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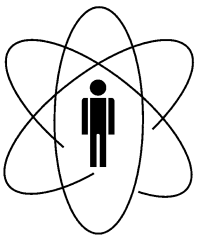

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Exactly solvable models for multiatomic molecular Bose-Einstein condensates

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Exactly solvable models for multiatomic molecular Bose-Einstein condensates

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Abstract. I introduce two family of exactly solvable models for multiatomic hetero-nuclear and homo-nuclear molecular Bose-Einstein condensates through the algebraic Bethe ansatz method. The conserved quantities of the respective models are also showed.

1. Introduction

One of the most interesting recent experimental achievements in physics is the one that led to realizations of Bose-Einstein condensates (BEC), by taking dilute alkali gases to extremely low temperatures [1, 2]. Since then, a great effort has been devoted to the comprehension of new phenomena involving this state of matter as well as its properties, either experimentally or theoretically. On the experimental side I could mention a molecular BEC compound that has been obtained combining different techniques [3], leading this kind of research also in the direction of a chemistry of BEC, where, for instance, by Feshbach resonances [4–6] or photo-association [7,8] the atomic constituents may form molecules.

Many compounds of diatomic homo-nuclear molecular BECs [9–11] have been produced since the first realization [12]. Also, diatomic hetero-nuclear molecular BECs have been detected using these techniques [13–19]. Actually, due to the rapid technological developments in the field of ultra-cold systems, it is believed that some of these experiments may be just the dawn of the study of multiatomic molecules [20, 21]. More recently the experimental evidence for Efimov states in an ultra-cold cesium gas [22–24] and a mixture of ultra-cold potassium and rubidium gases [25] provides a physical ground for the investigation of triatomic and tetratomic homo-nuclear and triatomic hetero-nuclear molecular BECs.

These results boosted the search for solvable models that could describe some of the BEC properties [26–42]. The rationale beneath these studies is that through exactly solvable models it is possible to fully take into account quantum fluctuations, going beyond the usual mean field approximations. Then, I expect that this approach may provide some impact in this area, as well as a contribution to the field of integrable

systems itself [43, 44]. In this paper I will use the algebraic Bethe ansatz method. The algebraic formulation of the Bethe ansatz, and the associated quantum inverse scattering method (QISM), was primarily developed by the group of mathematical physicists in St. Petersburg [45–49]. The QISM could be used to study the one-dimensional spin chains, quantum field theory in one-dimensional bosons interacting systems [50] and two-dimensional lattice models [51], systems of strongly correlated electrons [52, 53], conformal field theory [54], as well as precipitated the notion of quantum algebras (deformations of universal enveloping algebras of Lie algebras) [55–58]. For a pedagogical and historical review see [59].

Owing to recent insights in the understanding of the construction of Lax operators it is possible to obtain solvable models suitable for the effective description of the interconversion interactions occurring in the BEC. Inspired by some of these ideas I present, in the present paper, the construction of two complete family of Bethe-ansatz solvable models for both homo-nuclear and hetero-nuclear molecular BECs obtained through a combination of three Lax operators constructed using special realizations of the $su(2)$ Lie algebra and of the Heisenberg-Weyl Lie algebra, as well as a multibosonic representation of the $sl(2)$ Lie algebra, discussed recently in [60]. Notice that the models obtained through this construction do not have spatial degrees of freedom.

The paper is organized as follows: In Section 2, I will review shortly the algebraic Bethe ansatz method and present the Lax operators and the transfer matrix for both models. In Section 3, I will present a family of multiatomic homo-nuclear models and their solutions. In Section 4, I will present a family of multiatomic hetero-nuclear models and their solutions. In Section 5, I will make my remarks.

2. Algebraic Bethe ansatz method

In this section we will shortly review the algebraic Bethe ansatz method and present the transfer matrix used to get the solution of the models [28, 61]. We begin with the $gl(2)$-invariant R-matrix, depending on the spectral parameter u,

$$R(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1}$$

with $b(u) = u/(u+\eta)$ and $c(u) = \eta/(u+\eta)$. Above, η is an arbitrary parameter, to be chosen later. It is easy to check that $R(u)$ satisfies the Yang-Baxter equation

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v). \tag{2}$$

Here $R_{jk}(u)$ denotes the matrix acting non-trivially on the j-th and the k-th spaces and as the identity on the remaining space.

