Critical string wave equations and the QCD $(U(N_c))$ string. (Some comments)

Luiz C.L. Botelho
Departamento de Matemática Aplicada,
Instituto de Matemática, Universidade Federal Fluminense,
Rua Mario Santos Braga
24220-140, Niterói, Rio de Janeiro, Brazil
e-mail: botelho.luiz@superig.com.br

Abstract

We present a simple proof that self-avoiding fermionic strings solutions solve formally (in a Quantum Mechanical Framework) the $QCD(U(N_c))$ Loop Wave Equation written in terms of random loops.

Introduction

We aim, in this paper, to present a formal interacting string solution for the Migdal-Makeenko Loop Wave Equation for the colour group $(U(N_c))$ (Ref. [1] and references therein).

Our main tool to solve the Migdal-Makeenko Loop Wave Equation is based on the remark made in the Section 1 of this note, where we address the problem of solving critical string wave equations by string functional integral by applying simple rules of the operatorial calculus of Quantum Mechanics. We thus apply the results of Section 1 to present a string functional integral solution for the Migdal-Makeenko Loop Wave Equation for the colour group $U(N_c)$.

1 The critical area-diffusion string wave equations

Let us start this section by briefly reviewing our general procedure to write diffusion string wave equations for bosonic non-critical strings[2]. The first step is by considering the following fixed area string propagator in 2D induced quantum gravity string quantization framework.

$$\mathcal{G}[C^{\text{out}}, C^{\text{in}}, A] = \int D^c[g_{ab}] D^c[X_\mu] \times \delta\left(\int_D d\sigma d\tau \sqrt{g(\sigma,\tau)} - A\right) \times \exp(-I_0(g_{ab}, X_\mu, \mu^2 = 0)). \tag{1}$$

Here the string surface parameter domain is taken to be the rectangle $D = \{(\sigma, \tau). -\pi \leq \sigma \leq \pi, 0 \leq \tau \leq T\}$. The action $I_0(g_{ab}, X_\mu, \mu^2 = 0)$ is the Brink Di Vecchia-Howe covariant action with a zero cosmological term and the covariant functional measures $D^c[g_{ab}]D^c[X_\mu]$ are defined over all cylindrical string world sheets without holes and handles with the initial and final string configurations as unique non-trivial boundaries: i.e. $X_\mu(c, 0) = C^{\text{in}}$. $X_\mu(\sigma, T) = C^{\text{out}}$.

In order to write an area diffusion wave equation for Eq. (1), we exploit an identity which relates its area variation (the Mandelstam area derivative for strings) to functional variations on the conformal factor measure when one fixes the string diffeomorphism group in Eq. (1) by imposing the conformal gauge $g_{ab}(\sigma, \tau) = \rho(\sigma, \tau)\delta_{ab}$ (see Refs. [1], [2]). This procedure yields, thus, the following area diffusion string Euclidean wave equation

$$\frac{\partial}{\partial A}\mathcal{G}[C^{\text{out}}, C^{\text{in}}, A] = \int_{-\pi}^{\pi} d\sigma \biggl(-\frac{\delta^2}{2e_{\text{in}}^2(\sigma)\delta C_\mu^{\text{in}}(\sigma)\delta C_\mu^{\text{in}}(\sigma)} + \frac{1}{2}C_\mu^{\text{in}}(\sigma)^2 + \frac{26-D}{24\pi}\lim_{r\to 0^+}[R(\rho(\sigma,\tau)) + C_\infty]\biggr) \times \mathcal{G}[C^{\text{out}}, C^{\text{in}}, A] \tag{2}$$

At this point a subtle difficulty appears when the theory described by Eq. (1) is at its critical dimension $D = 26$ since the conformal field $\rho(\sigma, \tau)$ decouples from the theory, making it subtle to implement the fixed area constraint in Eq. (1). It is instructive to point out that for a cylinder surface without holes and handles with non trivial boundaries, the argument that the fixed area constraint is simply fixing the modulus λ of the (torus) conformal gauge $g_{ab}(\sigma, \tau) = \rho(\sigma, \tau)\left((d\sigma)^2 + \lambda^2(d\tau)^2\right)$ is insufficient to cover the case of "string creation" from the vacuum as we will need in Section 2. This is because in this case $\lambda = 0$ and the string world sheet still has a non-zero area. Note that the topology of this string world sheet creation process is now a hemisphere which again makes impossible the use of the modulus λ as an area parameter.

However, it makes sense to consider the limit of the parameter $D = 26$ directly in our string diffusion Eq. (2) which reproduces the usual critical string wave equations (Eq. (2) with $D = 26$ and $\rho(\sigma, \tau) = 1$).

In this short section we intend to show that the following critical string propagator:

$$\mathcal{G}[C^{\text{out}}, C^{\text{in}}, A] = \int D^F[X^\mu(\sigma,\tau)] \quad X^\mu(\sigma, 0) = C_\mu^{\text{in}}(\sigma), \quad X^\mu(\sigma, A) = C_\mu^{\text{out}}(\sigma) \quad \exp\left\{-\frac{1}{2}\int_0^A d\tau \int_{-\pi}^{\pi} d\sigma \left[\left(\frac{\partial X^\mu}{\partial\sigma}\right)^2 + \left(\frac{\partial X^\mu}{\partial\tau}\right)^2\right](\sigma,\tau)\right\} \tag{3}$$

where the intrinsic string time parameter A is identified with the area diffusion variable, satisfies the string critical diffusion wave equation.

To show this simple result we evaluate the A-derivative of Eq. (3) by means of Leibnitz's rule

$$\frac{\partial}{\partial A}\mathcal{G}[C^{\rm out},C^{\rm in},A]=\frac{-1}{2}\left\{\lim_{\tau\to A^-}\left\langle\int_{-\pi}^{\pi}d\sigma\left[\left(\frac{\partial X^\mu}{\partial\sigma}\right)^2+\left(\frac{\partial X^\mu}{\partial\tau}\right)^2\right](\sigma,\tau)\right\rangle\right\}, \tag{4}$$

where the surface average $\langle\ \rangle_s$ is defined by bosonic path-integral in Eq. (3).

