RELATIVISTIC THEORY OF SPINNING POINT PARTICLES

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(July 10, 1955)

In a recente paper[1] the classical (non quantum) relativistic theory of a point particle with spin was developed by Proca in a rather simple form using four component spinors. His equations are obtained from a variational principle which is, however, quite arbitrary.

In the present paper the relativistic theory of spinning particles is developed starting from the non relativistic theory which is modified in order to become invariant under Lorentz transformations.

In Part I the non relativistic theory of spinning particles[2,3] is analysed and reformulated in terms of two component spinors. A variational principle is established and written in a form which is appropriate to the passage to an invariant form.

In Part II the variation integral is made invariant in the usual way. Two alternate possibilities are obtained. In the first case, which will be analysed in a forth coming paper, the invariant Hamiltonian is quadratic in the momenta. In the second case the Hamiltonian is linear in the momenta. The theory obtained in this second case is essentially the same as the one developed by Proca.

Indeed the equations are the same but we have two supplementary conditions for the four spinor ψ as in the non relativistic theory. These conditions, $\bar{\psi}\psi = 1$ and $\bar{\psi}\frac{d\psi}{d\tau} = 0$ restrict somehow the allowed solutions of Proca equation. For instance in the case of free particles only those which satisfy the following condition are allowed:

$$0 \leq \sqrt{-p_\mu p^\mu} \leq |m|$$

In Part II the transition to quatum theory is performed with the usual rules. The irreducible representations of the obtained equation are, for spin 1/2 and 1 respectively, exactly the Dirac equation and the Kemmer - Proca - Yukawa equation.

PART I. NON RELATIVISTIC CLASSICAL THEORY OF SPINNING PARTICLES.

1. Hamiltonian formalism

The equations of motion for particles with an intrinsec angular momentum (3) discribed by a vector $\vec{\Sigma}$ can be obtained from the Hamiltonian

$$H = \vec{p}^{\,2}/2m + eV + U(\vec{\Sigma}, \vec{x}, t) \tag{1}$$

where

$$P_i = p_i - eA_i, \quad i = 1, 2, 3 \tag{2}$$

Now we write

$$\left.\begin{aligned} \Sigma_x &= I \sin\lambda \sin\theta \\ \Sigma_y &= -I \sin\lambda \cos\theta \\ \Sigma_z &= I \cos\lambda \\ \Sigma &= \left(\Sigma_x^2 + \Sigma_y^2 + \Sigma_z^2\right)^{1/2} = I \end{aligned}\right\} \tag{3}$$

where θ and λ are the Euler angles of the rotation which brings a system attached to the particle with its z axis pointing along the spin vector $\vec{\Sigma}$ and the x axis along the nodal line to coincide with the fixed reference frame.

Using equations (1)-(3) and the Hamilton equations

$$\begin{aligned} \frac{dx_i}{dt} &= \frac{\partial H}{\partial p_i}\,; & \frac{dp_i}{dt} &= -\frac{\partial H}{\partial x^i} \\ \frac{d\theta}{dt} &= \frac{\partial H}{\partial \Sigma_z}\,; & \frac{d\Sigma_z}{dt} &= -\frac{\partial H}{\partial \theta} \end{aligned} \tag{4}$$

we obtain the equations of motion viz

$$\begin{aligned} \frac{d\vec{x}}{dt} &= \left(\vec{p} - e\vec{A}\right)/m \\ m\frac{d^2\vec{x}}{dt^2} &= -e\left(\overrightarrow{\text{grad}}\, V + \frac{\partial \vec{A}}{\partial t}\right) + e\,\frac{d\vec{x}}{dt} \wedge \overrightarrow{\text{rot}}\,\vec{A} - \overrightarrow{\text{grad}}\, U \\ \frac{d\vec{\Sigma}}{dt} &= -\vec{\Sigma} \wedge \frac{\partial U}{\partial \vec{\Sigma}} \end{aligned} \tag{5}$$

From (4) it is clear that the z component of the spin, Σ_z, is the momentum conjugate to the angular variable. θ. So if we define the Poisson bracket (u, v) of two physical quantities

u and v which are functions of $\vec{x}$, $\vec{p}$, Σ_{γ}, θ and t as

$$(u,v) = \sum_{i=1}^{3} \left(\frac{\partial u}{\partial x^i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial x^i} \right) + \frac{\partial u}{\partial \theta} \frac{\partial v}{\partial \Sigma_{\gamma}} - \frac{\partial u}{\partial \Sigma_{\gamma}} \frac{\partial v}{\partial \theta} \tag{6}$$

we have

$$\frac{df}{dt} = (f, H) + \frac{\partial f}{\partial t} \tag{7}$$

Equations (5) are again obtained from (7) using (1) - (3).