Next we define the monodromy matrix $T(u)$,

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \tag{3}$$

that satisfy the Yang-Baxter algebra,

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v). \tag{4}$$

In what follows we will choose different realizations for the monodromy matrix $\pi(T(u)) = L(u)$ to obtain solutions of two family of models for multiatomic hetero-nuclear and homo-nuclear molecular BECs. In this construction, the Lax operators $L(u)$ have to satisfy the relation

$$R_{12}(u-v)L_1(u)L_2(v) = L_2(v)L_1(u)R_{12}(u-v). \tag{5}$$

where we use the notation,

$$L_1 = L(u) \otimes I \quad \text{and} \quad L_2 = I \otimes L(u). \tag{6}$$

Then, defining the transfer matrix, as usual, through

$$t(u) = tr\ \pi(T(u)) = \pi(A(u) + D(u)), \tag{7}$$

it follows from (4) that the transfer matrix commutes for different values of the spectral parameter; i. e.,

$$[t(u), t(v)] = 0, \qquad \forall\ u,\ v. \tag{8}$$

Consequently, the models derived from this transfer matrix will be integrable. Another consequence is that the coefficients $\mathcal{C}_k$ in the transfer matrix $t(u)$,

$$t(u) = \sum_k \mathcal{C}_k u^k, \tag{9}$$

are conserved quantities or simply c-numbers, with

$$[\mathcal{C}_j, \mathcal{C}_k] = 0, \qquad \forall\ j,\ k. \tag{10}$$

If the transfer matrix $t(u)$ is a polynomial function in u, with $k \geq 0$, it is easy to see that,

$$\mathcal{C}_0 = t(0) \quad \text{and} \quad \mathcal{C}_k = \frac{1}{k!}\left.\frac{d^k t(u)}{du^k}\right|_{u=0}. \tag{11}$$

We will use three solutions of the equation (5):

(i) **The $L^S(u)$ Lax operator:**

$$L^S(u) = \frac{1}{u}\begin{pmatrix} u - \eta S^z & -\eta S^+ \\ -\eta S^- & u + \eta S^z \end{pmatrix}, \tag{12}$$

in terms of the $su(2)$ Lie algebra with generators S^z and $S^\pm$ subject to the commutation relations

$$[S^z, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = 2S^z. \tag{13}$$

(ii) **The $L^j(u)$ Lax operator:**

$$L^j(u) = \begin{pmatrix} u + \eta N_j & j \\ j^\dagger & \eta^{-1} \end{pmatrix}, \tag{14}$$

in terms of the Heisenberg-Weyl Lie algebra with generators N_j, j, $j^\dagger$ and I, subject to the commutation relations

$$[N_j, j] = -j, \quad [N_j, j^\dagger] = +j^\dagger, \quad [j, j^\dagger] = I \quad and \quad [I, \star] = 0, \tag{15}$$

where $\star$ means $N_j,\ j$ or $\ j^\dagger$.

(iii) **The $L^A(u)$ Lax operator:**

$$L^A(u) = \begin{pmatrix} u + \frac{\eta}{2} A_0 & \eta A_- \\ -\eta A_+ & u - \frac{\eta}{2} A_0 \end{pmatrix}, \tag{16}$$

in terms of the $sl(2)$ Lie algebra with generators A_0 and $A_\pm$, subject to the commutation relations

$$[A_-, A_+] = A_0, \quad [A_0, A_\pm] = \pm 2 A_\pm. \tag{17}$$

Using the co-multiplication properties of the Lax operator and the $gl(2)$ invariance of the R-matrix, we can obtain different realizations for the monodromy matrix:

(i) **Multiatomic homo-nuclear:** For the multiatomic homo-nuclear molecular BEC model we choose

$$\pi(T(u)) = L(u) = \eta^{-1} G L^j(u - \delta - \eta^{-1}) L^A(u + \omega), \tag{18}$$

with, $G = diag(-, +)$, from which we find the following transfer matrix

$$\begin{aligned} t(u) = & -\eta^{-1}(u + \omega + \frac{\eta}{2} A_0)(u - \delta - \eta^{-1} + \eta N_b) \\ & + \eta^{-2}(u + \omega - \frac{\eta}{2} A_0) + b A_+ + b^\dagger A_-, \end{aligned} \tag{19}$$

with

$$t(0) = \eta^{-1}\omega(\delta + 2\eta^{-1}) - \omega N_b + \frac{1}{2}(\delta - \eta N_b) A_0 + b A_+ + b^\dagger A_-, \tag{20}$$

and, discarding c-number terms, the conserved quantities are,

$$\mathcal{C}_0 = t(0), \tag{21}$$

$$\mathcal{C}_1 = \frac{1}{2} A_0 + N_b. \tag{22}$$

(ii) **Multiatomic hetero-nuclear:** For the multiatomic hetero-nuclear molecular BEC models we choose

$$\pi(T(u)) = L(u) = \eta^{-1}u^{-}GL^{S}(u^{-})L^{A}(u^{+}), \tag{23}$$

with $u^{\pm} = u \pm \omega$, $G = diag(+,-)$, from which we find the following transfer matrix

$$t(u) = u^{-}A_0 - 2u^{+}S^{z} + \eta(S^{+}A_{+} + S^{-}A_{-}), \tag{24}$$

and the conserved quantities,

$$\mathcal{C}_0 = t(0), \tag{25}$$
$$\mathcal{C}_1 = A_0 - 2S^{z}. \tag{26}$$

In the next sections we will describe the models and its integrability by the algebraic Bethe ansatz method, using different realizations of the algebras (13), (15) and (17). The Hamiltonians are written in the Fock space using the standard notation. We are considering the coupling parameters real, such that the Hamiltonians are Hermitian. In the diagonal part of the Hamiltonians, the U_j parameters describe the atom-atom, atom-molecule and molecule-molecule S-wave scatterings and the μ_j parameters are the externals potentials. The operators N_j are the number operators of atoms or molecules. In the off diagonal part of the Hamiltonians the parameter Ω is the amplitude for interconversion of atoms and molecules.

3. Multiatomic homo-nuclear molecular models

In this section we present the integrability of a new family of Hamiltonians describing multiatomic homo-nuclear molecular BEC. The Hamiltonians that describes the interconversion of homogeneous molecules labelled by b with l atoms of type a are given by

$$\begin{aligned} H &= U_a N_a^2 + U_b N_b^2 + U_{ab} N_a N_b + \mu_a N_a + \mu_b N_b \\ &+ \Omega((a^{\dagger})^{l}\, b\, \alpha_{-}(N_a) + \alpha_{-}(N_a)\, b^{\dagger}\, (a)^{l}), \end{aligned} \tag{27}$$

where $\alpha_{-}(N_a)$ is a function of N_a that controls the amplitude of interconversion Ω. This indicates that the density of atoms N_a has some influence in the generation of a bound-state composed by l identical atoms. The $l = 3$ case was studied in [35]. The total number of particles $N = N_a + lN_b$ is a conserved quantity.

There is a multibosonic realization of the $sl(2)$ Lie algebra [60]

$$A_0 = \alpha_0(N), \quad A_{-} = \alpha_{-}(N)a^{l}, \quad A_{+} = (a^{\dagger})^{l}\alpha_{-}(N), \tag{28}$$

with

$$\alpha_0(N) = \frac{2}{l}(N-R) + \alpha_0(R), \tag{29}$$

$$\alpha_-(N) = \sqrt{\frac{N!}{(N+l)!}(\frac{1}{l}(N-R)+\alpha_0(R))(\frac{1}{l}(N-R)+1)}, \tag{30}$$