In order to translate the path integral relation Eq. (4) into a operator statement, we use the usual Heisenberg Commutation Relations for two-dimensional ($2D$) free fields on D (with the Bidimensional PläPlänck constant = Regge slope parameter set to the value one)

$$[\Pi_\mu(\sigma,\tau),X_\nu(\sigma',\tau)]=i\delta(\sigma-\sigma')\delta_{\mu\nu} \tag{5}$$

and its associated Schrödinger representation for $\tau=A$ (that are the quantum mechanial definition of the lopp derivatives operators ([1]).

$$\Pi_\mu(\sigma,A)=\lim_{\tau\to A}\left\langle\frac{\partial}{\partial\tau}X^\mu(\sigma,\tau)\right\rangle_s=+i\,\frac{\delta}{C_\mu^{\rm out}(\sigma)} \tag{6}$$

$$\left|\frac{dC_\mu^{\rm out}(\sigma)}{d\sigma}\right|^2=\lim_{\tau\to A}\left(\frac{\partial X^\mu(\sigma,\tau)}{\partial\tau}\right)^2 \tag{7}$$

After substituting Eqs. (6)-(7) into Eq. (4) we obtain the desired result

$$\frac{\partial}{\partial A}\mathcal{G}[C^{\rm out},C^{\rm in},A]=-\left(\int_{-\pi}^{\pi}d\sigma\left[-\frac{\delta^2}{2\delta C_\mu^{\rm out}(\sigma(\delta C_\mu^{\rm out}(\sigma)}+\frac{1}{2}\left|C_\mu^{\rm out}(\sigma)\right|^2\right]\right)\times\mathcal{G}[C^{\rm out},C^{\rm in},A]. \tag{8}$$

Let us point out that general string wave functionals (the Schrödinger representation for the theory's quantum states) may be formally expanded in terms of the eigenfunctions of the quantum string Hamiltonian (the string wave operator in Eq. (8))

$$-\Delta_c\psi_E[c]=-\left\{\int_{-\pi}^{\pi}d\sigma\left(-\frac{\delta^2}{2\delta C_\mu(\sigma)\delta C_\mu(\sigma)}+\frac{1}{2}\left|C'_\mu(\sigma)\right|^2\right)\right\}\psi_E(c)=E\psi_E[c] \tag{9}$$

$$\psi[c]=\sum_{\{E\}}\rho(E)\psi_E[c] \tag{10}$$

The functionals endowed with the (formal) inner product given by

$$\langle\psi[c]\mid\Omega[c]\rangle=\int D^F[c]\cdot\psi^*[c]\cdot\Omega[c] \tag{11}$$

constitute a Hilbert space where the string Laplacian $-\Delta_c$ is formally a Hermitean operator.

It is worth remarking that an explicit expression for the Green's Function

$$(-\Delta_c)^{-1}(C^{\text{out}}, C^{\text{in}}) = \sum_{\{E\}} \psi_E[C^{\text{out}}]\psi_E^*[C^{\text{in}}]/E$$

of the string Laplacean in terms of the cylindrical string propagator Eq. (3) may be easily obtained.

In order to deduce this expression we integrate both sides of Eq. (8) with respect to the A-variable. Considering now the Asymptotic Behaviors

$$\lim_{A\to\infty} \mathcal{G}[C^{\text{out}}, C^{\text{in}}, A] = 0 \tag{12}$$

$$\lim_{A\to 0} \mathcal{G}[C^{\text{out}}, C^{\text{in}}, A] = \delta^F(C^{\text{out}} - C^{inn}) \tag{13}$$

we obtain the relationship

$$\delta^F(C^{\text{out}} - C^{\text{in}}) = -\Delta_c \left(\int_0^\infty \mathcal{G}[C^{\text{out}}, C^{\text{in}}, A] \right) \tag{14}$$

leading thus to the following identiy

$$(-\Delta_c)^{-1}[C^{\text{out}}, C^{\text{in}}] = \left(\int_0^\infty \mathcal{G}[C^{\text{out}}, C^{\text{in}}, A] \right) \tag{15}$$

2 A bilinear fermion coupling on a self-interacting Bosonic Random surface as solution of $QCD(U(N_c))$ Migdal-Makeeko Loop Equation

Let us start this section by considering the (non-renormalized) Migdal-Makeenko Loop Equation satisfied by the Quantum Wilson Loop in the form of Ref. [3] for the colour group $U(N_c)$

$$\begin{aligned} -\Delta_c\langle W_{l\ell}[C_{X(-\pi)X(\pi)}\rangle &= (g^2N_c)\int_{-\pi}^{\pi} d\sigma \int_{-\pi}^{\pi} d\sigma' \,\frac{dX^\mu(\sigma)}{d\sigma}\cdot\frac{dX^\mu(\sigma')}{d\sigma'} \\ &\times \delta^{(D)}(X_\mu(\sigma) - X_\mu(\sigma')\langle W_{kp}[C_{X(-\pi)X(\sigma)}]\, W_{p\ell}[C_{X(\sigma)X(\pi)}]\rangle \end{aligned} \tag{16}$$

The Quantum Wilson Loop is given by

$$\langle W_{k\ell}[C_{X(-\pi)X(\pi)}, A_\mu(x)]\rangle = \frac{1}{N_c}\left\langle T_R^{(c)}\left(\exp - \int_{-\pi}^{\pi} d\sigma (A_\mu(X_\mu(\sigma))\cdot X'^\mu(\sigma)\right)\right\rangle_{k\ell}. \tag{17}$$

As usual, $A_\mu(x)$ denotes the usual $U(N)$ colour Yang-Milss field which possesses an additional, not yet specified intrinsic global "Flavor" group $O(M)$ represented by matrix indices (k,ℓ). The average $\langle\ \rangle$ is given by the $U(N)$-colour Yang-Milss field theory.

Let us consider the following critical non-linear interacting Fermionic String theory first considered in Ref. [4]

$$S[X_\mu(\sigma,\tau),\psi_{(k)}(\sigma,\tau)] = \frac{1}{2}\int_0^A d\tau \int_{-\pi}^{\pi} d\sigma \left[\left(\frac{\partial X^\mu}{\partial\tau}\right)^2 + \left(\frac{\partial X^\mu}{\partial\sigma}\right)^2\right](\sigma,\tau)$$
$$+ \int_0^A d\tau \int_{-\pi}^{\pi} d\sigma[\bar{\psi}_{(k)}(\gamma^a\partial_a)\psi_{(k)}](\sigma,\tau) + \frac{\beta}{2}\int_0^A d\tau \int_0^A d\tau' \int_{-\pi}^{\pi} d\tau' \int_{-\pi}^{\pi} d\sigma'$$
$$\times (\psi_{(k)}\bar{\psi}_{(k)})(\sigma,\tau) \times T^{\mu\nu}(X_\alpha(\sigma,\tau))\delta^{(D)}(X_\alpha(\sigma,\tau) - X_\alpha(\sigma',\tau'))T_{\mu\nu}(X_\alpha(\sigma',\tau')). \tag{18-a}$$

