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ON THE DIFFRACTION OF A PULSE AT THE OPEN END OF A PARALLEL-PLATE WAVEGUIDE *

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SUMMARY. Energy accumulation effects near the open end of a parallel-plate waveguide, due to the diffraction of an incident pulse travelling within the waveguide, are investigated. The behaviour of the reflected pulse at large distances from the open end provides information about the times taken for the formation and depletion of the energy reservoir near the edges. The effects of low-frequency and of high-frequency components are separately discussed. The low-frequency components are related with the retardation and width of the reflected pulse, whereas the high-frequency components determine its behaviour in the immediate neighbourhood of the wave front. The asymptotic behaviour of the reflection coefficient of the principal mode at high frequencies is derived. The effect of higher-order modes is briefly discussed.

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1. Introduction

Energy accumulation effects play an important role in transient phenomena associated with emission and scattering processes.[1,2] Energy can be temporarily stored in free space, in the neighbourhood of a curved surface. This process can be understood in terms of inertial forces.[1]

Effects of this kind also play an important role in the theory of diffraction of pulses. In this case one should expect the formation of energy reservoirs around the diffracting object, especially in the neighbourhood of sharp edges. The energy contained in these reservoirs must gradually leak out, giving rise to the diffracted pulse.

In the present paper, as well as in a companion one,[3] we shall study energy accumulation effects in diffraction theory. The emphasis will be placed on the description of these effects, rather than in their connection with inertial forces.

The problem that will be considered in this work is the diffraction of an electromagnetic pulse at the open end of a semi-infinite parallel-plate waveguide. Two different approaches to the solution of this problem are described in Section 2. The first one is the application of the Fourier method, starting from the well-known solution for the monochromatic case.[4,5,6] The other approach, which is due to Chester,[7] applies only to an incident pulse with a sharp wave front. Immediately after the arrival of the incident pulse at the open end, the two plates behave in dependently, each of them giving rise to a diffracted wave which can be derived from the well-known solution of the half-plane problem.[8] After the diffracted wave originating from each edge meets the opposite one, multiple diffraction effects must be taken into account.

The behaviour of the solution near the open end immediately after the arrival of the incident pulse, which leads to the formation of energy reservoirs around the edges, will not be discussed here; a detailed discussion of an entirely similar problem is given in another paper[3]. We shall be concerned only with the asymptotic behaviour of the reflected pulse in the neighbourhood of the wave front, at large distances from the open end. We shall employ chiefly the Fourier method, because this provides greater insight into the role of low-frequency and high-frequency components in the energy accumulation process. The main contribution to the reflected pulse arises from the reflection coefficient of the principal mode. In Chester's method, it suffices to consider the first few successive diffractions.

The low-frequency accumulation effects are discussed in Section 3. They give rise to a retardation and a width of the reflected pulse, which are related respectively to the times spent in forming and in depleting the energy reservoir.

The effect of high-frequency components is discussed in Section 4, by considering a delta-type incident pulse. The behaviour of the reflected pulse in the immediate neighbourhood of the wave front is determined by the asymptotic behaviour at high frequencies of the reflection coefficient of the principal mode. This behaviour is derived in section 5, directly from Sommerfeld's monochromatic solution[9] of the half-plane problem. The result indicates a capacitive effect of the open end, which is another manifestation of the energy accumulation process. The contributions from higher-order modes are briefly considered in section 6.

2. Formulation of the problem.

The coordinate system is shown in figure 1. The two perfectly conducting plates are represented by $x > 0$, $y = \pm a$. The incident pulse travels between them towards the open end, where it is diffracted, giving rise to a reflected pulse and to radiation into free space.

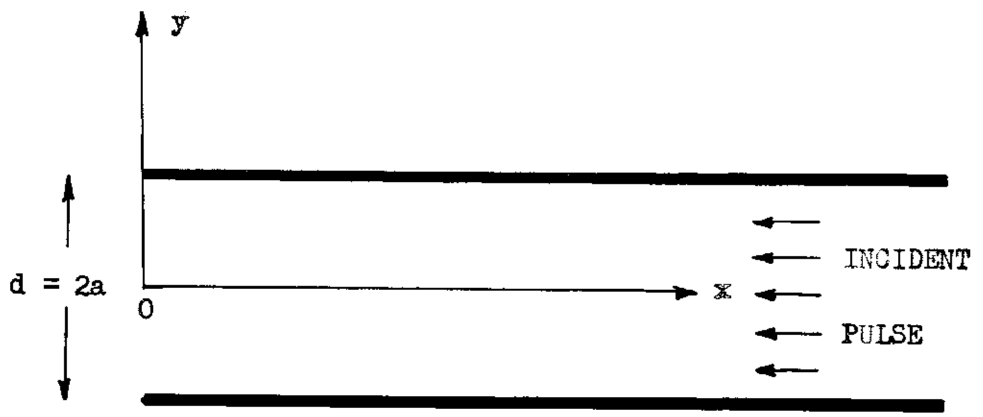

Fig. 1 - The coordinate system.

We shall consider only the case of transverse magnetic waves, which can be described by a scalar function $u(x,y,t)$.The field components are

$$\vec{H} = (0,\ 0,\ u), \quad \vec{E} = (E_x,\ E_y,\ 0), \tag{1}$$

with the boundary condition $E_x = 0$ for $x > 0$, $y = \pm a$.

The solution within the waveguide, in the monochromatic case, for the incident wave $\exp\left[-i(kx+\omega t)\right]$, is of the form

$$u_k(x,y) = \exp(-ikx) + a_o \exp(ikx) + \sum_{n=1}^{\infty} a_n \cos(k_{yn}\, y) \exp\left[i(k^2 - k_{yn}^2)^{\frac{1}{2}}\, x\right], \tag{2}$$

where $k_{yn} = n\pi/a$, $\mathcal{J}\left[(k^2 - k_{yn}^2)^{\frac{1}{2}}\right] \geqslant 0$, and the time factor $\exp(-i\omega t)$ has been omitted. The coefficients $a_n(k)$ $(n = 1,2\ldots)$ are the amplitudes of the modes excited by the incident wave. They comprise both travelling modes $(k_{yn} < k)$ and evanescent modes $(k_{yn} > k)$.