where $N = a^\dagger a$ and $l \in \mathbb{N}$. The operator R is

$$R = \begin{cases} 0 & \text{for} \quad l = 1, \\ \dfrac{l-1}{2} + \displaystyle\sum_{m=1}^{l-1} \frac{e^{-(2\pi m/l)N}}{e^{(2\pi m/l)}-1} & \text{for} \quad l > 1, \end{cases} \tag{31}$$

and acts on the states $\{|n\rangle\}$ as $R|n\rangle = n \; mod \; l|n\rangle$. The function $\alpha_0(R)$ is a positive function of the spectrum of R defined by initial conditions. For $n = r < l$, we have

$$\frac{1}{l}(N-R)|r\rangle = 0|r\rangle, \tag{32}$$

with $A_0 = \alpha_0(R)$ such that $\alpha_0(R)|r\rangle = \alpha_0(r)|r\rangle$ and $R|r\rangle = r|r\rangle$.

Now we will use this realization to show how to construct the Hamiltonian (27) from the transfer matrix (19) and present their exact Bethe ansatz solution. It is straightforward to check that the Hamiltonian (27) is related with the transfer matrix $t(0)$ (20), or with the conserved quantity $\mathcal{C}_0$ (21), through

$$H = \Omega \, t(0), \tag{33}$$

where we have the following identification

$$\eta = \frac{l^2 U_a - l U_{ab} + U_b}{\Omega}, \tag{34}$$

$$\theta = 2lU_{ab} - 4U_b, \qquad \xi = 2l^2\mu_a - 2l\mu_b, \tag{35}$$

$$2l\Omega(\omega+\delta) = (2\Omega\eta+\theta)N - \Omega\rho\eta + \xi, \tag{36}$$

$$\begin{aligned} 4l^2\Omega\omega\delta\eta + 8l^2\Omega\omega = \eta^2[&\Omega\rho^2\eta + 4U_bN^2 - \theta N\rho \\ &+ 4l(\Omega\omega+\mu_b)N + (2l\Omega\omega - \xi)\rho], \end{aligned} \tag{37}$$

with $\rho \equiv \rho(R) = l\alpha_0(R) - 2R$.

It is easy to see that ρ is a conserved quantity using the total number of atoms, N, to write the conserved quantity $\mathcal{C}_1$ (22) as,

$$\mathcal{C}_1 = \frac{1}{l}N + \frac{1}{2l}\rho. \tag{38}$$

We can apply the algebraic Bethe ansatz method, using as the pseudo-vacuum the product state ($|0\rangle = |0\rangle_b \otimes |r\rangle_A$, with $|0\rangle_b$ denoting the Fock vacuum state and $|r\rangle_A$ denoting the lowest weight state of the $sl(2)$ Lie algebra, where $r = 0,\ 1,\ ...\ ,\ l-1$, are the eigenvalues of R for $N = nl + r$, with $n \in \mathbb{N}$, to find the Bethe ansatz equations (BAE)

$$\frac{(1-\eta v_i+\eta\delta)(v_i+\omega+\frac{\eta}{2}\alpha_0(r))}{v_i+\omega-\frac{\eta}{2}\alpha_0(r)} = \prod_{i\neq j}^{M}\frac{v_i-v_j-\eta}{v_i-v_j+\eta}, \quad i,j=1,...,M, \quad (39)$$

and the eigenvalues of the Hamiltonian (27),

$$\begin{aligned} E = {} & \Omega\eta^{-1}(\delta+\eta^{-1})\left(\omega+\frac{\eta}{2}\alpha_0(r)\right)\prod_{i=1}^{M}\frac{v_i-\eta}{v_i} \\ & + \Omega\eta^{-2}\left(\omega-\frac{\eta}{2}\alpha_0(r)\right)\prod_{i=1}^{M}\frac{v_i+\eta}{v_i}. \end{aligned} \quad (40)$$