Finally we should mention that the Hamilton equations (4), and consequently the equations of motion (5), may be obtained from a variational principle(3).

$$\delta T = 0$$

with
$$T = \int_{t_0}^{t_1} \left(\vec{p} \cdot \frac{d\vec{x}}{dt} + \Sigma_{\gamma} \frac{d\theta}{dt} - H \right) dt \tag{8}$$

Here arbitrary variations are given to $\vec{x}$, $\vec{p}$, Σ_{γ} and θ.

2. Spinor equations.

Now we wish to show that the Hamilton equations (4) can be written in spinor form[4].

Let us consider the most general spinor.

$$\psi = R\, e^{i s/2}\, e^{i \sigma_{\gamma} \theta/2}\, e^{i \sigma_x \lambda/2}\, u \tag{9}$$

where R and S are new real quantities depending only of t and are the Pauli matrices and u a unit spinor

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ; \ u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{10}$$

It is easy to show that the components of the vector quantity

$$\vec{\Sigma} = I \, \psi^* \vec{\sigma} \, \psi / \psi^* \psi \tag{11}$$

are given exactly by the expressions (3).

Now it can be proved that the equation

$$i I \frac{d\psi}{dt} = \left[-E + \frac{\vec{p}^2}{2m} + eV + \mathcal{U}(\vec{x}, I\vec{\sigma}, t) \right] \psi = 0 \tag{12}$$

where E is the energy and $\mathcal{U}$ is obtained from U by the substitution of $I\vec{\sigma}$ for $\vec{\Sigma}$, is equivalent to the set of equations

$$\frac{d\theta}{dt} = \frac{\partial R^2 U}{\partial R^2 \Sigma_z} ; \quad \frac{d R^2 \Sigma_z}{dt} = - \frac{\partial R^2 U}{\partial \theta} \tag{13}$$

$$\frac{d\psi^*\psi}{dt} = \frac{dR^2}{dt} = 0 \tag{14}$$

$$I \frac{ds}{dt} + \Sigma_z \frac{d\theta}{dt} = \frac{1}{2iR^2} \left(\psi^* \frac{d\psi}{dt} - \frac{d\psi^*}{dt} \psi \right) = H - E \tag{15}$$

Equation (14) implies that R is a constant which should be different of zero:

$$R^2 = \psi^* \psi = \text{const.} \neq 0 \tag{16}$$

Equations (13) together with

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} \quad ; \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} \tag{17}$$

are thus precisely the Hamilton equations (4).

Finally equation (15) is only the definition of the funcion S. Now, for the actual motion of the particle we must have

$$H = E \tag{18}$$

So, in view of equation (15), we have to impose to the solution of equation (12) the supplementary conditiom

$$\psi^* \frac{d\psi}{dt} = 0 \tag{19}$$

We have proved therefore that equations (12) and (17) are equivalent to the Hamilton equations (4) if the supplementary condition (19) is imposed on the non vanishing solution of equation (12).

3. Variational principle in spinor notation.

It is easy to show that equations (12) and (17) result from the following variational principle

$$\delta T = 0$$

$$T = \int_{\tau_1}^{\tau_2} \left[-E\frac{dt}{d\tau} + \vec{p}\,\frac{d\vec{x}}{dt} + (E-H)\psi^*\psi - iI\left(\psi^*\frac{d\psi}{d\tau} - \frac{d\psi^*}{d\tau}\psi\right)\right] d\tau \tag{20}$$

with

$$H = \vec{P}^2/2m + eV + \psi^* \mathcal{U} \psi / \psi^* \psi$$

In this principle τ is taken as an independent variable and arbitrary variations are given to t, E, $\vec{x}$, $\vec{p}$, ψ and ψ^*

