}_j \, [\Theta^k{}_\ell]$$

Using a theorem by Cayley on matrix, we can re-write (2) in the form

$$\Pi^{i}{}_{j} = A\ \delta^{i}{}_{j} + B\ \Theta^{i}{}_{j} + C\ \Theta^{i}{}_{k}\ \ \Theta^{k}{}_{j}$$

in which A, B and C are polynomials in the principal invariants of the matrix $\Theta^{i}{}_{j}$.

We now introduce the concept of non-stokesian fluid. Let us define the symmetric trace-free matrices A_{ij} *and* Ω_{ij} *in terms of the acceleration vector* $a_i \equiv \dot{V}_i$ *and the vorticity* ωi *of the fluid, through the expression*

$$\Omega^{i}{}_{j} = \omega^{i}\ \omega_{j} - \frac{\omega^2}{3}\ \delta^{i}{}_{j}$$

$$A^{i}{}_{j} = a^{i}\ a_{j} - \frac{a^2}{3}\ \delta^{i}{}_{j}$$

in which $a^2 = a^{i}\ a_{i}$ *and* $\omega^2 = \omega^{i}\ \omega_{i}$.
We will say that a fluid is of a non-stokesian type if it has anisotropic stress which can be described by a functional either of matrix $\Omega^{i}{}_{j}$ *or* $A^{i}{}_{j}$. *The main purpose of the present paper is to present examples of such behavior. We will show that Einstein's equation can be of great help into such search.*

II - QUADRATIC STOKESIAN FLUID

An example of a fluid which behaves as a quadratic stokesian fluid is given by the source of the Novello-Soares (NS) cosmological model. It is understood that in the present paper the relation between the geometry and the matter is given through Einstein's equation. There are a whole class of NS geometries but we will limit our analysis here only to the steady-state case. We write the fundamental lenght under the form:

$$ds^2 = dt^2 - 2\, e^{Mt}\, dx\, dt - e^{2Jt}\,(dy^2 + dz^2) \tag{3}$$

in which M and J are arbitrary parameters constrained to satisfy the unique requirement $J(J - M) < 0$.

Let us choose a tetrad frame $e^{(\alpha)}_{A}$ given by:

$$e^{(0)}_{0} = 1$$
$$e^{(0)}_{1} = - e^{Mt}$$
$$e^{(1)}_{1} = - e^{Mt}$$
$$e^{(2)}_{2} = e^{Jt}$$
$$e^{(3)}_{3} = e^{Jt}$$

In this tetrad frame the unique non null components of the contracted Riemann tensor are:

$$R_{00} = R_{11} = - R_{01} = 2J(J - M)$$

The stress-energy tensor is given by the general expression (in tetrad frame)

$$T^{A}{}_{B} = \rho\, V^{A}\, V_{B} - p\, (\eta^{A}{}_{B} - V^{A}\, V_{B}) + q_{(A}\, V_{B)} + \Pi_{AB}$$

in which $\eta_{AB} = diag\ (+---)$. We set $V^{A} = \delta^{A}{}_{0}$.

The heat flux q_A and the anisotropic pressure Π_{AB} are related to the kinematical quantities through the phenomenological equations of state:

$$q_A = \xi \dot{V}_A$$

$$\Pi^A{}_B = \phi \left(\Theta^A{}_C \, \Theta^C{}_B - \frac{1}{3} \delta^A{}_B (\Theta^M{}_N \, \Theta^N{}_M) \right)$$

in which ξ *and* ϕ *are the coefficients to be evaluated. The expansion* Θ *, the dilatation tensor* $\Theta^i{}_j$ *and the acceleration* $\dot{V}_A$ *are given by*

$$\Theta = M + 2J$$

$$\Theta^1_1 = M$$

$$\Theta^2_2 = J$$

$$\Theta^3_3 = J$$

$$\dot{V}_A = (0, M, 0, 0)$$

Using these values, Einstein's equations will be satisfied if we have

$$\xi = \frac{2J(J - M)}{M}$$

$$\phi = \frac{-2J}{J + M}$$

$$p = \frac{1}{3} \rho$$

Indeed, in this case, the unique non-null terms of the stress-energy tensor $T^A{}_B$*, are:*

$$T_{00} = T_{11} = -T_{01} = \rho$$

Thus, we obtain

$$\rho = -2J(J - M).$$

This completes our proof: a quadratic stokesian fluid with constitutive relations characterized by equations (11) is indeed a source of the geometry (3).

Now one should ask: is this the unique possibility of description of such source ?

In general, to answer a question like that for an arbitrary geometry may turn into a very difficult problem. There are some authors which have claimed that the knowledge of the geometry induces the knowledge of its source (via Einstein's equations). Unformately things are not that easy in general. For the case treated here for instance, it has been shown by Novello and Soares[(2)] *that the source of the NS geometry (3) can be interpreted as a neutrino field with a current* $j^{u} = \overline{\Psi} \ \gamma^{u} \ \Psi$ *directed in the X-direction.*

We can unify both interpretations by treating the neutrino field in such geometry as a realization of a quadratic stokesian fluid.