A rigorous expression for these coefficients, in the form of an infinite product, has been given by Vajnshtejn[4]. This expression lends itself to computation only for values of $q = ka/\pi = d/\lambda$ not much larger than unity. The absolute value of a_o and a_1 and the phase of a_o $(a_o = |a_o|\exp(i\,\varphi_o))$, for $0 < q < 2$, are represented graphically in figures 2 and 3, respectively. For $0 < q < 1$, $|a_o| = \exp(-\pi q)$.

Let us now consider the solution for an arbitrary incident pulse $F(x+ct)$. The solution can be derived from (2) by expanding F into a Fourier integral. A specially simple case is that of the pulses $F_s(x+ct)$ having Fourier coefficients $C_s(k)$ given by

$$C_s(k) = 1 \text{ for } -s\pi/a < k < s\pi/a, \quad = 0 \text{ for } |k| > s\pi/a, \tag{3}$$

where s is an integer. The corresponding incident pulses are given by

$$F_s(x+ct) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} C_s(k)\exp\left[-ik(x+ct)\right]dk = \frac{1}{\pi}\,\frac{\sin[s\pi(x+ct)/a]}{x+ct}. \tag{4}$$

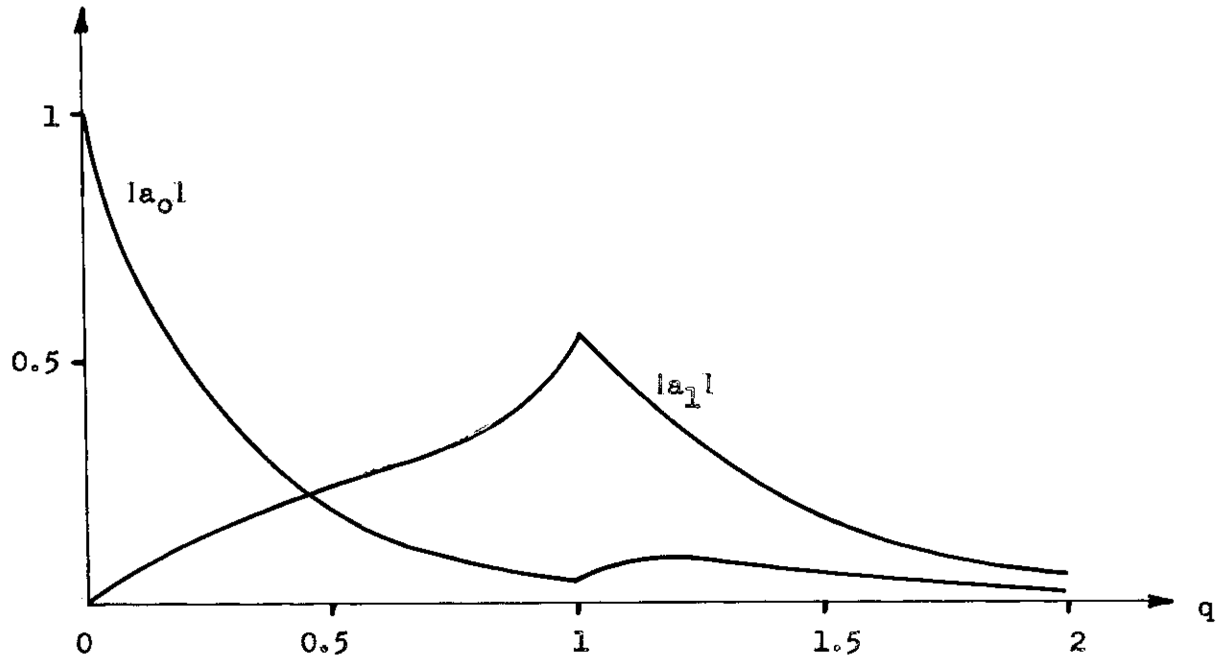

Fig. 2 - The absolute values of a_o and a_1 as a function of q.

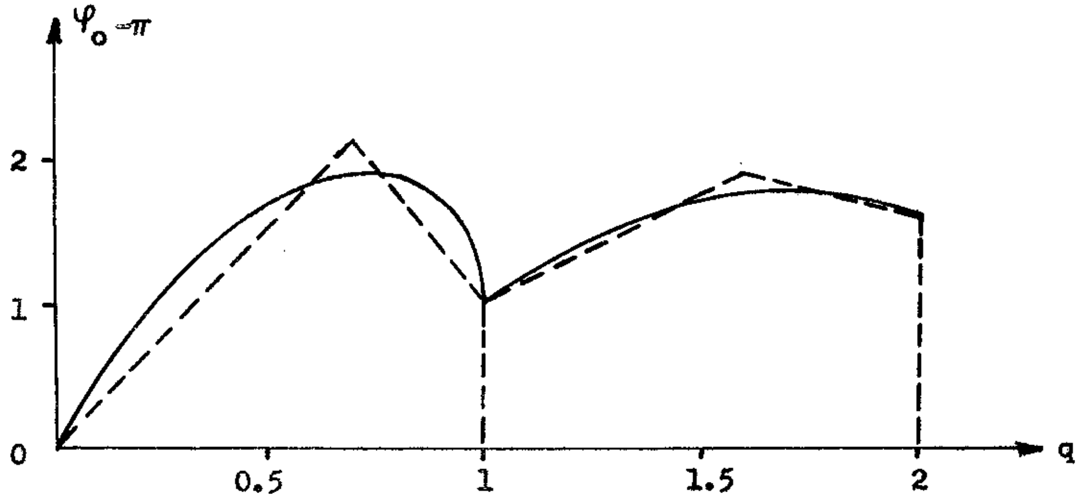

Fig. 3 – Behaviour of $\varphi_o - \pi$ as a function of q, where φ_o is the phase of a_o.
———————— exact solution.
- - - - - - - - approximation employed in the calculation.

The form of these pulses for s = 1 and s = 2 is shown in figures 4 and 5, respectively.

The reflected pulse is obtained by replacing exp(-ikx), in the integral in (4), by the last two terms on the right-hand side of (2). The modes for which $n > s$ are all evanescent, whereas, for $n < s$, they are travelling for $k_{yn}^2 < k^2 < k_{ys}^2$ and evanescent for $k^2 < k_{yn}^2$.

