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ON WAVE EQUATIONS IN DE - SITTER SPACE

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1. INTRODUCTION

Gűrsey and Lee [1] have written the Dirac [2] equation for spin 1/2 particles in the de - Sitter space in the following convenient invariant form [3]

$$(\beta^i \partial i - \frac{2}{\rho} \beta^5)\psi + m\psi = 0 \tag{1}$$

where ψ is a 4 - component spinor field, ρ is equal to the radius of the de - Sitter pseudosphere, the real number m is the mass of the particle, and

$$\begin{aligned} \beta^\mu &\equiv \frac{\partial x^\mu}{\partial \xi^\nu} \gamma^\nu \\ \beta^5 &= \frac{1}{\rho} (\xi_\mu \gamma^\mu) \\ \beta^5 \beta^i + \beta^i \beta^5 &= 0 \\ \beta^i \beta^j + \beta^j \beta^i &= g^{ij} \\ g^{ij} &= \left(\frac{\partial x^i}{\partial \xi^\mu}\right)\left(\frac{\partial x^j}{\partial \xi_\mu}\right) \end{aligned} \tag{2}$$

Also,

$$\gamma^{\mu}\gamma^{\nu}+\gamma^{\nu}\gamma^{\mu}=2\eta^{\mu\nu} \tag{3}$$

i.e., the five constant (4 × 4) Hermitian matrices γ^{μ} satisfy the relation (3), where $\eta^{\mu\nu}$ = (1, 1, 1, - 1, 1) diagonal.

Equation (1) becomes the ordinary Dirac equation if we restrict the de-Sitter space to be the neighborhood of a point as [1]

$$(\xi^1, \xi^2, \xi^3, \xi^4, \xi^5) = (0,0,0,0,R) \tag{4.1}$$
$$\rho = R, \ \xi^1 = x^1$$

so that

$$\beta^{\mu} = \gamma^{\mu} \tag{4.2}$$

in the usual space-time. The de - Sitter space goes over into the last one when R tends to infinity.

To describe free particles of arbitrary half-integral spin, we may consider the generalization of equation (1) that follows

$$\left[\left(\frac{\partial x^1}{\partial \xi^{\mu}}\gamma^{\mu}\right)_{a'a} \frac{\partial}{\partial x^1} - \left(\frac{2}{\rho^2}\xi_{\mu}\gamma^{\mu}\right)_{a'a} + m \ \delta_{a'a}\right]\psi_{abc\ldots} = 0 \tag{5}$$

where [3] $\psi_{abc\ldots}$ is a symmetric spinor, i.e., equations(5) are the Dirac-Gürsey-Lee equation (1) in each of the spinor indices (a,b,c,...). They constitute an extension of the Bergmann-Wigner[4] equations to the de - Sitter space, and become the usual ones when the conditions (4) are imposed.

The purpose of the present note is to formulate equations (5) in tensor, and spin - tensor forms for the case of free particles of spins 1, 2, and 3/2 respectively. This will be done by following a similar treatment to that employed in previous papers [5, 6]. Then, the wave functions will be constructed with the aid of the ten Dirac symmetric operators and the original spinors $\psi_{abc,\ldots}$. This enables us to obtain a formulation such that in the limit (4) when $R \rightarrow \infty$, it reduces to that developed in references 5 and 6. Since the latter 4-dimensional formulation is equivalent [5] to ordinary free particle formalisms [7] for spins higher than one half, we can use the simplest tensors and spin-tensors required for a suitable description of the particles. This we proceed to do.

WAVE FUNCTIONS [3, 8]

The spin -1- wave functions are the vector and antisymmetric tensor defined by

$$\phi^{\mu} = (\bar{C}\,\gamma^{\mu})_{ab}\ \psi_{ab} \tag{6.1}$$

$$F^{[\mu\nu]} = (\bar{C}\,\gamma^{\mu}\gamma^{\nu})_{ab}\ \psi_{ab} \tag{6.2}$$

where ψ_{ab} is the original symmetric spinor.