Indeed the resulting equations are

$$\frac{dt}{d\tau} = \psi^* \psi \tag{21}$$

$$\frac{d\vec{x}}{dt} = \frac{\partial H}{\partial \vec{p}} \psi^* \psi ; \quad \frac{d\vec{p}}{dt} = - \frac{\partial H}{\partial \vec{x}} \psi^* \psi \tag{22}$$

$$-2iI \frac{d\psi}{d\tau} = \left(-E + eV + \vec{P}^2/2m + \mathcal{U} \right) \psi \tag{23a}$$

$$2iI \frac{d\psi^*}{d\tau} = \psi^* \left(-E + eV + \vec{P}^2/2m + \mathcal{U} \right) \tag{23b}$$

$$\frac{dE}{d\tau} = \frac{\partial H}{\partial t} \psi^* \psi \tag{24}$$

Taking (21) in (22) we obtain equations (17). Equations (23a - b) are equivalent and have as consequences

$$\frac{d}{d\tau} \psi^* \psi = 0 \tag{25}$$

$$iI \left(\psi^* \frac{d\psi}{d\tau} - \frac{d\psi^*}{d\tau} \psi \right) = (E - H) \psi^* \psi \tag{26}$$

From (25) $\psi^* \psi$ is an integral of motion, and if we assume

$$\psi^* \psi = 1 \tag{27}$$

equation (23a) becomes identical with equation (12) in view of (21).

Equation (24) is a consequence of the remaining ones, if E is the energy

$$E = H$$

Finally we should notice that if we take into account equations (9), (21) and (27) the integral (20) become equal to

$$T = \int_{t_1}^{t_2} \left(\vec{p}\,\frac{d\vec{x}}{dt} + \Sigma_{\gamma}\frac{d\theta}{dt} + \Sigma\,\frac{ds}{dt} - H \right) dt \qquad (28)$$

with

$$\Sigma = I$$

This differs from the variation integral (8) only by the term in $\Sigma\,\frac{ds}{dt}$ which do not give any contribution to the variational equations because it is an exact differencial as Σ is constant. Expression (28) is, however, more appropriate than (8) for a theory which allow for a variation of Σ. If H depends also on s we obtain[3], besides equations (4) the following Hamilton equations:

$$\frac{ds}{dt} = \frac{\partial H}{\partial \Sigma} \quad ; \quad \frac{d\Sigma}{dt} = -\frac{\partial H}{\partial s} \qquad (29)$$

PART II. RELATIVISTIC THEORY OF SPINNING PARTICLES (CLASSICAL)

1. Relativistic invariant formalism.

We shall now obtain the relativistic equations starting from the non relativistic theory in spinor form. We shall start from the variational principle (20) and make Lorentz invariant the integral T.

If τ is a scalar the terms

$$-E\,\frac{dt}{d\tau}+\vec{p}\,\frac{d\vec{x}}{d\tau}=p_\mu\,\frac{dx^\mu}{d\tau} \tag{30}$$

are already invariant. Repeated upper and lower indices mean summation from 0 to 3. Here

$$\left.\begin{array}{l} x^0=ct\,,\ p^0=E/c\,,\ p_\mu=g_{\mu\nu}\,p^\nu \\ g_{\mu\nu}=0\,,\quad \mu\neq\nu\ ;\ g_{ii}=-g_{00}=1 \end{array}\right\} \tag{31}$$

If we use instead of the two component spinors four component ones

$$\overline{\Psi}\cdot\frac{d\Psi}{d\tau}$$

will be invariant, with

$$\overline{\Psi}=\Psi^*\beta\quad,\quad\beta^2=1 \tag{32}$$

if the matrix β has the property

$$\gamma^+_{\mu\nu}\,\beta=\beta\,\gamma_{\mu\nu} \tag{33}$$

Here $\frac{1}{2}\gamma_{\mu\nu}=-\frac{1}{2}\gamma_{\nu\mu}$ are the infinitesimal operators of the Lorentz group and have the commutation relations

$$\left[\gamma_{\mu\nu},\gamma_{\rho\sigma}\right]=2i\left(g_{\mu\rho}\gamma_{\nu\sigma}-g_{\nu\rho}\gamma_{\mu\sigma}+g_{\nu\sigma}\gamma_{\rho\mu}-g_{\mu\sigma}\gamma_{\rho\nu}\right) \tag{34}$$