The parameters δ and ω are arbitrary and can be chosen conveniently. In the limit without scatterings, $U_j \to 0$, the BAE (39) can be write as,

$$\sum_{i=1}^{M}\frac{1}{v_i+\omega} = \frac{1}{\alpha_0(r)}\sum_{i=1}^{M}v_i - \frac{M}{\alpha_0(r)}\delta, \quad (41)$$

and for $\omega = 0$ the eigenvalues (40) become,

$$E = \left(\frac{1}{2}\alpha_0(r)+M\right)\Omega\delta - \Omega\sum_{i=1}^{M}v_i. \quad (42)$$

Now, the relation between the interconversion parameter and the externals potentials is simply,

$$\Omega = \frac{l\mu_a-\mu_b}{\delta}. \quad (43)$$

4. Multiatomic hetero-nuclear molecular models

In this section we present the integrability of a new family of Hamiltonians describing multiatomic hetero-nuclear molecular BEC. The Hamiltonians that describes the interconversion of heterogeneous molecules labelled by c with l atoms of type a and one atom of type b are given by

$$\begin{aligned} H = {} & U_aN_a^2 + U_bN_b^2 + U_cN_c^2 + U_{ab}N_aN_b + U_{ac}N_aN_c + U_{bc}N_bN_c \\ & + \mu_aN_a + \mu_bN_b + \mu_cN_c \\ & + \Omega((a^\dagger)^l\ b^\dagger\ c\ \alpha_-(N_a) + \alpha_-(N_a)\ c^\dagger\ b\ (a)^l), \end{aligned} \quad (44)$$

where $\alpha_-(N_a)$ is a function of N_a that controls the amplitude of interconversion Ω. In the same way of the Hamiltonians (27), this indicates that the density of atoms N_a has some influence in the generation of a bound-state composed by l identical atoms.

The imbalance between the number of atoms a and the number of atoms b,

$$\mathcal{J}_{ab} = N_a - lN_b, \tag{45}$$

is a conserved quantity and the total number of atoms, $N = N_a + N_b + (l+1)N_c$, can be writes with the other two conserved quantities

$$\mathcal{I}_{ac} = N_a + lN_c, \tag{46}$$

$$\mathcal{I}_{bc} = N_b + N_c. \tag{47}$$

Using $\mathcal{J}_{ab}$ and N, the S-wave diagonal part of the Hamiltonian (44) can be writte as,

$$\alpha\mathcal{J}_{ab}^2 + \beta N^2 + \gamma N\mathcal{J}_{ab}, \tag{48}$$

where we have used the following identification for the coupling constants

$$U_a = \alpha + \beta + \gamma, \qquad U_b = \alpha l^2 + \beta - \gamma l, \qquad U_c = \beta(l+1)^2, \tag{49}$$

$$U_{ab} = -2l\alpha + 2\beta - \gamma(l-1), \qquad U_{ac} = 2\beta(l+1) + \gamma(l+1), \tag{50}$$

$$U_{bc} = 2\beta(l+1) - \gamma l(l+1). \tag{51}$$

Now, using the following realization for the $su(2)$ Lie algebra,

$$S^+ = b^\dagger c, \quad S^- = c^\dagger b, \quad S^z = \frac{N_b - N_c}{2}, \tag{52}$$

and the multibosonic realization of the $sl(2)$ Lie algebra (28),

$$A_0 = \alpha_0(N), \quad A_- = \alpha_-(N)a^l, \quad A_+ = (a^\dagger)^l\alpha_-(N), \tag{53}$$

with

$$\alpha_0(N) = \frac{2}{l}(N-R) + \alpha_0(R), \tag{54}$$

$$\alpha_-(N) = \sqrt{\frac{N!}{(N+l)!}(\frac{1}{l}(N-R) + \alpha_0(R))(\frac{1}{l}(N-R)+1)}, \tag{55}$$

where $N = a^\dagger a$ and $l \in \mathbb{N}$, it is straightforward to check that the Hamiltonian (44) is related with the transfer matrix $t(u)$ (24), or with the conserved quantity $\mathcal{C}_0$ (25) if $u = 0$, through

$$H = \sigma + \alpha\mathcal{J}_{ab}^2 + \beta N^2 + \gamma N\mathcal{J}_{ab} + t(u), \tag{56}$$

where the following identification has been made for the parameters

$$\mu_a = 2\frac{u^-}{l}, \qquad \mu_c = -\mu_b = u^+, \qquad \Omega = \eta, \qquad \sigma = -\frac{u^-}{l}\rho, \tag{57}$$

with $\rho \equiv \rho(R) = l\alpha_0(R) - 2R$.