The notation is as follows: the string vector position is described by the $2D$-fields $X_\mu(\sigma,\zeta)$ with the Dirichlet boundary condition $X_\mu(\sigma,A) = C_{X(-\pi),X(\pi)}$; i.e., the surface $S = \{X_\mu(\sigma,\zeta)$, $-\pi \le \sigma \le \pi$, $0 \le \zeta \le A)$ has a unique boundary the fixed Lopp $C_{X(-\pi)X(\pi)}$ of Eq. (16). The surface orientation tensor where it is defined is given by

$$T_{\mu\nu}(X_\mu(\sigma,\tau)) = \frac{\varepsilon^{ab}}{\sqrt{2}}\,\frac{\partial_a X^\mu \partial_b X^\nu}{\sqrt{h}} \tag{18-b}$$

with $h = \det\{h_{ab}\}$ and $h_{ab}(\sigma,\tau) = \partial_a X^\mu \partial_b X^\mu$. Note that S possesses self-intersecting lines such that $X_\mu(\sigma,\tau) = X_\mu(\sigma',\tau')$ with $0 < \{\tau,\tau'\} < A$ has non-trivial self-intersecting lines solutions. For $\tau = A = \tau'$, $X_\mu(\sigma,A) = C_\mu^{\text{out}}(\sigma)$ posseses solely simple isolated self-intersections points (eights loops), however with $T^{\mu\nu}(X_\alpha(\sigma,A))T_{\mu\nu}(X_\beta(\sigma',A)) = 0$ for $\sigma \neq \sigma'$ (the Fermion Exclusion Pauli Principle). Additionally we have introduced a set of single-valued intrinsic Majorana $2D$-spinors on the surface domain parameter $D = \{(\sigma,\tau),\ 0 \le \tau \le A;\ -\pi \le 0 \le \tau\}$. They are chosen to belong to a real representation of the flavor group $O(22)$ since for this group we have cancelled exactly the theory's conformal anomaly ($26 = 4 + 22$), which in turn leads to the vanishing of the kinetic term associated to the conformal factor $\rho(\sigma,\zeta)$ see Ref. [1]). We further impose as a boundary condition on these Fermions the vanishing of the Fermion energy-tensor projected on the Loop $C_{X(-\pi),X(\pi)}$. Let us point out that the Weil symmetry makes sense to speak in conformal anomaly in our theory Eq. (16) which preservation at quantum level by its turn will determine the string flavor group to be the "String" Weinberg-Salam group $O(22)$ (see Ref. [1]).

Associated to the non-linear string's theory Eq. (18) we consider the following Fermionic propagator for a fixed string world sheet $\{X_\mu(\sigma,\tau)\}$

$$\overline{Z}_{k\ell}[C_{X(-\pi)X(\pi)};X_\mu(\sigma,\zeta),A] = \int D^F[\psi_k(\sigma,\tau)](\psi_{(k)}(-\pi,A)\bar{\psi}_{(\ell)}(\pi,A)$$
$$\times \exp\left\{-\int_0^A d\tau \int_{-\pi}^{\pi} d\sigma(\bar{\psi}_{(k)}(\gamma^a\partial_a)\psi_{(k)})(\sigma,\tau)\right\}$$
$$\times \exp\left\{-\frac{\beta}{2}\int_0^A d\tau \int_{-\pi}^{\pi} d\sigma \int_0^A d\tau' \int_{-\pi}^{\pi} d\sigma'(\bar{\psi}_{(k)}\psi_{(k)})(\sigma,\tau)\right.$$
$$\left.\times T^{\mu\nu}(X_\alpha(\sigma,\tau))\delta^{(D)}(X_\alpha(\sigma,\tau)-X_\alpha(\sigma',\tau'))T_{\mu\nu}(X_\alpha(\sigma,\tau'))\right\}. \quad (19)$$

The basic idea of our string solution for $QCD(U(N_c))$ is a technical improvement of Ref. [1] and consists in showing that the surface averaged propagator Eq. (19)

$$\langle \overline{Z}_{k\ell}[C_{X(-\pi)X(\pi)},X_\mu(\sigma,\tau),A]\rangle_s = \mathcal{G}_{k\ell}(C_{X(-\pi)X(\pi)},A),$$

when integrated with respect to the A-parameter as in Eq. (15), now satisfies the full $U(N_c)$ non-linear Migdal-Makeenko Loop Equation (16) instead of the factorized Loop equations associated to the T'Hooft limit $N_c \to \infty$.

The surface average $\langle\ \rangle$, is defined by the free bosonic action piece of Eq. (18) as in Section 1. In this context we consider $\mathcal{G}_{k\ell}(C_{X(-\pi)X(\pi)},A)$ as the non-linear string propagator describing the "creation" of the Loop $C_{X(-\pi)X(\pi)} = C^{\rm out}$ from the string vacuum, which is represented here by a "collapsed" point - like string initial configuration $C^{\rm in} \equiv (x)$ (x denotes an arbitrary point of the surface which may be considered as such initial string configuration).

Let us thus, evaluate the A-derivative of $\mathcal{G}(C_{X(-\pi)X(\pi)},A)$

$$\frac{\partial}{\partial A}\langle Z_{k\ell}(C_{X(-\pi)X(\pi)},A\rangle_s$$
$$= \int D^F[X^\mu(\sigma,\tau)]\int D^F[\psi(k)(\sigma,\tau)]\exp[(-S[X_\mu(\sigma,\tau),\psi_{(k)}(\sigma,\tau))]\times\psi_{(k)}(-\pi,A)\bar{\psi}_{(\ell)}(+\pi,A)$$
$$(-1)\times\frac{\partial}{\partial A}\left\{\int_0^A d\tau\int_{-\pi}^{\pi}d\sigma\left[\frac{1}{2}(\partial X^\mu)^2+\bar{\psi}_{(k)}(\gamma^a\partial_s)\psi_{(k)}\right](\sigma,\tau)\right.$$
$$+\frac{\beta}{2}\int_0^a d\tau\int_{-\pi}^{\pi}d\sigma\int_0^A d\tau'\int_{-\pi}^{\pi}d\sigma'(\psi_{(k)}\bar{\psi}_{(k)}(\sigma,\tau)T^{\mu\nu}(X_\alpha(\sigma,\tau))$$
$$\left.\times\,\delta^{(D)}(X_\alpha(\sigma,\tau)-X_\alpha(\sigma',\tau')T_{\mu\nu}(X_\alpha(\sigma',\tau')\right\} \quad (20)$$