Instead of (11) we use for the expression of the intrinsec angular momentum of the particle

$$S_{\mu\nu} = I\ \bar{\Psi}\gamma_{\mu\nu}\Psi \big/ \bar{\Psi}\Psi \tag{35}$$

Now any four component spinor can be written in the form

$$\Psi = R\, e^{i s_1/2}\, e^{i\gamma_5 \phi/2}\, L\, e^{i\beta \chi/2}\, \mu$$

$$L = e^{i\gamma_{12}\theta_1/2}\, e^{i\gamma_{30}\theta_2/2}\, e^{i\gamma_{23}\lambda_1/2}\, e^{i\gamma_{10}\lambda_2/2} \tag{36}$$

where R, S, ϕ, θ_1, θ_2, λ_1, λ_2 and χ are real numbers; γ_5 is a matrix with the properties

$$\gamma_5^2 = 1\,, \quad \gamma_5\beta = -\beta\gamma_5\,, \quad \gamma_5\gamma_{\mu\nu} = \gamma_{\mu\nu}\gamma_5 \tag{37}$$

and u a basic spinor defined by

$$\gamma_{12}\mu = \gamma_5\mu = \mu; \qquad \mu^*\mu = 1 \tag{38}$$

If we use the representation

$$\gamma_{ij} = \begin{pmatrix}\sigma_k & 0\\ 0 & \sigma_k\end{pmatrix};\quad \gamma_{0k} = i\begin{pmatrix}\sigma_k & 0\\ 0 & -\sigma_k\end{pmatrix};\quad \beta = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix};\quad \gamma_5 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{39}$$

then

$$\underline{\mu} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

Now, if we form the complex spin vector [5]

$$\Sigma_i = S_{jk} + i S_{k0} \tag{40}$$

we find

$$\begin{aligned}\Sigma_x &= \Sigma \sin\Lambda \sin \Theta \\ \Sigma_y &= -\Sigma \sin\Lambda \cos\Theta \\ \Sigma_z &= \Sigma \cos\Lambda \\ \Sigma &= I(1 + i \cot g\, \phi)\end{aligned} \tag{41}$$

where

$$\begin{aligned}\Lambda &= \lambda_1 + i\lambda_2 \\ \Theta &= \theta_1 + i\theta_2\end{aligned} \tag{42}$$

Whe have also

$$P = \bar{\Psi}\Psi = R^2 \sin\chi \sin\phi$$

$$\Omega = \frac{1}{i}\bar{\Psi}\gamma_5\Psi = R^2 \sin\chi\cos\phi$$

and

$$I\left(\bar{\Psi}\frac{d\Psi}{d\tau} - \frac{d\bar{\Psi}}{d\tau}\Psi\right) =$$

$$= \bar{\Psi}\Psi\left(\frac{1}{2}\Sigma_z\frac{d\Theta}{d\tau} + \frac{1}{2}\Sigma_z^*\frac{d\Theta^*}{d\tau} + \frac{1}{2}\Sigma\frac{ds}{d\tau} + \frac{1}{2}\Sigma^*\frac{ds^*}{d\tau}\right) \tag{45}$$

In Expression (45)

$$S = S_1 + i \lg \operatorname{tg} \frac{\chi}{2} \tag{46}$$

represents the intrinsic rotation (complex) conjugate to the modulus of the complex spin, for $\overline{\Psi}\psi = 1$

It is very interesting that in the relativistic case the expression (45) for the gyroscopic energy allows for a variation of the modulus of the spin $\vec{\Sigma}$, in opposition to the non relativistic one.