It would be very difficult to derive the form of the reflected pulse for all values of time by this method, owing to

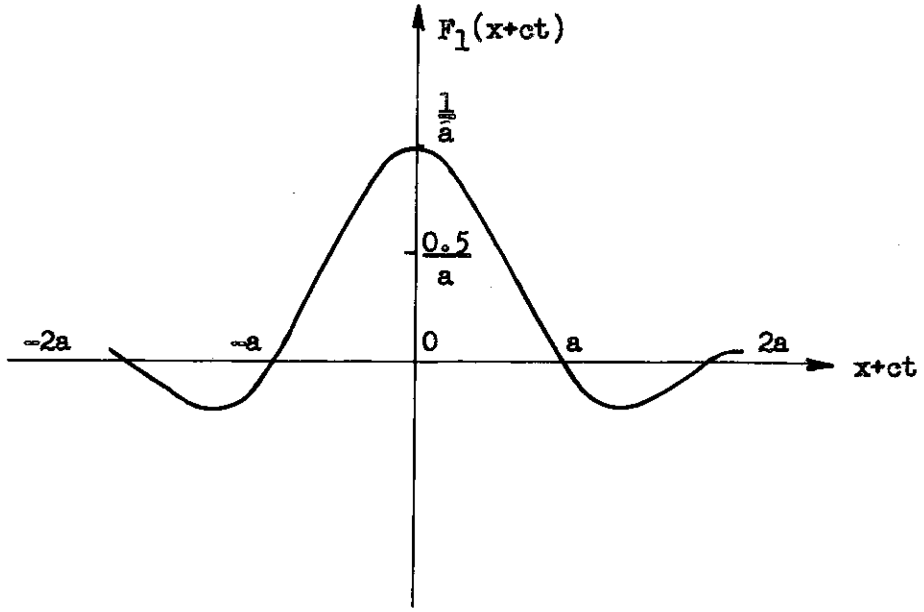

Fig. 4 - The incident pulse $F_1(x+ct)$.

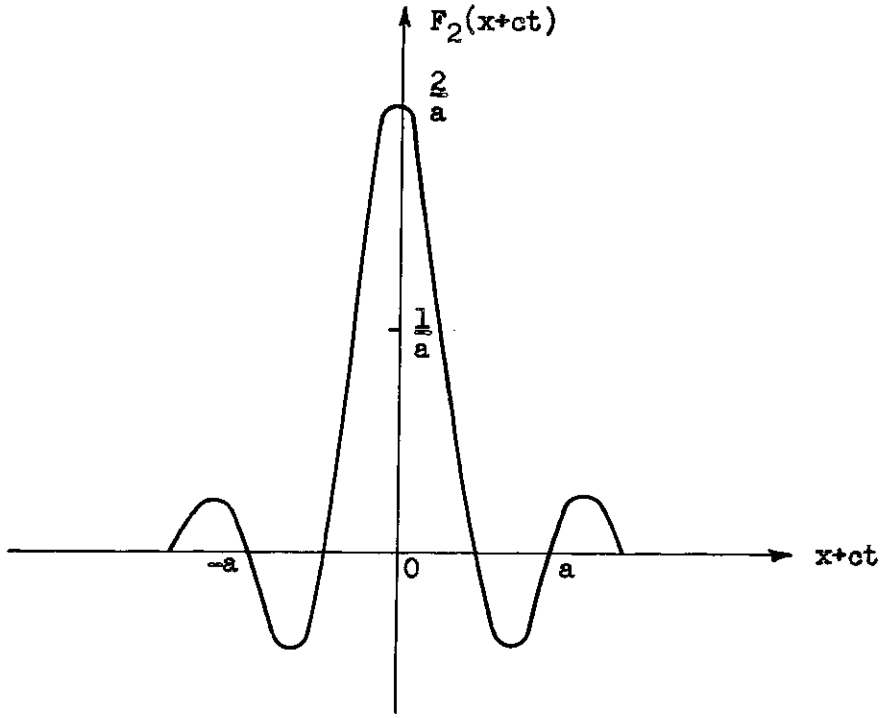

Fig. 5 - The incident pulse $F_2(x+ct)$.

the complicated expressions of the coefficients $a_n(k)$. However, some characteristic effects, due to the energy accumulation near the open end, can be studied without following the complete evolution of the reflected pulse. It suffices to consider its asymptotic behaviour for $x \sim ct \gg a$. Under these conditions, we can disregard the contribution from the evanescent part of all modes, as well as that of the travelling part for $n \neq 0$, since the group velocity is then smaller than c. The only remaining contribution is that of the principal mode (n = 0), so that the reflected pulse in the domain under consideration is approximately given by

$$G_s(x - ct) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} C_s(k)\, a_0(k)\, \exp\left[ik(x - ct)\right] dk, \qquad (5)$$

which is a plane wave, as ought to be expected. The application of these results to incident pulses of the form (4) will be discussed in the next section.

A different method of solution, for incident pulses of the form $H(x + ct)\, F(x + ct)$, where $H(x)$ is the Heaviside step function, $H(x) = 1$ for $x > 0$, $H(x) = 0$ for $x < 0$, has been given by Chester[7]. This method consists in considering first the diffraction of the incident pulse by each plate separately. This provides the correct solution for $0 < ct < 2a$. In fact, for these values of the time, the two plates behave independently, one plate not yet having had time to feel the presence of the other. For $ct = 2a$, the diffracted pulse originated at the edge of each plate reaches the opposite one, where it is reflected and dif fracted. Thus, to obtain the correct solution for $2a < ct < 4a$, two new reflected and diffracted waves must be added. This procedure is repeated again and again, the solution being modified whenever a diffracted wave reaches one of the plates.

For $ct \gg a$, the envelope of each diffracted wave and its successive reflections within the waveguide will be plane. Thus, the contributions from each diffraction will be limited by two plane wave fronts separated by a distance 2a. If we want to know the form of the reflected pulse for $-2a < x - ct < 0$, only the contributions of the first two diffracted waves (one for each edge) and their reflections at the plates must be taken into account; there are no multiple-diffraction effects. For $-4a < x - ct < -2a$, we must take into account the effect of one additional diffraction at each edge, and so on.