We also can use the conserved quantities $\mathcal{J}_{ab}$ and $\mathcal{I}_{ac}$ to write the conserved quantity $\mathcal{C}_1$ (26) as,

$$\mathcal{C}_1 = \frac{1}{l}(\mathcal{J}_{ab} + \mathcal{I}_{ac}) + \frac{1}{l}\rho, \tag{58}$$

showing that ρ is also a conserved quantity.

We can apply the algebraic Bethe ansatz method, using as the pseudo-vacuum the product state ($|0\rangle = |r\rangle_A \otimes |\phi\rangle$, with $|r\rangle_A$ denoting the lowest weight state of the $sl(2)$ Lie algebra where $r = 0,\ 1,\ ...\ ,\ l-1$, are the eigenvalues of R for $N = nl + r$, with $n \in \mathbb{N}$ and $|\phi\rangle$ denoting the highest weight state of the $su(2)$ Lie algebra with weight m_z), to find the Bethe ansatz equations (BAE)

$$-\frac{(v_i - \omega - \eta m_z)(v_i + \omega + \frac{\eta}{2}\alpha_0(r))}{(v_i - \omega + \eta m_z)(v_i + \omega - \frac{\eta}{2}\alpha_0(r))} = \prod_{i\neq j}^{M} \frac{v_i - v_j - \eta}{v_i - v_j + \eta}, \quad i, j = 1, ..., M, \tag{59}$$

and the eigenvalues of the Hamiltonian (44)

$$\begin{aligned} E = {} & \sigma + \alpha\mathcal{J}_{ab}^2 + \beta N^2 + \gamma N\mathcal{J}_{ab} \\ & + (u - \omega - \eta m_z)(u + \omega + \frac{\eta}{2}\alpha_0(r)) \prod_{i=1}^{M} \frac{u - v_i + \eta}{u - v_i} \\ & - (u - \omega + \eta m_z)(u + \omega - \frac{\eta}{2}\alpha_0(r)) \prod_{i=1}^{M} \frac{u - v_i - \eta}{u - v_i}. \end{aligned} \tag{60}$$

The eigenvalues (60) are independent of the spectral parameter u and of the parameter ω, that are arbitrary.

5. Summary

I have introduced two new family of multiatomic molecular BEC models for homo-nuclear and hetero-nuclear molecules and derived the Bethe ansatz equations and the eigenvalues. The conserved quantities are also derived. The multiatomic homo-nuclear and hetero-nuclear molecular BEC models were obtained through a combination of Lax operators constructed using special realizations of the $su(2)$ Lie algebra and Heisenberg-Weyl Lie algebra, as well as a multibosonic representation of the $sl(2)$ Lie algebra. The dependence of the parameters with the size of the molecules is explicit.