Finally we have to make invariant the remaining terms of expression (20), viz

$$(E-H)\psi^*\psi = -\left(P_0 c + \frac{\vec{p}^2}{2m}\right)\psi^*\psi - \psi^* \mathcal{U} \psi \tag{47}$$

with $P_0 c = -E + eV$

The last term in (47), the interaction term, is made invariant by the usual procedure. For instance

$$\begin{aligned} \vec{H}\cdot\psi^*\vec{\sigma}\psi &\longrightarrow \frac{1}{2} F_{\mu\nu}\overline{\Psi}\gamma^{\mu\nu}\psi \\ \psi^*\psi &\longrightarrow \overline{\Psi}\psi \end{aligned} \tag{48}$$

However other invariant interactions which have no simple counter part in the non relativistic theory can be formed with the others covariant quantities

$$V^\mu = \frac{1}{i}\overline{\Psi}\gamma^\mu\Psi \quad \text{(vector)} \tag{49}$$

$$\omega^\mu = \frac{1}{i}\overline{\Psi}\gamma_5\gamma^\mu\Psi \quad \text{(pseudovector)} \tag{50}$$

where $\gamma^0 = i\beta; \quad \gamma^k = i\beta\gamma^5\gamma^i\sigma$

Now the remaining term in (47) is linear in P_0 but quadratic in P_i.

So we have two possibilities in order to make it becomes relativistically invariant: either make it quadratic or linear both in P_0 and P_i.

In the firs case we make the substitution

$$\Psi^*\Psi\left(P_0 c + \frac{P^2}{2m}\right) \longrightarrow P_\mu P^\mu \bar{\Psi}\Psi/2m \tag{51}$$

We shall not consider in the present paper the relativistic theory resulting from this substitution.

In the second case we make the substitution

$$\Psi^*\Psi\left(P_0 c + \frac{P^2}{2m}\right) \longrightarrow \frac{1}{i} P_\mu \bar{\Psi}\gamma^\mu\Psi \tag{52}$$

The resulting theory is essentially equivalent to the recent Proca theory[1] and will be studied in the next section.

2. Proca theory.

We start with the linearized variation integral

$$T = \int \left[p_\mu \frac{dx^\mu}{d\tau} - K - iI\left(\bar{\Psi}\frac{d\Psi}{d\tau} - \frac{d\bar{\Psi}}{d\tau}\Psi\right)\right] d\tau \tag{53}$$

$$K = \frac{1}{i} P_\mu \bar{\Psi}\gamma^\mu\Psi + m\bar{\Psi}\Psi + \bar{\Psi}\mathcal{U}\Psi$$

where $\mathcal{U}$ is the interaction invariant term, formed with the covariant operators and functions of x^μ We put $c = 1$

Now the variational principle

$$\delta T = 0$$

where p^μ, x^μ, ψ and $\overline{\psi}$ are given arbitrary variations, leads to the equations:

$$\frac{dx^\mu}{d\tau} = \frac{\partial K}{\partial p_\mu} = V^\mu \; ; \; \frac{dp_\mu}{d\tau} = -\frac{\partial K}{\partial x^\mu} \tag{54}$$

$$-2iI \frac{d\psi}{d\tau} = \left(\frac{1}{\lambda}\gamma^\mu P_\mu + m + U\right)\psi \tag{55}$$

$$2iI \frac{d\overline{\psi}}{d\tau} = \overline{\psi}\left(\frac{1}{\lambda}\gamma^\mu P_\mu + m + U\right) \tag{56}$$

If we put $\xi = e^{im\tau/2I}\psi$ and $2I = 1$ we see that these are exactly Proca equations[1]. However we have here a normalization condition, similar to the corresponding non relati~~vistic~~ one

$$\overline{\psi}\psi = 1 \tag{57}$$

consistent with equations (55) and (56).

Besides we shall find a further supplementary condition, similar to equation (19) for the non relativistic case. If we multiply equation (55) by $\overline{\psi}$ we find

$$-2iI\,\overline{\psi}\frac{d\psi}{d\tau} = P^\mu V_\mu + m + \overline{\psi}U\psi = K \tag{58}$$

Now it is an immediate consequence of (54) - (56) that K is an integral of motion

$$\frac{dK}{d\tau} = 0 \; ; \quad K = \text{const.} \tag{59}$$

There is no loss of generality if we take this constant as zero. So we have the condition*

$$\bar{\Psi}\,\frac{d\Psi}{d\tau} = 0 \tag{60}$$

In order to have a better understanding of the fundamental solutions and of our supplementary conditions we shall consider now the case of free particles.