The free Bosonic term in the right-hand side of Eq. (20) leads to the string Laplacean as in Eq. (4) of Section 1. The free Fermion term

$$\lim_{\tau\to A^-}\bar{\psi}_{(k)}(\sigma,\tau)(\gamma^a\partial_a)\psi_{(k)}(\sigma,\tau)$$

vanishes as a consequence of our imposed vanished energy - momentum tensor boundary conditions on the intrinsic Fermion field. The evaluation of the boundary limit on β-term requires explicitly that the surface $\{X_\mu(\sigma,\tau)\}$ **does not possesses self-intersections of the type** $X_\mu(\sigma,A) = C_\mu^{\text{out}}(\sigma) = X_\mu(\sigma',\tau')$**. The result of this boundary limit evaluation is given explicitly by the expression below (Ref. [1] - Appendix B)**

$$\int D^F[X^\mu(\sigma,A)] \int D^F[\psi_k(\sigma,\tau)] \left\{ \int_{-\pi}^{\pi} d\sigma \int_{-\pi}^{\pi} d\sigma' \delta^{(D)}(X_\mu(\sigma) - X_\mu(\sigma') \right\}$$
$$\times \frac{dX^\mu(\sigma)}{da} \cdot \frac{dX^\mu(\sigma')}{da'} \left[\sum_{p=1}^{22} \psi_k(-\pi,A).\bar{\psi}_\ell(+\pi,A) \times (\psi_p(\sigma,0)\bar{\psi}_p(\sigma,0)) \right] \exp(-S[X_\mu(\sigma,\tau),\psi_k(\sigma,\tau)])$$
$$= \beta \int_{-\pi}^{\pi} do \int_{-\pi}^{\pi} d\sigma' \delta^{(D)}(X_\mu(\sigma) - X_\mu(o') \, \frac{dX^\mu(\sigma)}{da} \cdot \frac{dX^\mu(\sigma')}{da'}$$
$$\times \langle \overline{Z}_{kp}[C_{X(-\pi)X(\sigma)}, X_\mu(\sigma,\tau)] \times \overline{Z}_{p\ell}[C_{X(\sigma)X(\pi)}, X_\mu(\sigma,\tau), A] \rangle_s \tag{21}$$

Note that either crucial result below:

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \left[\int_0^A d\tau \int_0^A d\tau' \int_{-\pi}^{\sigma-\varepsilon} d\xi \int_{\sigma+\varepsilon}^{\pi} d\xi' [(\bar{\psi}_k\psi_k)(\xi,\tau)] T_{\mu\nu}(X(\xi,\tau)) \times \delta^{(D)}(X(\xi,\tau) - X(\xi',\tau')) T^{\mu\nu}(X(\xi',\tau')) \right] \equiv 0, \tag{22-B}$$

since our orientation tensor strings world-sheet $\{X(\xi,\tau)\}$ is such that for $\xi \neq \xi'$ and $\zeta,\zeta' \in [0,A]$

$$T^{\mu\nu}(X(\xi,\tau)) \cdot T_{\mu\nu}(X(\xi',\tau')) \equiv 0 \tag{22-C}$$

(these "string fixed-time" loop $X_\mu(\xi,\hat{\zeta}) \equiv e_\mu^{(\hat{\zeta})}(\xi)$ possesses solely simple isolated self-intersections points ("eights" loops) where the non-trivial tangent lines at theses self-intersect points are supposed to be always orthogonal to each other: $T_{\mu\nu}(X(\xi,\zeta))T^{\mu\nu}(X(\xi',\zeta)) \equiv 0$ as a remnant of the Fermion Exclusion Pauli Principle still acting for these bosonic pieces $C_\mu(\xi)$ of the full fermionic "quark trajectores").

$$\sum_{p=1}^{22}\int\left(\prod_{\substack{-\pi\le\xi\le\pi\\0\le\tau\le A}} d\psi(\xi,\tau)\right)\exp\left\{\int_0^A d\tau\int_{-\pi}^{\pi} d\xi(\bar{\psi}_k(\gamma^a\partial_a)\psi_k)(\xi,\tau)\right\}$$
$$\times\exp\left\{-\frac{\beta}{2}\int_0^A d\tau\int_0^A d\tau'\int_{-\pi}^{\pi} d\xi\int_{-\pi}^{\pi} d\xi'(\bar{\psi}_k\psi_k)(\xi,\tau)T_{\mu\nu}(\xi,\tau))\right.$$
$$\left.\times\,\delta^{(D)}(X_\alpha(\xi,\tau)-X_\alpha(\xi',\tau'))T^{\mu\nu}(X(\xi',\tau'))\right\}$$
$$\times\,(\psi_k(-\pi,A)\bar{\psi}_\ell(\pi,A)\psi_p(\sigma,A)\bar{\psi}_p(\sigma,A)$$
$$=\sum_{p=1}^{22}\left[\int\left(\prod_{\substack{-\pi\le\xi<\sigma\\0\le\tau\le A}} d\psi(\xi,\tau)\right)\exp\left\{\int_0^A d\tau\int_{-\pi}^{\sigma}\right.\right.$$
$$\left.\times\, d\xi(\bar{\psi}_k(\gamma^a\partial_a)\psi_k)(\xi,\tau)\right\}(\psi_k(-\pi,A)\bar{\psi}_p(\sigma,A))$$
$$\times\exp\left\{-\frac{\beta}{2}\int_0^A d\tau\int_0^A d\tau'\int_{-\pi}^{\sigma} d\xi\int_{-\pi}^{\sigma} d\xi'(\bar{\psi}_k\psi_k)(\xi,\tau)T_{\mu\nu}(X(\xi,\tau))\right.$$
$$\left.\left.\times\,\delta^{(D)}(X(\xi,\tau)-X(\xi',\tau'))T^{\mu\nu}(X(\xi',\tau'))\right\}\right]$$
$$\times\left[\int\left(\prod_{\substack{\sigma<\xi<\pi\\0\le\tau\le A}} d\psi(\xi,\tau)\right)\exp\left\{\int_0^A d\tau\int_{\sigma}^{\pi} d\xi(\bar{\psi}_k(\gamma^a\partial_a)\psi_k)(\xi,\tau)\right\}\right.$$
$$\times\exp\left\{-\beta\int_0^A d\tau\int_0^A d\tau'\int_{\sigma}^{\pi} d\xi\int_{\sigma}^{\pi} d\xi'(\bar{\psi}_k\psi_k)(\xi,\tau)T_{\mu\nu}(X(\xi,\tau))\right.$$
$$\left.\left.\times\,\delta^{(D)}(X(\xi,\tau)-X(\xi',\tau'))T^{\mu\nu}(X(\xi',\tau'))\right\}\times(\psi_p(\sigma,0)\bar{\psi}_\ell(\pi,0))\right] \tag{22}$$