III - NON STOKESIAN FLUID - I

Some years ago Bertotti (and independently Robinson) have presented a conformally flat static Universe generated by an electromagnetic field. We will analyse such geometry here in order to show that it is possible to interpret its source in terms of a non-stokesian fluid of the accelerated type, that is, $\Pi^{i}{}_{j} \sim A^{i}{}_{j}$.

The BR geometry can be written in the form

$$ds^{2} = (Qr)^{-2} \left[dt^{2} - dr^{2} - r^{2} \, d\Theta^{2} - r^{2} \sin^{2}\Theta \, d\phi^{2}\right] \qquad (13)$$

in which Q is a constant.

Let us choose a local Minkowskian tetrad frame $e^{A}_{(\alpha)}$ *and define the one-forms* $\Theta^{A} = e^{A}_{(\alpha)}\, dx^{\alpha}$ *by the expressions*

$$\Theta^{0} = (Qr)^{-1}\, dt$$

$$\Theta^{1} = (Qr)^{-1}\, dr$$

$$\Theta^{2} = Q^{-1}\, d\Theta$$

$$\Theta^{3} = \Theta^{-1}\, \sin\Theta\, d\phi$$

The unique non-null components of the contracted Riemann tensor in such frame are:

$$R_{00} = -R_{11} = R_{22} = R_{33} = -\,Q^{2}$$

Let us take the fluid velocity as $V_{A} = \delta^{0}_{A}$. *The kinematical quantities of the associated congruence vanishes excluding the aceleration which is given by*

$$\dot{V}_{A} = (0, Q, 0, 0).$$

We can then construct the accelerated matrix $A^{i}{}_{j}$:

$$A^{i}{}_{j} = \frac{Q^{2}}{3}\begin{pmatrix} -2 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

The source of the geometry is a fluid with density of energy ρ, *pressure p and anisotopic pressure* $\Pi^{i}{}_{j}$ *of the accelerated non-stokesian type:*

$$\Pi^{i}{}_{j} = \phi\; A^{i}{}_{j}$$

Then,

$$T^{A}{}_{B} = \begin{pmatrix} \rho & & & \\ & -p-\frac{2}{3}\,\phi\,Q^{2} & & \\ & & -p+\frac{\phi}{3}\,Q^{2} & \\ & & & -p+\frac{\phi}{3}\,Q^{2} \end{pmatrix}$$

Einstein's equations imposes

$$p = \frac{1}{3}\,\rho$$

$$\phi = -2$$

$$\rho = Q^{2}$$

This proves that the source of BR geometry is a non-stokesian fluid of the accelerated type. Now, Bertotti has arrived at the BR model by looking for solution of Einstein's equations in an Universe filled solely with (static) electromagnetic field. Thus, we conclude that the BR electromagnetic field is a realization of an accelerated non-stokesian fluid of the form discussed above.

IV - NON STOKESIAN FLUID - II

In the preceding section we have considered a non stokesian fluid of the accelerated type. We now turn to the consideration of a vortex dominated non stokesian fluid - that is one which has a linear relation between the anisotropic pressure and the $\Omega^{i}{}_{j}$ *matrix .*

The simplest model of such behavior can be obtained by a slight modification of the static rotating Universe of Gödel[4,5]*. Let us choose a cylindrical coordinate system and write the fundamental element of lenght in the form*

$$ds^2 = dt^2 - dr^2 - dz^2 + g(r)\ d\phi^2 + 2h(r)\ d\phi\ dt$$

Gödel solution is obtained by setting $g(r) = \sinh^4 r - \sinh^2 r$ and $h(r) = \sqrt{2}\ \sinh^2 r$. The source of Gödel geometry is a perfect fluid with constant density ρ, without pressure. We will now generate new solutions, that is, new functions for $g(r)$ and $h(r)$ by enlarging the behavior of the matter.

Let us first discuss the fundamentals of the geometry. We define the 1-forms Θ^A by setting

$$\Theta^0 = dt + h(r)\ d\phi$$

$$\Theta^1 = dr$$

$$\Theta^2 = \Delta(r)\ d\phi$$

$$\Theta^3 = dz$$

in which $\Delta(r) \equiv \sqrt{h^2 - g}$

The geometry is given by

$$ds^2 = \eta_{AB}\ \Theta^A\ \Theta^B = (\Theta^0)^2 - (\Theta^1)^2 - (\Theta^2)^2 - (\Theta^3)^2$$

The unique non-null Riemann contracted tensors are the following:

$$R_{00} = -\frac{1}{2}\left(\frac{h'}{\Delta}\right)^2$$

$$R_{11} = \frac{\Delta''}{\Delta} - \frac{1}{2}\left(\frac{h'}{\Delta}\right)^2$$

$$R_{22} = R_{11}$$

$$R_{02} = \frac{1}{2}\ \frac{h'\Delta'}{\Delta^2} - \frac{1}{2}\ \frac{h''}{\Delta}$$

where $h' \quad \frac{dh}{dr}$.