3. Retardation and width.

Let us consider the time behaviour of the solution corresponding to the incident pulses shown in fig. 4 and fig. 5. The main part of the diffraction process will take place after the crest of the pulse reaches a distance of the order of a from the open end. As the pulse travels past the edge of each half-plane, part of its energy becomes attached to it, giving rise to an energy res ervoir of a quasi-electrostatic type. The energy accumulated in this reservoir is then reemitted, giving rise both to the reflected wave (backward radiation) and to the wave diffracted around the edge. This process is studied in detail elsewhere[3] for the entirely similar case of perpendicular incidence of a pulse on a half-plane.

The effect of the accumulation process on the shape of the reflected pulse at large distances from the open end can be derived by inserting (3) into (5). The form of the reflected pulse for $x \sim ct \gg a$, and for $s = 1$ and 2, is shown in figures 6 and 7, respectively. To derive these results, the exact expression for $|a_o|$ for $0 < q < 1$ quoted in section 2 was employed, together with a ladder-type approximation for $1 < q < 2$, while the phase φ_o was approximated by the polygonal line shown dashed in fig. 3. The curve shown in fig. 6 represents roughly the

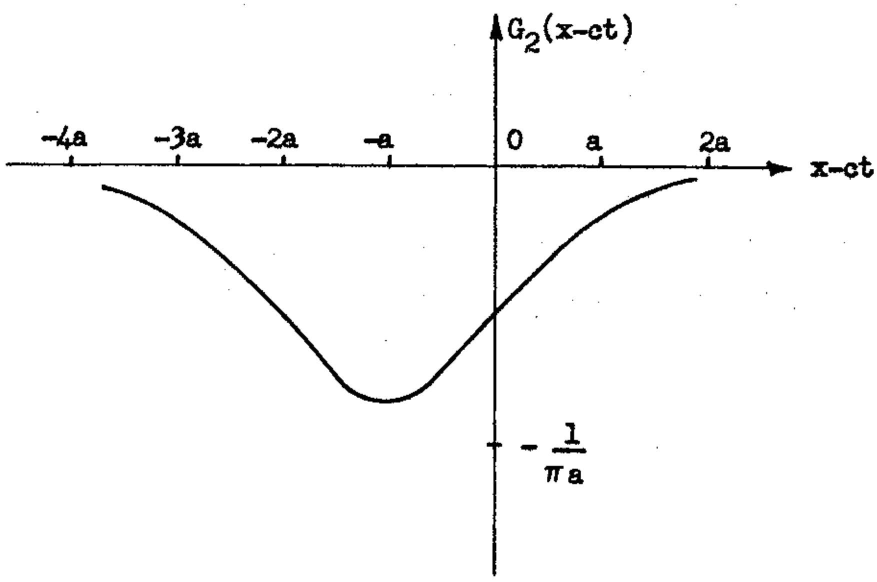

Fig. 6 - The reflected pulse $G_1(x - ct)$.

function $-(a/\pi)\left[a^2 + (x - ct + a)^2\right]^{-1}$.

The shape of the reflected pulse differs from that of the incident one, the amount of distortion depending on the behaviour of $a_o(k)$ in the domain where the Fourier transform of the incident pulse differs significantly from zero (cf. (5)). As $|a_o(k)|$ varies from 1 to 0 when k goes from 0 to ∞, the longest wave lengths in the Fourier spectrum are mostly reflected, whereas the shortest ones are mostly transmitted.

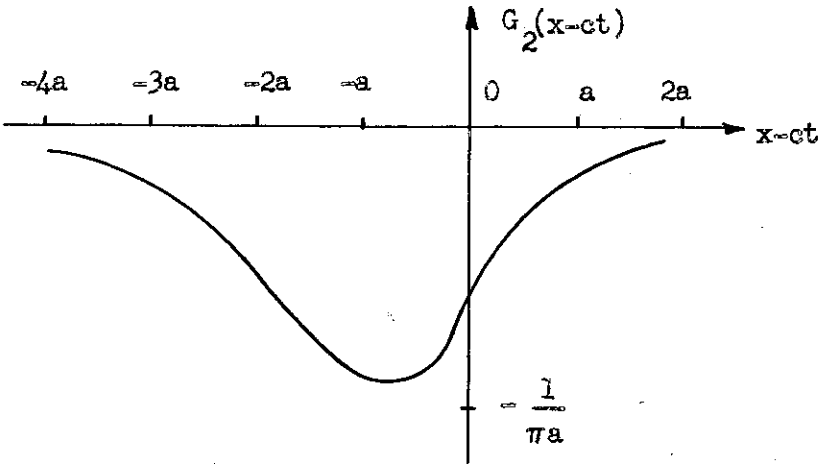

Fig. 7 - The reflected pulse $G_2(x - ct)$.

The form of the reflected pulse, shown in figures 6 and 7, suggests the introduction of two concepts related with the energy accumulation near the edges: <u>retardation</u> and <u>width</u>. The retardation may be defined as the distance (for $ct \gg a$) between the maximum of the reflected pulse, and the maximum of a virtual reflected pulse, which would be obtained if a perfect mirror were placed at the open end of the waveguide.

The retardation is a measure of the time delay between the arrival of the maximum of the incident pulse at the open end and its reemission. It should be emphasized that this is only an asymptotic description: the behaviour of the field near the open end can only be obtained by taking into account the contributions of all travelling and evanescent modes. However, in so far as the asymptotic form of the reflected pulse is concerned, we can imagine that the incident pulse arrives at the open end with velocity c, is trapped there during a time of the order of a/c, and then reemitted again with velocity c.

On account of the deformation of the reflected pulse, the concept of retardation is significant only if its value is at least a large fraction of the width of the incident pulse. This is so in figs 6 and 7, where the retardation is of the order of a for an incident pulse of width a or 2a, respectively. The value of the retardation is essentially given[10] by the derivative of the phase φ_0 with respect to k for $ka \ll 1$, which corresponds to the slope of the dashed line through the origin in fig. 3. The fact that the retardation is approximately the same for two incident pulses of different widths indicates that this effect is largely due to the geometry of the waveguide itself.