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References

[1] E. A. Cornell and C. E. Wieman, *Rev. Mod. Phys.* **74** (2002) 875.
[2] J. R. Anglin and W. Ketterle, *Nature* **416** (2002) 211.
[3] P. Zoller, *Nature* **417** (2002) 493.
[4] S. Inouye, M. R. Andrews, J. Stenger, H.-J. Miesner, D. M. Stamper-Kurn and W. Ketterle, *Nature* **392** (1998) 151.
[5] T. Kohler, K. Goral, and P. S. Julienne, *Rev. Mod. Phys.* **78** (2006) 1311.
[6] Cheng Chin, Rudolf Grimm, Paul Julienne and Eite Tiesinga, *Rev. Mod. Phys.* **82**, (2010) 1225.
[7] K. M. Jones, E. Tiesinga, P. D. Lett, and P. S. Julienne, *Rev. Mod. Phys.* **78** (2006) 483.
[8] C. R. Menegatti, B. S. Marangoni and L. G. Marcassa, *Laser Physics* **18** (2008) 1305.
[9] J. Herbig et al, *Science* **301** (2003) 1510.
[10] S. Durr et al, *Phys. Rev. Lett.* **92** (2004) 020406.
[11] C. Chin, T. Kraemer, M. Mark, J. Herbig, P. Waldburger, H.-C. Nägerl and R. Grimm, *Phys. Rev. Lett.* **94** (2005) 123201.
[12] E. A. Donley, N. R. Claussen, S. T. Thompson, and C. E. Wieman, *Nature* **417**, (2002) 529.
[13] B. Damski, L. Santos, E. Tiemann, M. Lewenstein, S. Kotochigova, P. Julienne, and P. Zoller, *Phys. Rev. Lett.* **90** (2003) 110401.
[14] M.W. Mancini, G. D. Telles, A. R. L. Caires, V. S. Bagnato and L. G. Marcassa, *Phys. Rev. Lett.* **92** (2004) 133203.
[15] C. Haimberger, J. Kleinert, M. Bhattacharya and N. P. Bigelow, *Phys. Rev. A* **70** (2004) 021402(R).
[16] J. M. Sage, S. Sainis, T. Bergeman, and D. DeMille, *Phys. Rev. Lett.* **94** (2005) 203001.
[17] S. B. Papp and C. E. Wieman, *Phys. Rev. Lett.* **97** (2006) 180404.
[18] C. Weber, G. Barontini, J. Catani, G. Thalhammer, M. Inguscio and F. Minardi, *Phys. Rev. A* **78** (2008) 061601R.
[19] K. Pilch, A. D. Lange, A. Prantner, G. Kerner, F. Ferlaino, H.-C. Nägerl and R. Grimm, *Phys. Rev. A* **79** (2009) 042718.
[20] E. Braaten, H.-W. Hammer, *Annals of Physics* **322** (2007) 120.
[21] B. D. Esry and C. Greene, *Nature* **440** (2006) 289.
[22] T. Kraemer, M. Mark, P. Waldburger, J. G. Danzl, C. Chin, B. Engeser, A. D. Lange, K. Pilch, A. Jaakkola, H.-C. Nagerl, R. Grimm, *Nature* **440** (2006) 315.
[23] S. Knoop, F. Ferlaino, M. Berninger, M. Mark, H.-C. Nägerl and R. Grimm, *Journal of Physics: Conf. Ser.* **194** (2009) 012064.
[24] F. Ferlaino, S. Knoop, M. Berninger, W. Harm, J. P. D'Incao, H.-C. Nägerl and R. Grimm, Phys. Rev. Lett. **102** (2009) 140401.
[25] G. Barontini, C. Weber, F. Rabatti, J. Catani, G. Thalhammer, M. Inguscio and F. Minardi, *Phys. Rev. Lett.* **103** (2009) 043201.
[26] H.-Q. Zhou, J. Links, M. Gould and R. McKenzie, *J. Math. Phys.* **44** (2003) 4690.
[27] H.-Q. Zhou, J. Links, R. H. McKenzie, *Int. Jour. Mod. Phys. B* **17** (2003) 5819.
[28] J. Links, H.-Q. Zhou, R. H. McKenzie and M. D. Gould, *J. Phys. A* **36** (2003) R63.
[29] A. Foerster, J. Links, H.-Q. Zhou, in *Classical and quantum nonlinear integrable systems: theory and applications*, edited by A. Kundu (IOP Publishing, Bristol and Philadelphia, 2003) pp. 208-233.
[30] J. Dukelsky, G. Dussel, C. Esebbag and S. Pittel, *Phys. Rev. Lett.* **93** (2004) 050403.
[31] G. Ortiz, R. Somma, J. Dukelsky and S. Rombouls, *Nuclear Physics B* **707** (2005) 421.