and its unity normalization condition

$$\overline{Z}_{pp}[C_{X(\sigma)X(\sigma)},X_\mu(\sigma,\xi),A]=1. \tag{23}$$

By imposing the identification $\boxed{g^2N_c=\beta}$ between the $QCD(U(N_c))$ gauge coupling constant and our non linear stgring theory described by Eq. (3) we obtain the identification between the $QCD(U(N_c))$ Wilson Loop Eq. (17) and the surface averaged Fermion Propagator Eq. (19)

$$\left\langle W_{k\ell}[C_{X(-\pi)X(\pi)},A_\mu(x)]\right\rangle_{\text{Yang Mills }U(N_c)}=\int_0^\infty dA\left\langle\overline{Z}_{k\ell}[C_{x(-\pi')X(\pi)},X_\mu(\sigma,\xi),A]\right\rangle_s. \tag{24}$$

The above equation is the main result of this note and generalizes to the case of $U(N_c)$ colour group our previous studies made for the T'Hooft limit of Ref. [1].

Finally we remark that by considering an ultra-violet cut-off on the space-time, $\Delta X^\mu(\sigma,\tau) \geq 1/\Lambda$, our proposed self-avoiding string theory Eq. (18) in the case of non dynamical $2D$-Fermions ($\langle(\psi_k\bar{\psi}_k\rangle = \mu =$ constant) produces the extrinsic string with the topological invariant of string world-sheet self-intersection number as an effective string theory for the proposed QCD string as conjectured in the first Ref. [5] (see Ref. [6] for this study and the enclosed appendix).

Finally it is worth re-write eq. (19) in a form where appears an interaction with an external white-noise Gaussian auxiliary anti-symetric tensor field as suggested in ref. [1]. Namely:

$$\begin{aligned}
&\exp\left\{-\frac{\beta}{2}\int_0^A d\tau\int_0^A d\tau'\int_{-\pi}^{\pi}d\xi\int_{-\pi}^{\pi}d\tau(\bar{\psi}_{(k)}(\xi,\tau)\psi_{(k)}(\xi,\tau))\times T^{\mu\nu}(X_\alpha(\xi,\tau))\right\}\\
&\times\delta^{(D)}(X(\xi,\tau)-X(\xi',\tau'))T_{\mu\nu}(X_\beta(\xi',\tau'))=\\
&=\int D^F B_{\mu\nu}(x)\exp\left\{-\frac{1}{2}\int B^2_{\mu\nu}(x)d^Dx\right\}\times\exp\left\{i\int B_{\mu\nu}(x)J^{\mu\nu}(x,S)\right\}
\end{aligned}\tag{25}$$

with the dynamical string world-sheet current

$$J^{\mu\nu}(x,S)=\beta\left[\int_{-\pi}^{\pi}d\xi\int_0^A d\tau\, T^{\mu\nu}(X(\xi,\tau))\left(\sum_{p=1}^{22}\bar{\psi}_{(p)}\psi_{(p)}\right)^{\frac{1}{2}}\times\delta^{(D)}(x-X(\xi,\tau))\right]\tag{26}$$

References

1. Luiz C.L. Botelho, Methods of Bosonic and Fermionic Path Integral, representations - continuum random geometry in quantum field theory, Nova Science Publisher, 208, N.Y., USA.

2. Luiz C.L. Botelho, Phys. Rev. **D40**, 660 (1989), Phys. Rev. **D41**, 3283 (1990).

3. Luiz C.L. Botelho, Phys. Lett. **B152**, 358 (1985).

4. Luiz C.L. Botelho, Rev. Bras. Fis. 16, 279 (1986).

5. A. Polyakov, Nucl. Phys. **B268**, 406 (1986); L. Peliti and S. Leibler, Phys. Rev. Lett. 54, 1690 (1980); B. Duplantier, Commun. Math. Phys. 85, 221 (1982).

6. A.I. Karanikas and C.H. Ktorides, Phys. Letters **B235**, 90 (1990).

APPENDIX A

A reduced covariant string model for the extrinsic string

Our aim in this appendix is very modest: we write a covariant action for the elastic string and quantize in the Polyakov's path integral framework a truncated version of the covariant writen theory.

Let us start our study by considering the classical action for the elastic string in the conformal gauge.

$$S_0 = \frac{1}{2\pi\alpha'}\int d^2z\rho + \gamma\int d^2z\rho\left[\left(-\frac{1}{\rho}\,\partial^2 X\right)^2 + i\,\frac{\lambda_{ab}}{\rho}\,(\partial_a X\partial_b X - g_{ab})\right] \tag{1}$$

The string surface is described by $X = X(z)$, where X is the surface vector position in D Euclidean dimensionas; z_a $(a = 1, 2)$ are the coordinates of the world sheet. The first term in eq. (1) is the Nambu term with the string tension equal to $1/2\pi\alpha$. The second term is the square of the extrinsic curvature with the rigidity coupling constant denoted by γ ($\gamma = \lim_{N_c\to\infty}(g^2N_c)!$) and $\lambda^{ab}(z)$ is a Lagrange multiplier which insures that the metric (g_{ab}) coincides with the intrinsic metrics $(\partial_a X\partial_b X)$.

Let us consider a covariant version of action eq. (1) by promoting $\rho(z) = g_{ab}(z)$ to be a dynamical field. This procedure yields the following action

$$\begin{aligned} S_1[X)z), g_{ab}(z), \lambda_{ab}(z)] &= \frac{1}{2\pi\alpha}\int d^2z\sqrt{g} \\ &+ \int d^2z\sqrt{g}[\gamma(-\Delta_g X)^2 + i\lambda_{ab}(g_{ab} - \partial_a X\partial_b X)] \end{aligned} \tag{2}$$

Here $\sqrt{g(z)} = \text{Det}(g_{ab}(z))$ and $-\Delta g = -\frac{1}{\sqrt{g}}\,\partial_a(g^{ab}\partial_b)$ is the Laplace-Beltrami operator associated to the intrinsic metric $g_{ab}(z)$.

In the Polyakov's path integral quantization effective framework the partition functional for the theory eq. (1) should be given by

$$Z = \int D^c[g_{ab}]D^c[X]D^c[\lambda_{ab}] \times \exp -S_1[X(z), g_{ab}(z), \lambda_{ab}(z)] \tag{3}$$

where the functional measures are the De-witt covariant functional measures ([1]).