After the reflected energy has reached its maximum, there still remains near the edge a certain amount of energy, which will gradually vanish with the time. This gives rise to a tail in the reflected pulse, in addition to that which is due to the reflection of the tail of the incident one. The width of the reflected pulse is therefore related to the "lifetime" of the energy reservoir. This interpretation can be applied only if the width of the reflected pulse is appreciably larger than that of the incident one. This is again true in the case of figures 6 and 7 (compare with figures 4 and 5).

There is no clear-cut distinction between the concepts of retardation and width, the separation being more or less clear according to the type of incident pulse which is considered.

4. Delta-type incident pulse

We shall now consider the limiting case $s \rightarrow \infty$, in (3). According to (4), this corresponds to taking an incident pulse

$$F_{\infty}(x+ct) = \delta(x+ct) . \qquad (6)$$

The corresponding solution $u(x, y, t)$ follows from (2):

$$u(x,y,t) = \delta(x+ct) + \frac{1}{2\pi}\left\{ \int_{-\infty}^{+\infty} a_o(k) \exp\left[ik(x-ct)\right] dk + \right.$$

$$\left. + \sum_{n=1}^{\infty} \cos(k_{yn}y) \int_{-\infty}^{+\infty} a_n(k) \exp\left[i(k^2-k_{yn}^2)^{\frac{1}{2}} x - ikct\right] dk \right\} . \qquad (7)$$

We can now consider the values of (5) for s = 1 and s = 2, which are shown in figs. 6 and 7, as two successive approximations to the first integral on the right-hand side of (7). This integral represents the main contribution to the reflected pulse for $x \sim ct \gg a$.

The solution must obviously satisfy the causality condition

$$u(x,y,t) = 0 \quad \text{for} \quad x > ct. \tag{8}$$

As the second member of (7) is a Fourier series in y, this condition must hold for each of its terms separately, and in particular for the second term $G_{\infty}(x-ct)$ (See (5)).

The fact that the curves in figures 6 and 7 fail to vanish for $x > ct$ is of course due to the cut-off in the Fourier spectrum introduced in (3). The causality condition will be fulfilled only in the limit as $s \longrightarrow \infty$.

The correct form of the asymptotic solution ($x \sim ct \gg a$) for the incident pulse (6) can be derived from the results contained in Chester's paper[7]. We get:

$$G_{\infty}(x-ct) = -\frac{1}{4a} H(ct-x) + \frac{1}{2\pi a} \cos^{-1}\left(\frac{2a}{ct-x}\right) H(ct-x-2a) + \ldots, \tag{9}$$

where the first two terms give the correct asymptotic solution for $-4a < x - ct < 0$. These two terms are shown in fig. 8, where Chester's results have been employed to represent the asymptotic solution for $x - ct > -6a$.

On comparing fig. 8 with figures 6 and 7, we can see that,

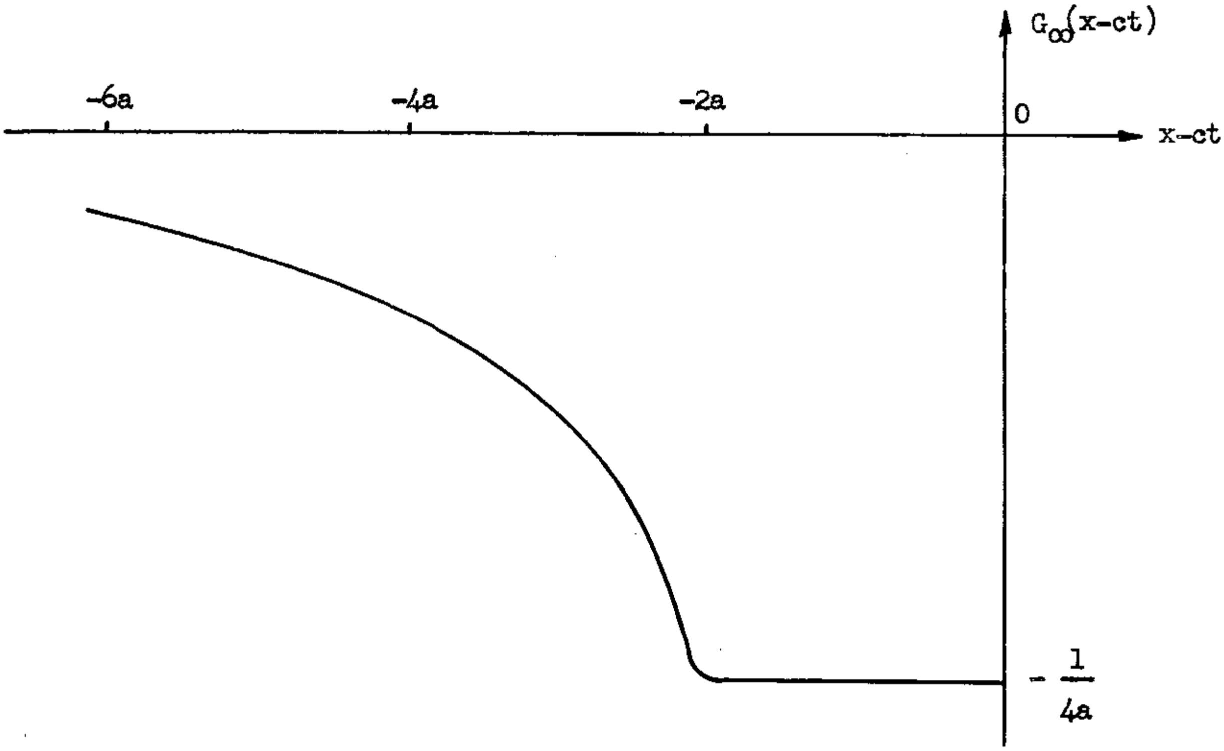

Fig. 8 - The reflected pulse $G_\infty(x - ct)$.

besides leading to the fulfilment of the causality condition (8), the effect of high-frequency components is to flatten the peak at $x - ct \approx - a$, giving rise to a plateau which extends from $x - ct = 0$ to $x - ct = - 2a$. A tendency to this effect can already be noticed in the transition from fig. 6 to fig. 7. The flattening of the peak of course introduces some arbitrariness in the definition of the retardation; however, the value of the retardation can still be defined as the value of $ct - x$ at the center of the plateau region, which again leads to a retardation $ct - x = a$.