[32] A. Kundu, *Theoretical and Mathematical Physics* **151** (2007) 831.
[33] A. Foerster and E. Ragoucy, *Nuclear Physics B* **777** (2007) 373.
[34] Jon Links, Angela Foerster, Arlei Prestes Tonel and Gilberto Santos, *Ann. Henri Poincaré* **7** (2006) 1591.
[35] G. Santos, A. Foerster, I. Roditi, Z. V. T. Santos and A. P. Tonel, *J. Phys. A: Math. Theor.* **41** (2008) 295003.
[36] G. Santos, A. Tonel, A. Foerster and J. Links, *Phys. Rev. A* **73** (2006) 023609.
[37] G. Santos, A. Foerster, J. Links, E. Mattei and S. R. Dahmen, *Phys. Rev. A* **73** (2010) 023609.
[38] A. P. Tonel, C. C. N. Kuhn, G. Santos, A. Foerster, I. Roditi and Z. V. T. Santos, *Phys. Rev. A* **79** (2009) 013624.
[39] L. Zhou, W. Zhang, H. Y. Ling, L. Jiang, and H. Pu, *Phys. Rev. A* **75** (2007) 043603.
[40] L. Zhou, J. Qian, H. Pu, W. Zhang, and H. Y. Ling, *Phys. Rev. A* **78** (2008) 053612.
[41] A. Vardi, V. A. Yurovsky and J. R. Anglin, *Phys. Rev. A* **64** (2001) 063611.
[42] M. Duncan, A. Foerster, J. Links, E. Mattei, N. Oelkers and A. Tonel, *Nuclear Physics B* **767** [FS] (2007) 227.
[43] M. Héritier, *Nature* **414** (2001) 31.
[44] M. T. Batchelor, *Physics Today* **60** (2007) 36.
[45] L. D. Faddeev, E. K. Sklyanin and L. A. Takhtajan, *Theor. Math. Phys.* **40** (1979) 194.
[46] P. P. Kulish and E. K. Sklyanin, *Lect. Notes Phys.* **151** (1982) 61.
[47] L. A. Takhtajan, *Lect. Notes Phys.* **370** (1990) 3.
[48] V. E. Korepin, N. M. Bogoliubov and A. G. Izergin, *Quantum inverse scattering method and correlation functions* (Cambridge University Press, Cambridge, 1993).
[49] L. D. Faddeev, *Int. J. Mod. Phys. A* **10** (1995) 1845.
[50] A. G. Izergin and V. E. Korepin, *Lett. Math. Phys.* **6** (1982) 283.
[51] A. G. Izergin and V. E. Korepin, *Nuc. Phys. B* **205** (1982) 401.
[52] F. H. L. Essler and V. E. Korepin (eds.) *Exactly solvable models of strongly correlated electrons* (World Scientific, Singapore, 1994).
[53] Fabian H. L. Essler, Holger Frahm, Frank Göhmann, Andreas Klümper and Vladimir E. Korepin, *The one-dimensional Hubbard Model* (Cambridge University Press, Cambridge, 2005).
[54] V. Bazhanov, S. Lukyanov and A. B. Zamolodchikov, *Commun. Math. Phys.* **177** (1996) 381.
[55] M. Jimbo, *Lett. Math. Phys.* **10** (1985) 63.
[56] M. Jimbo, *Lect. Notes Phys.* **246** (1986) 335.
[57] V. G. Drinfeld, Quantum groups *Proc. Int. Congress of Mathematicians* ed A M Gleason (Providence, RI: American Mathematical Society) (1986) 798.
[58] N. Yu Reshetikhin, L. A. Takhtajan and L. D. Faddeev, *Leningrad Math. J.* **1** (1990) 193.
[59] L. D. Faddeev, 40 Years in Mathematical Physics *World Scientific Series in 20th Century Mathematics*, vol. 2, (World Scientific Publishing Co. Pte. Ltd., Singapore, 1995).
[60] T. Goliński, M. Horowski, A. Odzijewicz, A. Sliżewska, *Jour. Math. Phys.* **48** (2007) 023508.
[61] I. Roditi, *Brazilian Journal of Physics* **30** (2000) 357.