Following Proca we find for the general solution of equation (55)

$$\Psi = \Psi_1\, e^{i(m-p)\tau/2I} + \Psi_2\, e^{i(m+p)\tau/2I} \tag{61}$$

$$\Psi_1 = \left(\frac{1}{i}\,\gamma^{\mu} p_{\mu} - p\right)\varphi \;;\; \Psi_2 = \left(\frac{1}{i}\,\gamma^{\mu} p_{\mu} + p\right)\varphi \tag{62}$$

where
$$p^{\mu} = \text{const.} \;;\; p = +\sqrt{-p^{\mu} p_{\mu}} \;;\; \frac{d\varphi}{d\tau} = 0$$

Now, in view of $\bar{\Psi}_1 \Psi_2 = \bar{\Psi}_2 \Psi_1 = 0$

we find

$$-2iI\,\bar{\Psi}\,\frac{d\Psi}{d\tau} = \bar{\Psi}_1 \Psi_1 (m-p) + \bar{\Psi}_2 \Psi_2 (m+p) \tag{63}$$

In view of condition (60) this should vanish. On the

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* We have introduced the term $m\psi$ in equation (55) in order that supplementary condition (60) would lead in the case $P_{\mu} = m V_{\mu}$ to particles of definite rest mass m.

other hand the normalization condition (57) gives

$$\bar{\Psi}_1 \Psi_1 + \bar{\Psi}_2 \Psi_2 = 1$$

So we find

$$\bar{\Psi}_1 \Psi_1 = \frac{m+p}{2p} \quad ; \quad \bar{\Psi}_2 \Psi_2 = \frac{p-m}{2p} \tag{64}$$

Now it is a result from (62) if $m > 0$

$$\bar{\Psi}_1 \Psi_1 > 0 \ , \ \bar{\Psi}_2 \Psi_2 < 0 \quad \text{if} \quad p_0 > 0 \tag{65}$$

$$\bar{\Psi}_1 \Psi_1 < 0 \ , \ \bar{\Psi}_2 \Psi_2 > 0 \quad \text{if} \quad p_0 < 0 \tag{66}$$

This can be casily proved in the reference frame for which $\vec{p} = 0$

Now, in view of (64) these inequalities imply that

$$0 < p \equiv + \sqrt{-p^\mu p_\mu} \leq |m| \tag{67}$$

Inequality (67) is a consequence of our supplementary conditions (57) and (60).

Solutions for p = o are excluded because condition (60) would imply that

$$\bar{\Psi} \Psi = 0$$

If $p = |m|$ we obtain Proca's "harmonic solutions" for which

$$p_\mu = m \frac{dx_\mu}{ds}$$

PART III. QUANTUM THEORY

1. Non relativistic schrödinger equation.

The non relativistic theory of spinning particles can be formulated in a generalised Hamiltonian form using the null "hamiltonian".

$$K = -E + H = -E + eV + \frac{(\vec{P} - e\vec{A})^2}{2m} + U \tag{68}$$

Indeed all the equations of motion are obtained, using the Poisson Bracket formalism:

$$\frac{df}{dt} = (f, K)$$

where

$$(u, v) = \sum_{i=1}^{3} \left(\frac{\partial u}{\partial x^i} \frac{\partial v}{\partial p^i} - \frac{\partial u}{\partial p^i} \frac{\partial v}{\partial x^i} \right) + \left(\frac{\partial u}{\partial E} \frac{\partial v}{\partial t} - \frac{\partial u}{\partial t} \frac{\partial v}{\partial E} \right) + \left(\frac{\partial u}{\partial \theta} \frac{\partial v}{\partial \Sigma_z} - \frac{\partial u}{\partial \Sigma_z} \frac{\partial v}{\partial \theta} \right) \tag{69}$$

The last term of (69) can be written as

$$\frac{i}{2I} \sum_{r=1}^{2} \left(\frac{\partial u}{\partial \psi_r} \frac{\partial v}{\partial \psi_r^*} - \frac{\partial u}{\partial \psi_r^*} \frac{\partial v}{\partial \psi_r} \right) \tag{70}$$

this is what one should expect from the fact that $-2iI\psi^*$ is the conjugate momentum of ψ as can be seen from the variation integral (20). The following Poisson brackets are consequences of (69) and (70).