The discontinuity at $x = ct$ in fig. 8 is due to the strongly singular character of the incident pulse (6). It has been shown by Chester[7] that, for an incident pulse $H(x+ct)\,F(x+ct)$, the reflected pulse for $ct \gg a$, $-2a < x - ct$, is given by

$$-\frac{H(ct-x)}{4a}\int_0^{ct-x} F(\xi)\,d\xi, \qquad (10)$$

which leads to the first term of (9), in the case (6). If we had taken the incident pulse $H(x+ct)$, as was done by Chester, the reflected pulse would present a linear increase from $x - ct = 0$ to $x - ct = -2a$, and thereafter it would decrease.

The form of the result (10) shows that the open end of the waveguide has an integrating effect on the pulse, which can be likened to that of a capacitor in electric circuit theory. This again reflects the existence of a quasi-electrostatic energy accumulation near the open end. This effect is closely related with the high-frequency behaviour of $a_0(k)$, as will be shown in the next section. In the case of a delta-type pulse, the effect of high frequencies is enhanced, because of the equal weight given to all frequencies in the incident pulse.

5. Asymptotic behaviour of $a_o(k)$ at high frequencies.

According to (3) and (5), we have

$$G_\infty(x-ct) \approx \frac{1}{2\pi} \int_{-\infty}^{+\infty} a_o(k) \exp\left[ik(x-ct)\right] dk, \qquad (11)$$

for $x \sim ct \gg a$. It follows from (9), on the other hand, that G_∞ has a discontinuity of $-1/4a$ at $x = ct$. It is well known in the theory of Fourier integrals that this behaviour is directly related with the asymptotic behaviour of $a_o(k)$ at high frequencies.

Although (11) is no longer valid at large distances from the wave front, we can write

$$a_o(k) \approx \int_{-\infty}^{+\infty} G_\infty(\xi) \exp(-ik\xi)\, d\xi = \int_{-\infty}^{0} G_\infty(\xi) \exp(-ik\xi) d\xi, \qquad (12)$$

because G_∞ is a rapidly decreasing function, so that the contribution from large distances is very small. The last equality in (12) follows from the causality condition (8).

In order to derive from (12) the asymptotic behaviour of $a_o(k)$ for large values of ka, it suffices to apply partial integration:

$$a_o(k) \approx \frac{G_\infty(0^-)}{-ik} = \frac{1}{4ika} \quad (\text{for } ka \gg 1), \qquad (13)$$

where the last equality follows from (9).

Conversely, the behaviour of G_∞ in the neighbourhood of the wave front is determined by the behaviour of $a_o(k)$ for $ka \gg 1$. It will now be shown that this aymptotic behaviour can be derived directly from Sommerfeld's well-known solution for the diffraction of a monochromatic plane wave by a half-plane,[9] without making use of Chester's results.

For this purpose, let us consider the monochromatic solu tion (2) for x = 0:

$$u_k(0,y) - 1 = a_o + \sum_{n=1}^{\infty} a_n \cos(n\pi y/a).$$

It follows that

$$a_o(k) = \frac{1}{2a} \int_{-a}^{a} \left[u_k(0,y) - 1\right] dy. \qquad (14)$$

This integral cannot be evaluated, in general, because the exact solution $u_k(0,y)$ is unknown. However, we are only interested in the behavior of $a_o(k)$ for $ka \gg 1$. This corresponds to a waveguide, the plates of which are far apart (as compared with the wavelength). Under these conditions, it may be expected that each plate diffracts the incident wave independently of the other one, so that multiple--diffraction effects can be neglected.

It was shown by Vajnshtejn[4] that, for an arbitrary incident mode $\cos(k_{yn}y)\exp(-ik_{xn}x)$, the angular distribution of the radiation emitted into free space, for $ka \gg 1$, is approximately given by the superposition of two Sommerfeld-type solutions: one of them representing the diffraction of the plane wave $\frac{1}{2}\exp\left[i(k_{yn}y - k_{xn}x)\right]$ by the upper plate, and the other one the diffraction of the plane wave $\frac{1}{2}\exp\left[-i\,(k_{yn}y + k_{xn}x)\right]$ by the lower plate. In view of the well-known reciprocal relation between angular distribution and aperture distribution,[11] the same must hold for the aperture distribution $u_k(0,y)$.

It follows that $u_k(0,y)$, for $ka \gg 1$, must be approximately given by the superposition of Sommerfeld's solutions for the diffraction of the plane wave $\frac{1}{2}\exp(-ikx)$ by the upper plate and for the diffraction of the plane wave $\frac{1}{2}\exp(-ikx)$ by the lower plate. Each half-plane gives rise to a diffracted wave corresponding to its excitation by one half of the incident mode. The factor 1/2 may appear strange at first sight, for one might expect that the amplitude of the diffracted wave arising from each edge would correspond to an excitation by the full amplitude of the incident mode. However, it must be kept in mind that the incident mode $\exp(-ikx)$ is a limiting case of an incident mode $\cos(k_{yn}y)\exp(-ik_{xn}x)$, as the "angle of propagation" $\theta_n = \tan^{-1}(k_{yn}/k_{xn})$ tends to zero. The splitting is quite natural for $\theta_n \neq 0$, and the transition to the case $\theta_n = 0$ can be rendered practically continuous by taking the diameter of the waveguide sufficiently large, in which case the discrete spectrum of the modes approaches a continuous one.

If we introduce two coordinate systems, (x', y') and (x'', y''), with origins at the two edges (fig. 9), we can therefore replace (14), for $ka \gg 1$, by

$$a_o(k) \approx \frac{1}{2a}\left\{\int_0^{2a}\left[\frac{1}{2}\,u_k^s(0,y') - \frac{1}{2}\right]dy' + \int_0^{2a}\left[\frac{1}{2}\,u_k^s(0,y'') - \frac{1}{2}\right]dy''\right\},$$

where $u_k^s(x',y')$ is Somerfeld's solution for an incident wave $\exp(-ikx')$.