The spin 3/2 wave function is given by the spin-vector

$$\phi^{\mu}_{c} = (\bar{C}\,\gamma^{\mu})_{ab}\ \psi_{abc}\ , \tag{7}$$

while the symmetric second rank tensor

$$\phi^{\mu\nu} = (\bar{C}\gamma^{\mu})_{ab}\ (\bar{C}\gamma^{\nu})_{cd}\ \psi_{abcd} \tag{8}$$

describes spin 2.

The inverse transformations of (6), (7) and (8) are

$$\psi_{ab} = \frac{S_1}{4} (\gamma_\mu \ C)_{ab} \ \phi^\mu \tag{9}$$

$$\psi_{ab} = \frac{S_2}{8} (\gamma_\mu \gamma_\nu \ C)_{ab} \ F^{[\mu\nu]}$$

$$\psi_{abc} = \frac{S_3}{4} (\gamma_\mu \ C)_{ab} \ \phi^\mu_c , \tag{10}$$

and

$$\psi_{abcd} = \frac{S_4}{16} (\gamma_\mu \ C)_{ab} \ (\gamma_\nu \ C)_{cd} \ \phi^{\mu\nu} \tag{11}$$

where S_i are the symmetrization operators acting on the spinor indices.

WAVE EQUATIONS [3]

The wave equations for spin 1 particles are

$$\partial\mu \ F^{[\mu\nu]} = -\left(\frac{2}{\rho}\beta^5 + m\right) \phi^\nu \tag{12.1}$$

$$\left(\frac{2}{\rho}\beta^5 + m\right) F^{[\mu\nu]} = \partial^\mu \phi^\nu - \partial^\nu \phi^\mu \tag{12.2}$$

which have been obtained from equations (5) for spin 1, and (6). They are the tensor form of (5), and become the usual Proca equa tions in the ordinary space-time. It can be seen also that, because of (9), equations (12) can be easily rewritten in the spinor form (5).

The set of equations

$$\left(\beta^i \partial i - \frac{2}{\rho}\beta^5 + m\right) \phi^\mu_c = 0 \tag{13.1}$$

$$\partial\mu \ \phi^\mu_c = 0 \tag{13.2}$$

$$\beta_\mu \, \phi_c^\mu = 0 \qquad (13.3)$$

follow from (5) for spin 3/2, and (7). They are the spin - tensor form of equation (5). These equations may be considered as an extension of the Rarita-Schwinger [7] equations to the de - Sitter space. Thus, they reduce to the usual ones if conditions (4) are imposed. It can also be noted that equation (5) for spin 3/2 follows immediately from the set (13) if ψ_{abc} is defined by (10). An equivalent description to (13) in terms of $F_c^{[\mu\nu]}$ can also be formulated.

The spin-2-wave equations are

$$\left[\Box - \left(\frac{2}{\rho}\beta^5\right)^2 + m^2\right]\phi^{\mu\nu} = 0 \qquad (14.1)$$

$$\partial_\mu \, \phi^{\mu\nu} = 0 \qquad (14.2)$$

which in the 4-dimensional flat space become the ordinary Fierz-Pauli equations. In this case also the set of equations (14) implies (5) for spin 2 if ψ_{abcd} is given by (11).

Further, introducing the tensors $F^{[\mu\nu]\sigma}$ and $F^{[\mu\nu][\sigma\eta]}$ we may obtain two equivalent sets of equations which also imply (14), and vice-versa. Therefore we conclude that the all three sets of equations are equivalent for free particles.

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3. Greek indices take the values 1, 2, 3, 4, 5, latin indices 1, 2, 3, 4 or as specified. All repeated indices are to be summed over. We use by convenience $\partial\mu = \frac{\partial x^i}{\partial \xi^\mu}\frac{\partial}{\partial x^i}$, $\partial i \equiv \frac{\partial}{\partial x^i}$. Here $\hbar = c = 1$.

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8. The matrix C is such that

$$\gamma_\mu^{\,T} = -\bar{C}\,\gamma_\mu\,C \qquad (T = \text{Transpose})$$

$$C^T = -C,\ \bar{C} \equiv C^{-1}.$$

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