Let us suppose that the constraint field in approximated by the intrinsic metric $\lambda_{ab}(z) = \langle\lambda\rangle g_{ab}(z)$. (The covariant version of the usual mean field approximation $\lambda_{ab}(z) = i\langle\lambda\rangle\delta_{ab}$ with

$\langle\lambda\rangle$ a positive fixed value). As a consequence of this hypothesis we get the truncated theory

$$Z_{(T)} = \int D^c[g_{ab}]D^c[X] \exp - S^{(T)}[g_{ab}(z), X(z)] \tag{4}$$

where the truncated action theory is written as

$$S^{(T)}[g_{ab}, X] = \frac{1}{2\pi\alpha'}\int d^2z\sqrt{g(z)} + \gamma\int d^2z[(-\Delta_g X)^2] \\ + \langle\lambda\rangle\int d^2z\sqrt{g}g^{ab}\partial_a X\partial_b X + \langle\lambda\rangle\int d^2z\sqrt{g(z)}. \tag{5}$$

For the evaluation of the X-functional integral in eq. (4) we consider the non-local variable change

$$X_\mu(z) = (-i(\Delta_g)^{1/2}\,\vartheta_\mu)(z) \;\; \mu = 1,\ldots,D.$$

Here $-i(\Delta_g)^{-1/2}$ is a well defined self-adjoint (pseudo-differential) operator. The truncated action takes the following form similar to a massive scalar field in the z domain:

$$S^{(T)}[g_{ab},\vartheta] = \left(\frac{1}{2\pi\alpha'} - \langle\lambda\rangle\right)\int d^2z\sqrt{g} + 2\langle\lambda\rangle\int d^2z\,\frac{1}{2}\,\vartheta^2 + 2\gamma\int d^2z\,\frac{1}{2}\,(\sqrt{g}\vartheta(-\Delta_g)\vartheta)(z). \tag{6}$$

The change in the (covariant) functional measure $D^c[x]$ is given by

$$D^c[x] = (\mathrm{Det}(-\Delta_g)^{-1})^{D/2} \times D^c]\vartheta]. \tag{7}$$

The main step in our calculation is to define the above written functional determinant as $\mathrm{Det}^{-D/2}(-\Delta_g)$. By choosing the conformal gauge $g_{ab} = e^\varphi\,\delta_{ab}$ and evaluating the covariant Gaussian ϑ-functional integral we obtain the partial result ([1])

$$Z_{(T)} = \int D[\vartheta]\exp\left(-\frac{26-D}{48\pi}\int\left[\frac{1}{2}\,(\partial_a\varphi)^2 + \mu_R^2\,e^\varphi\,d^2z\right]\right)\mathrm{Det}^{-D/2}(-2\gamma\Delta_g + 2\langle\lambda\rangle) \tag{8}$$

where

$$\mu_R^2 = \lim_{\varepsilon\to 0}\frac{2-D}{4\pi\varepsilon} + \frac{1}{2\pi\alpha} = \langle\lambda\rangle$$

may be though as a renormalization of the bare string thension $1/2\pi\alpha'$.

We analyze now the unrenormalized functional determinant

$$\exp - S_{EFF}[\varphi] = \mathrm{Det}^{-D/2}\left(-\Delta_g + \frac{\langle\lambda\rangle}{\gamma}\right).$$

By defining it by a propertime prescription we obtain the counterterms of the above written action. Explicitly

$$S_{EFF}[\varphi] = \lim_{\varepsilon\to 0} -\frac{D}{2}\int_\varepsilon^\infty\frac{dT}{T}\,\mathrm{Tr}\left(\exp - T\left(-\Delta_g + \frac{\langle\lambda\rangle}{\gamma}\right)\right). \tag{9}$$

Now it is well known that the counterterms of $S_{EFF}[\varphi]$ are determined by the asymptotic expansion of the diagonal part of massive Laplace-Beltrami operator which is tabulated

$$\lim_{T\to 0} \mathrm{Tr}\left(\exp\left(-T\left(-\Delta_g + \frac{\langle\lambda\rangle}{\gamma}\right)\right)\right)$$
$$= \int d^2z \left\{ \frac{e^{\varphi}}{2\pi} \lim_{T\to 0^+}\left(\frac{1}{T}\right) - \frac{1}{2\pi}\,\Delta\varphi + \frac{1}{2\pi}\, e^{\varphi}\cdot \frac{\langle\lambda\rangle}{\gamma} \right\}(z). \tag{10}$$

By substituting eq. (10) in to eq. (9) we get straightforwardly the following counterterms associated to the two-dimensional intrinsic "mass" $\langle\lambda\rangle/\gamma$

$$\frac{D}{2}\cdot\frac{1}{2\pi}\cdot\frac{\langle\lambda\rangle}{\gamma}\,\mathrm{lg}\left(\frac{1}{\varepsilon}\right)\int d^2z\, e^{\varphi(z)}. \tag{11}$$

So, on the basis of the counter term eq. (11) we have the following renormalization law for the inverse of the rigidity $\beta = 1/\gamma$ (by choosing $\langle\lambda\rangle = 1$)

$$\frac{1}{\beta_R} = \frac{1}{\beta_0} - \frac{D}{2}\cdot\frac{1}{2\pi}\,\mathrm{lg}(\varepsilon). \tag{12}$$

Eq. (12) yields the intrinsic two-dimensional momentum dependence of the running coupling constant β.

$$\beta_R(p^2) = \beta_0(p^2=0)/1 - \frac{D}{2}\,\frac{\beta_0}{2\pi}\cdot\mathrm{lg}\left(\frac{\varepsilon}{p^2}\right). \tag{13}$$

It is instructive point out the $D/2$ factor in eq. (13) which appears in a natural way in our calculations.

Since it is naively expected that the string perturbative phase p^2-small ($p \in R^2$) would corresponds to the underlying QCD field theory at its non-perturbative phase $k^2 \to +\infty$ ($k \in R^4$), one can see that eq. (13) suggests a natural explanation from the QCD's String Representation for the "strange" QCD field theory description of the asymptotic behavior for the coupling constant at large N_c, namely

$$\lim_{k^2\to\infty}\left(\lim_{N_c\to\infty}(g^2N_c)_{\mathrm{ren}}(k^2)\right) = 0. \tag{14}$$

As a general conclusion, one can see that still exists a great deal of not completely understood phenomena in QCD out of non-analytical field theoretic continuum aproaches-lattice approximations.