Now let us discuss the matter terms. In the co-moving frame $V_A = \delta^0_A$ the non-null vorticity vector ω^A points in the z-direction:
$\omega^A = (0,0,0,-\frac{1}{2}\frac{h'}{\Delta})$. The expansion, the shear and the acceleration are null. The vortex matrix constructed with ω^A gives

$$\Omega^1_1 = \Omega^2_2 = -\frac{1}{2}\Omega^3_3 = \frac{1}{3}\Omega^2$$

Einstein's equations reduce to the following:

$$R_{00} = \frac{-\rho}{2} + \Lambda - \frac{3}{2} p$$

$$R_{11} = \frac{-\rho}{2} + \frac{p}{2} - \alpha - \Lambda$$

$$R_{22} = \frac{-\rho}{2} + \frac{p}{2} + (\alpha+\beta) - \Lambda$$

$$R_{02} = 0$$

in which we have written the anisotropic pressure as $\Pi^1{}_1 = -\alpha$; $\Pi^2{}_2 = -\beta$; $\Pi^3{}_3 = \alpha+\beta$.
From $R_{22} = R_{11}$ we obtain that $\alpha = \beta$. Thus Π_{AB} and Ω_{AB} are proportional. We write

$$\Pi_{AB} = -\gamma^2 \Omega_{AB}$$

Condition $R_{02} = 0$ implies that

$$\frac{h'}{\Delta} = 2\Omega = constant$$

Let us set

$$\Delta = \sin (mr)$$

m is a constant.

Then, the above set of equations reduce to

$$\rho = - \Lambda + (1 + \gamma^2) \Omega^2$$

$$p = \Lambda + (1 - \frac{\gamma^2}{3}) \Omega^2$$

$$m^2 = (\gamma^2 - 2) \Omega^2$$

Positivity of energy and pressure imposes that the cosmological constant is negative (or vanishes) and that the possible range of γ is limited by the inequality

$$2 < \gamma^2 < 3$$

The complete solution is easily evaluated and we find

$$g(r) = \left(\frac{\gamma^2 + 2}{\gamma^2 - 2}\right) \cos^2 mr - 1$$

$$h(r) = - \frac{2}{(\gamma^2 - 2)^{1/2}} \cos mr$$

This solves Einstein's equations for the non-stokesian fluid of vortex type.

We will now prove that such fluid is not a hypothetical one but that it is simple to make a realization of it in terms of a perfect fluid coupled to an electromagnetic field.

We write the source of the above geometry as a perfect fluid plus an electromagnetic field.

$$T_{AB} = \rho\, V_A\, V_B - p\,(\eta_{AB} - V_A\, V_B) + T_{AB}^{(EM)}$$

In the comoving frame the electric and the magnetic vectors are parallel to the vorticity. We set

$$E^A = (0,0,0,A)$$
$$H^A = (0,0,0,B)$$

Maxwell's equations fixes the value of A and B:

$$A = e \sin 2\,\Omega\, z$$
$$B = e \cos 2\,\Omega\, z$$

in which e is a constant. In the above frame the Poynting vector vanishes. The anisotropic pressure of the electromagnetic field reduces to:

$$\Pi_{MN}^{(em)} = - E_M\, E_N + \frac{1}{3} E^2\, \delta_{MN} - H_M\, H_N + \frac{1}{3} H^2\, \delta_{MN}$$

in which

$$H^2 = H_A\, H^A = - B^2$$
$$E^2 = E_A\, E^A = - A^2$$

Thus, in the tetrad frame $\Pi_{AB}^{(em)}$ *is diagonal:*

$$\Pi_{11}^{(em)} = \Pi_{22}^{(em)} = - \frac{1}{2}\quad \Pi_{33}^{(em)} = \frac{e^2}{3}$$

Consequently we can write

$$\Pi^{(em)}_{AB} = - \gamma^2 \quad \Omega_{AB}$$

in which $\gamma^2 = e^2 \ \Omega^{-2}$.

The set of Einstein's equations reduce to:

$$\rho_0 + \frac{e^2}{2} = - \Lambda + (1+\frac{e^2}{\Omega^2}) \ \Omega^2$$

$$p_0 + \frac{1}{6} \ e^2 = \Lambda + (1-\frac{e^2}{3 \ \Omega^2}) \ \Omega^2$$

$$m^2 = \left(\frac{e^2}{\Omega^2} - 2\right) \Omega^2$$

Positivity of the energy ρ_0 and the pressure p_0 are guaranteed by restricting the range of possible values of the cosmological constant to the limits given by:

$$\Omega^2 \left(\frac{\gamma^2}{2} - 1\right) < \Lambda < \Omega^2 \left(\frac{\gamma^2}{2} + 1\right)$$

This ends our proof: the electromagnetic field coupled to a perfect fluid (ρ_0, p_0) generate a realization of a non-stokesian vortex dominated fluid. We conclude from the above examples that Einstein's equations can be used to detect more general fluid behavior than those described until now. Such behavior is a consequence of coupling of long range fields and perfect fluids with gravitation. The most direct consequence of this should be presented in the form of a question: Could such non-stokesian fluid represent the dynamical contents of matter in some epochs of the history of our Universe ? The answer to this will certainly introduce new features to the cosmological question.

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