The two integrals are obviously equal, so that, finally,

$$a_o(k) \approx \frac{1}{2a} \int_0^{2a} \left[u_k^s(0,y') - 1\right] dy'. \tag{15}$$

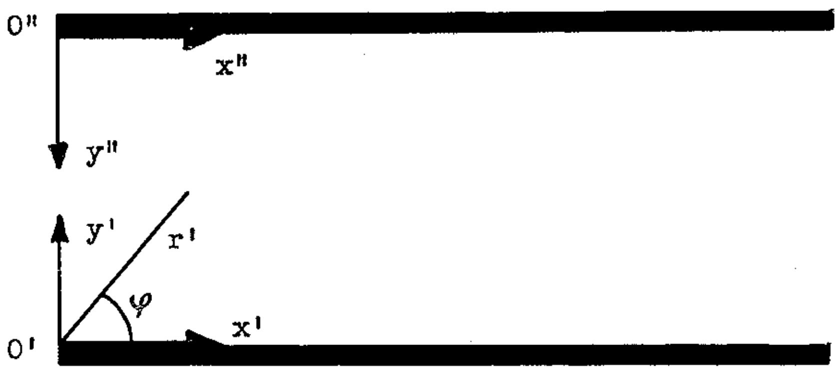

Fig. 9 - Coordinate system with origins at the edges.

In polar coordinates, $x' = r' \cos \varphi'$, $y' = r' \sin \varphi'$ (cf. fig. 9), we have[9]

$$u_k^s(r',\varphi') = \tfrac{1}{2}(1-i) \exp(-ikr' \cos \varphi') \left\{ F\left[2(kr'/\pi)^{\frac{1}{2}} \cos \varphi'/2\right] - F(-\infty) \right\} , \tag{16}$$

where

$$F(W) = \int_0^W \exp(i \tfrac{\pi}{2} \tau^2)\, d\tau \tag{17}$$

is Fresnel's integral.

Taking $\varphi' = \pi/2$, we get

$$u_k^s(0,y') = \frac{1}{2}(1-i)\left\{F\left[(2ky'/\pi)^{\frac{1}{2}}\right]-F(-\infty)\right\}.$$

Noting that $F(\infty) = \frac{1}{2}(1+i) = -F(-\infty)$, we obtain

$$u_k^s(0,y') - 1 = -\frac{1}{2}(1-i)\left\{F(\infty)-F\left[(2ky'/\pi)^{\frac{1}{2}}\right]\right\}. \tag{18}$$

Substituting (18) into (15) and making $ky' = \eta$, we get, taking into account (17),

$$a_o(k) \approx -\frac{1-i}{4ka}\int_0^{2ka} d\eta \int_{(2\eta/\pi)^{\frac{1}{2}}}^{\infty} \exp(i\pi/2\,\tau^2)\,d\tau. \tag{19}$$

The asymptotic expansion of (19) for $ka \gg 1$ will be carried out in the Appendix. The result is, according to (A 3):

$$a_o(k) = \frac{1}{4ika} + \underset{=}{O}\left[(ka)^{-3/2}\right], \tag{20}$$

in exact agreement with (13).

The behaviour of the reflected pulse in the neighbourhood of the wave front is closely connected with (20), as we can see on comparing it with (10), and noting that $(ik)^{-1}$ corresponds to the integration operator for Fourier transforms. The validity of (10) in the domain $-2a < x-ct < 0$, where multiple diffraction does not appear, is directly related with the approximation (15).

6. Contributions from the other modes.

Let us now consider the contribution from the modes $n = 1, 2, \ldots$ in (7), for $ct \gg a$. It suffices to consider a typical case, such as that of $n = 1$:

$$u_1 = \frac{1}{2\pi} \cos(\pi y/a) \int_{-\infty}^{+\infty} a_1(k) \exp\left[i \left(k^2 - \frac{\pi^2}{a^2} \right)^{\frac{1}{2}} x - ikct \right] dk. \qquad (21)$$

The integrand of (21) corresponds to a travelling wave for $q = ka/\pi > 1$, and to an evanescent one for $0 < q < 1$. Let us first consider the contribution from $q > 1$. As $ct \gg a$, we can apply the stationary-phase method[12]. The stationary-phase point is $q = ct(c^2t^2 - x^2)^{-\frac{1}{2}}$. In the evaluation of the integral, we have considered only the contribution from $1 < q < 2$, on account of the rapid decrease of $|a_1|$ for $q > 2$. Thus, the stationary-phase point falls outside of the domain of integration for $x > \frac{\sqrt{3}}{2} ct$, and the contribution from (21) is then much smaller. For $x < \frac{\sqrt{3}}{2} ct$, we get

$$u_1 \approx \frac{4}{a} (a/ct)^{\frac{1}{2}} \cos(\pi y/a)\, f(x/ct) \cos\left\{ -\frac{\pi}{a} ct \left[1 - (x/ct)^2 \right]^{\frac{1}{2}} + \theta(x, ct) + \frac{\pi}{4} \right\}, \qquad (22)$$

where

$$f(\xi) = \xi(1-\xi)^{-1}(1-\xi^2)^{-3/4} \exp\left[-\frac{\pi}{2} \left(\frac{1+\xi}{1-\xi} \right)^{\frac{1}{2}} \right] \qquad (23)$$

and $\theta(x, ct)$, the contribution from the phase of $a_1(k)$, is a complicated, but slowly-varying function, as compared with $(\pi ct/a)\left[1 - (x/ct)^2\right]^{\frac{1}{2}}$. The behaviour of $f(\xi)$ as a function of ξ is shown in fig. 10. The last cosine factor in (22) is a rapidly oscilating function, because $ct \gg a$. Taking into account the factor $(a/ct)^{\frac{1}{2}}$, we see that the contribution of

(22) is much smaller than that of the principal mode.

The contribution of (21) for $0 < q < 1$ is even smaller: it is of the order of a/ct, as can easily be shown by Laplace's method[12].