APPENDIX B

The Loop Space Program in the bosonic $\lambda\phi^4 - O(N)$-field theory and the QCD triviality for R^D. $D > 4$

Let us start our study by considering the (bare) generating functional of the Green's functions of the $O(N)$ (symmetric phase) $\lambda\phi^4$ field theory in a D-dimensional Euclidean space-time

$$Z[J^a(x)] = \int \prod_{a=1}^{N} d\mu[\Phi^a(x)] \cdot \exp\left\{ -\frac{\lambda_0}{4} \int d^D x \left(\sum_{a=1}^{N} \Phi^a(x)^2 \right)^2 (x) \right. \\ \left. - \int d^D x \left(\sum_{a=1}^{N} J^a(x)\Phi^a(x) \right) \right\} \tag{1}$$

where $\Phi^a(x)$ denotes a N-component real scalar $O(N)$ field, (μ_0, λ_0) the (bare) mass and coupling parameters and the Gaussian functional measure in eq.(1) is

$$\prod_{a=1}^{N} d\mu[\Phi^a(x)] = \prod_{a=1}^{N} \left[\left(\prod_{x\in R^D} d\Phi^a(x) \right) \exp\left\{ -\frac{1}{2} \int d^D x \left(\sum_{a=1}^{N} (\partial_\mu \Phi^a) \right)^2 (x) \right. \right. \\ \left. + \mu_0^2 \sum_{a=1}^{N} \left(\Phi^a\right)^2(x) \right\}. \tag{2}$$

Now, in order to get an effective expression for the functional integrand eq.(2), where we can evaluate the Φ^a functional integrations, we write the intersection $\lambda\phi^4$ term in the following form

$$\exp\left\{ -\frac{\lambda_0}{4} \int d^D x \left(\sum_{a=1}^{N} \left(\Phi^a(x)\right)^2 \right)^2 \right\} = \int d\mu[\sigma] \cdot \exp\left\{ -i \int d^D x\, \sigma(x) \left(\sum_{a=1}^{N} \left(\phi^a(x)\right)^2 \right) \right\} \tag{3}$$

where $\sigma(x)$ is an auxiliary scalar field and the σ functional measure in eq.(3) is given by

$$d\mu[\sigma] = \left(\prod_{x\subset R^D} d\sigma(x) \right) \exp\left\{ -\frac{1}{2} \int d^D x \, \frac{2}{\lambda_0} \, \sigma^2(x) \right\} \tag{4}$$

with covariance

$$\langle \sigma(x_1)\sigma(x_2) \rangle_\sigma = \int d\mu[\sigma]\sigma(x_1)\sigma(x_2) = \frac{\lambda_0}{2} \, \delta^{(D)}(x_1 - x_2). \tag{5}$$

The last result alows us to consider the $\delta(x)$ field as a random gaussian potential with noise's strenght $\frac{\lambda_0}{2}$ ·

After substitution of eq.(4) into eq.(2), we can evaluate explicitly the Φ-functional integrations since they are of gaussian type. We, thus, get the result

$$Z[J^a(x)] = \int d\mu[\sigma]\,\mathrm{Det}^{-N/2}(-\Delta + \mu_0^2 - 2i\sigma)$$
$$\exp\left\{\frac{1}{2}\int d^D x\, d^D y\, J^a(x)(-\Delta + \mu_0^2 - 2i\sigma)\delta_{ab}\, J^B(y)\right\}. \tag{6}$$

At this point of our study we implement the main idea: by following Symanzik's analysis, we express the σ-functionals integrands in eq.(6) as functional defined on the Feynman-Kac-Wiener space of Random paths by making use of the well known random path representation for the non-relativistic euclidean propagator of a particle of mass μ_0 in the presence of the external random guassian potential $\sigma(x)$:

$$(-\Delta + \mu_0^2 - 2i\sigma)^{-1}(x,y) = \int_0^\infty d\zeta\, G(x,y,\sigma)(\zeta) \tag{7}$$
$$\log\,\mathrm{Det}(-\Delta + \mu_0^2 - 2i\sigma) = -\int_0^\infty \frac{d\zeta}{\zeta}\int d^D x\, G(x,x,\sigma)(\zeta) \tag{8}$$

where the non-relativistic propagator is given by

$$G(x,y,\sigma)(\zeta) = \int d\mu\{w_{xy}^{(\zeta)}\}\, e^{i\int d^D\,\sigma(z) j\left(w_{wy}^{(\zeta)}(z)\right)} \tag{9}$$

with the Feynman-Kac-Wiener path measure

$$d\mu\big[w_{xy}^{(\zeta)}\big] = \underset{\substack{0<\alpha<\zeta \\ w(0)=x \\ w(0)=y}}{(\Pi dw[\alpha])}\exp\left\{-\frac{1}{2}\int_0^\zeta \left(\frac{dw}{d\alpha}\right)^2 - \frac{1}{2}\mu_0^2\zeta\right\} \tag{10}$$

ane the (random) world-line currents defined by

$$j\big(w_{xy}^{(\zeta)}\big)(z) = \int_0^\zeta \delta^D\big(z - w_{xy}^{(\zeta)}(\alpha)\big)d\alpha. \tag{11}$$

So, we obtain the proposed reformulation of $\lambda\phi^4 O(N)$-theory as a theory of random paths $\big\{w_{xy}^{(\zeta)}(\alpha)\big\}$ in the presence of a random gaussian potential

$$Z[J^a(x)] = \int d\mu[\sigma]\cdot\exp\left\{\frac{N}{2}\int_0^\infty \frac{d\zeta}{\zeta}\int d^D x\int d\mu\big[w_{xx}^{(\zeta)}\big]\right.$$
$$\left.\exp\left(i\int d^D z\,\sigma(z) j\big(w_{xx}^{(\zeta)}\big)(z)\right)\right\}\exp\left\{\frac{1}{2}\int d^D x^D y\sum_{s=1}^{N} J_a(x)\left[\int_0^\infty d\zeta\int d\mu\big[w_{xy}^{(\zeta)}\big]\right.\right.$$
$$\left.\left.\exp\left(i\int d^D z\,\sigma(z) j\big(w_{wy}^{(\zeta)}\big)\right)\right]\delta_{ab}\cdot J_b(y)\right\} \tag{12}$$

We shall use the random path formulation eq.(12) to analyse the correlation functions of the $\lambda\phi^4$ theory. As a useful remark, we note by using eq.(12) that the general k-point (bare) correlation function possesses the general structure

$$\langle \Phi_{i_1}(x_1)\ldots\Phi_{i_k}(x_k)\rangle_\Phi = \begin{cases} 0 & \text{if } k=2j+1 \\ \sum\limits_{(2j+1)} \langle \Phi_{i_1}(x_{\ell_1})\Phi_{i_2}(x_{\ell_2})\rangle_\Phi \ldots \langle \Phi_{i_{k-1}}(x_{\ell_{2j-1}})\Phi_{i_k}(x_{\ell_{2j}})\rangle_\Phi & \\ \ell-\text{pairings} & \text{if } k=2j \end{cases} \tag{13}$$

where the quantum averages $\langle\ \rangle_\Phi$ in eq.(13) are defined by the $\lambda\phi^4$ partition functional $Z[0]$ (see eq.(1) with $J^a(x)\equiv 0$).