$$(x_i, p_j) = \delta_{ij} \; ; \; (x_i, x_j) = (p_i, p_j) = 0$$

$$(E, t) = 1; \; (E, x_i) = (E, p_i) = (t, x_i) = (t, p_i) = 0 \quad (71)$$

$$(\Sigma_i, \Sigma_j) = -\Sigma_k \, ; \, (\Sigma_i, x_j) = (\Sigma_i, p_j) = (\Sigma_i, t) = (\Sigma_j, E) = 0$$

The quantum theory is obtained substituting the equation

$$K(\vec{x}, t, \vec{p}, E, \vec{\Sigma}) = 0 \quad (71)$$

by the equation

$$K \Psi = 0 \quad (72)$$

where the classical quantities are substituted by operators acting on the wave function Ψ with commutation relations obtained from the Poisson brackets (71) by

$$[u, v] = -i\hbar \, (u, v) \quad (73)$$

We only mention here that the irreducible representation discribing spin 1/2 particles of equation (72) is just Pauli equation:

$$-i\hbar \frac{\partial}{\partial t} + eV + (\hbar/i \; \overrightarrow{grad} + e\vec{A})^2/2m + u(\vec{x}, t, \hbar\, \vec{\sigma}/2)\Psi = 0$$

2. Relativistic equations for the quantum theory of spinning particles.

Following the same procedure we should start from the "null Hamiltonian" (58) and substitute the equation

$$K(x^{\mu}, p^{\nu}, V^{\lambda}, S^{\rho\sigma}, \omega^{\alpha}, \Omega) = 0$$

by the equation (72) with the commutation relations (73) for the operator discribing the physical quantities. The Poisson brackets for the classical relativistic theory are obviously given by

$$(u,v) = \frac{\partial u}{\partial x^{\mu}}\frac{\partial v}{\partial p_{\mu}} - \frac{\partial u}{\partial p^{\mu}}\frac{\partial v}{\partial x_{\mu}} + \\ + \sum_{\alpha=1}^{4} \frac{i}{2I}\left(\frac{\partial u}{\partial \psi_{\alpha}}\frac{\partial v}{\partial \bar{\psi}_{\alpha}} - \frac{\partial u}{\partial \bar{\psi}_{\alpha}}\frac{\partial v}{\partial \psi_{\alpha}}\right) \qquad (74)$$

as the conjugate variable to ψ_{α} is $-2iI\bar{\psi}_{\alpha}$

We obtain, for instance, the following Poisson Brackets

$$(V^{\mu}, V^{\nu}) = -\frac{1}{I^2}S^{\mu\nu}$$
$$(S^{\mu\nu}, S^{\rho\sigma}) = g^{\mu\sigma}S^{\rho\nu} - g^{\nu\sigma}S^{\rho\mu} + g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} \qquad (75)$$
$$(V^{\mu}, S^{\rho\sigma}) = g^{\mu\rho}V^{\sigma} - g^{\mu\sigma}V^{\rho}$$

So, for the case $\mathcal{U} =$ we obtain the quantum equation

$$\left[\vartheta^{\mu}\left(\frac{\hbar}{i}\frac{\partial}{\partial x^{\mu}} + eA_{\mu}\right) + m\right]\psi = 0 \qquad (76)$$

The two irreducible representations of equations (76) which discribe particles of spin 1/2 and 1 are, respectively Dirac equation and Kemmer-Proca-Yukawa equation, as can be easily verified.

1. A. Proca, J. de Phys. Radium, 15, 65, 1954

2. K. A. Kramers, Handbuch der Chemischen Physik, Band I, Leipzig, 1938

3. M. Shönberg, Thesis, Universidade de São Paulo, 1943

4. For the use of spinors in the hydrodinamics of a fluid with intrinsic spin see: D. Bohm, R. Schiller and J. Tiomno, Nuovo Cimento, I, 48, 1955; see also D. Bohm and R. Schiller, Nuovo Cimento, 1, 67, 1955

5. C. M. Lattes, M. Shönberg and W. Schützer, An. Acad. Bras. Ci. , XIX, 193, 1947