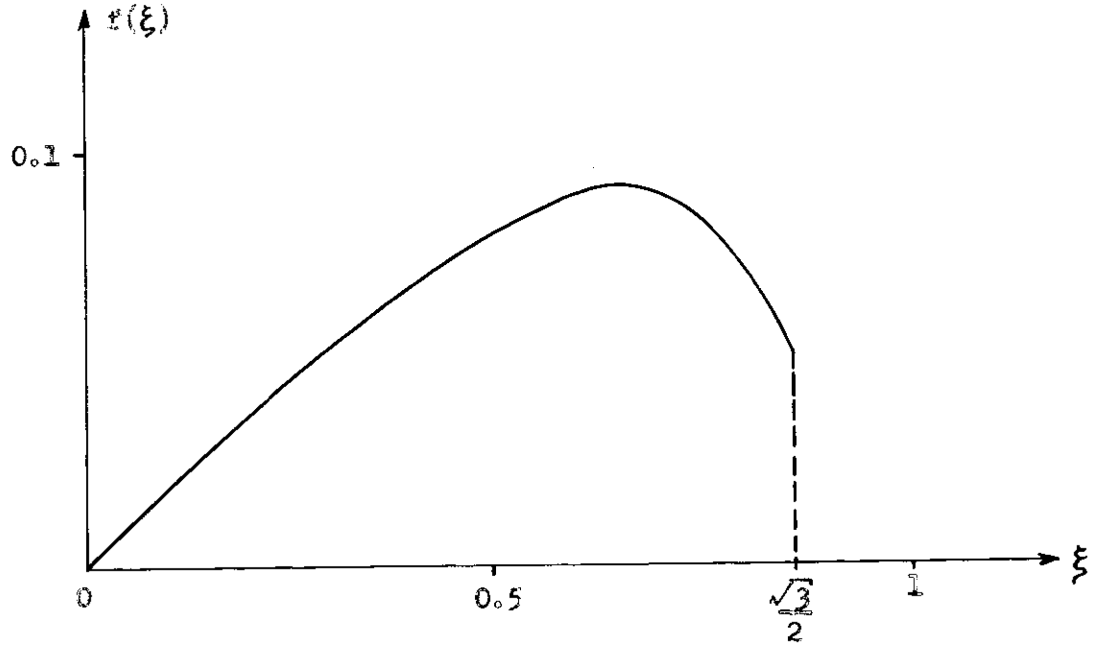

Fig. 10 - Behaviour of $f(\xi)$ as a function of ξ.

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APPENDIX

Asymptotic expansion of $a_o(k)$.

The domain of integration in the (τ, η) plane associated with the double integral (19) is shaded in fig. 11. Inverting the order of integration, we get:

$$a_o(k) = -\frac{1-i}{4ka}\left[\frac{\pi}{2}\int_0^{(4ka/\pi)^{\frac{1}{2}}} \tau^2 \exp(i\pi/2\ \tau^2)\, d\tau + 2ka\int_{(4ka/\pi)^{\frac{1}{2}}}^{\infty} \exp(i\pi/2\ \tau^2) d\tau\right].$$

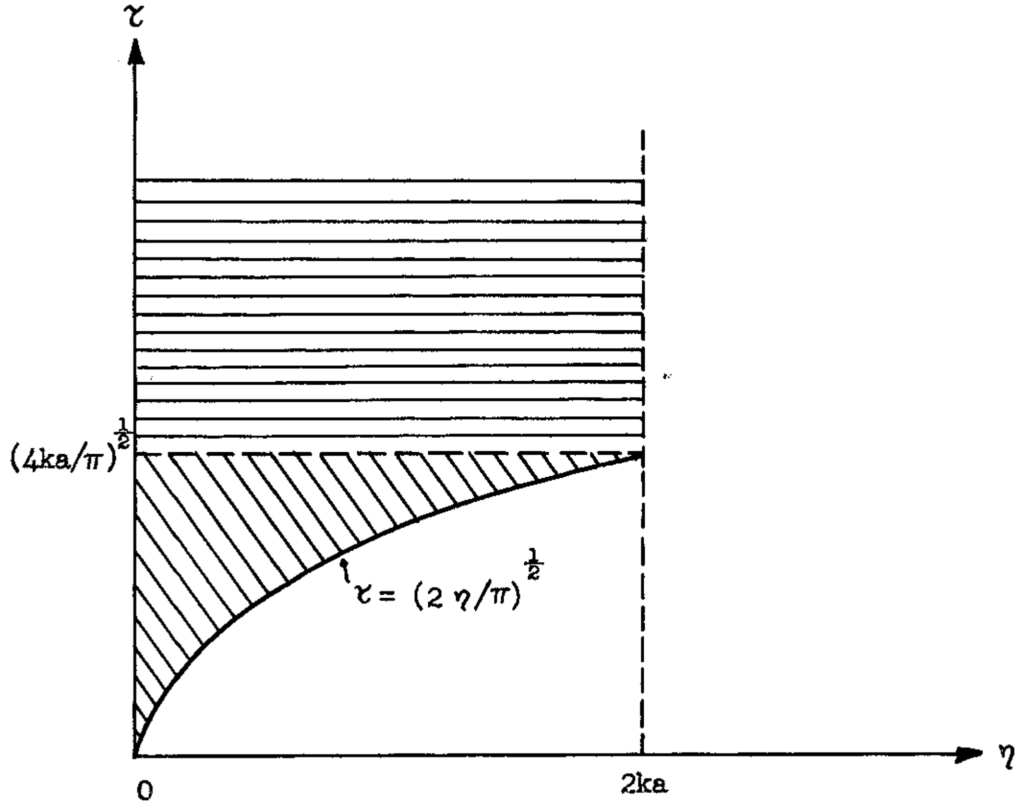

Fig. 11 - The domain of integration.

Integrating the first integral by parts, we are led to:

$$a_o(k) = \frac{1+i}{(4\pi ka)^{\frac{1}{2}}} \exp(2ika) - \frac{1+i}{4ka} F\left[(4ka/\pi)^{\frac{1}{2}}\right] - \frac{1}{2}(1-i)\left\{F(\infty) - F\left[(4ka/\pi)^{\frac{1}{2}}\right]\right\}. \tag{A-1}$$

For $ka \gg 1$, we can employ the asymptotic expansion of the Fresnel integral[9]

$$F(W) = \frac{1}{2}(1+i) + \frac{\exp(i\pi/2\ W^2)}{i\pi W}\left[1 + \frac{1}{i\pi W^2} + \dots\right] \quad (|W| \gg 1). \tag{A-2}$$

Substituting this result in (A-1) and expanding up to terms of order $(ka)^{-3/2}$, we finally get

$$a_o(k) = \frac{1}{4ika} + O\left[(ka)^{-3/2}\right]. \tag{A-3}$$

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