Because of this result, we have solely to study the properties of the 2-point correlation function

$$\langle \Phi_{i_1}(x_1)\Phi_{i_2}(x_2)\rangle_\Phi = \delta_{i_1 i_2}\langle \int_0^\infty d\zeta\, d\mu\big[w^{(\zeta)}_{x_1x_1}\big]\cdot \exp\left\{ i\int d^Dz\,\sigma(z)\cdot j\big(w^{(\zeta)}_{x_1x_2}\big)(z)\right\}$$
$$\exp\left\{\frac{N}{2}\int_0^\infty \frac{d\zeta}{\zeta}\int d^Dx\int d\mu\big[x^{(\zeta)}_{xx}\big]\cdot\exp\big[i\int d^Dz\,\sigma(z) j\big(x^{(\zeta)}_{xx}(z)\big)\big]\right\}\rangle_\sigma \tag{14}$$

Let us evaluate the σ-functional averages $\langle\ \rangle_\sigma$ in eq.(14) (see eq.(4) and eq.(5)). For this task we expand the "close path term" in powers of N. Explicitly

$$\langle \Phi_{i_1}(x)\Phi_{i_2}(y)\rangle = \delta_{i_1 i_2}\sum_{k=0}^{\infty}\left(\frac{N}{2}\right)\left\{\prod_{\ell=1}^{k}\int_0^\infty \frac{d\zeta_\ell}{\zeta_\ell}\int d^Dx_\ell\right.$$
$$\left.\int d\mu\big[w^{(\zeta_\ell}_{x_\ell x_\ell}\big]\cdot\int_0^\infty d\zeta\cdot\int d\mu\big[w^{(\zeta)}_{xy}\big]\right\}\cdot\langle\exp\left\{ i\sum_{\ell=1} k\int d^Dz_\ell\,\sigma(z_\ell)\cdot j\big(w^{(\zeta_\ell)}_{x_\ell x_\ell}\big)(z_\ell)\right.$$
$$\left. + i\int d^D\,\sigma(z) j\big(w^{(\zeta)}_{xy}\big)(z)\right\}\rangle \tag{15}$$

and since the σ-average in eq.(15) is of the gaussian type we can perform it exactly. The result reads:

$$\langle \Phi_{i_1}(x)\Phi_{i_2}(y)\rangle = \delta_{i_1 i_2} \sum_{k=0}^{\infty} \binom{N}{2} \Big\{ \prod_{\ell=1}^{k} \int_0^\infty \frac{d\zeta_\ell}{\zeta_\ell} \int d^D x_\ell \int d^D x_\ell \int d\mu\big[w^{(\zeta_\ell)}_{x_\ell x_\ell}\big]$$

$$\int_0^\infty d\zeta \cdot \int d\mu\big[w^{(\zeta)}_{xy}\big] \cdot \exp\Big\{ -\frac{\lambda}{4}\,\big[\big(2\cdot \sum_{\ell\neq\ell'}^{k} \int_0^{\zeta_\ell} d\alpha_\ell \cdot \int_0^{\zeta_{\ell'}} d\alpha_{\ell'} \cdot \delta^{(D)}\big(w^{(\zeta_\ell)}_{x_\ell x_\ell}(\alpha_\ell) - w^{(\zeta_{\ell'})}_{x_{\ell'} x_{\ell'}}(\alpha_{\ell'})\big)$$

$$+ \left(\sum_{\ell=\ell'}^{k} \int_0^{\zeta_\ell} d\alpha_\ell \int_0^{\zeta_\ell} \delta^{(D)}\big(w^{(\zeta_\ell)}_{x_\ell x_\ell}(\alpha_\ell) - w^{(\zeta_\ell)}_{x_\ell x_\ell}(\alpha_{\ell'})\big) \right)$$

$$+ \left(2\cdot \sum_{\ell=1}^{k} \int_0^{\zeta_\ell} d\alpha_\ell \cdot \int_0^{\zeta} d\alpha\, \delta^{(D)}\big(w^{(\zeta_\ell}_{x_\ell x_\ell}(\alpha_\ell)) - w^{(\zeta)}_{xy}(\alpha) - w^{(\zeta)}_{xy}(\alpha))\big) \right)$$

$$+ \left(\int_0^{\zeta} d\alpha \int_0^{\zeta} d\alpha'\, \delta^{(D)}\big(w^{(\zeta)}_{xy}(\alpha) - w^{(\zeta)}_{xy}(\alpha 1)\big) \right) \big] \Big\}. \tag{16}$$

The above expression is the two-point correlation function of the $\lambda\Phi^4 - O(N)$-theory expressed as a system of interacting random paths with a repulsive self-interaction at these points where they crosses themselves.

Now we can offer a topological explanation for the theory triviality phenomenon for $D > 4$. At first, we note that the correlation function eq.(17) will differ from the free one, namely

$$\langle \Phi_{i_1}(x)\Phi_{i_2}(y)\rangle_{\text{FREE}} = \delta_{i_1 i_2} \left(\int_0^\infty d\zeta d\mu\big[w^{(\zeta)}_{xy}\big] \right) \tag{17}$$

if the path intersections implyed by the delta functions in eq.(16) are non-empty sets in the R^D space-time. We intend to argument that those intersection sets are empty for space-time with dimensionality greater than four. At first we recall some well-known concepts of topology: the topological Hausdorff dimension of a set A embedded in R^D is d (with d being a real number) if the minimum number of D-dimensional spheres of radius γ needed to cover it, grow like γ^{-d} when $r \to 0$. The rule for (generical) intersections for sets A and B (both are embeddeds in R^D) is given by

$$d(A \cap B) = d(A) + d(B) - D \tag{18}$$

where a negative Hausdorff dimension means no (generical) intersection or equivalently the set $A \cap B$ is empty.

As is well know the Hausdorff dimension of the random paths in eq.(16) is 2. A direct application of the rule eq.(18) gives us that the intersection sets in eq.(16) possesses a Hausdorff dimension $4 - D$. So, for $D > 4$ these sets are empty and leading to the triviality phenomenon (see eq.(17)).

Finally we make some comments on the analyses of the divergencies in the random path expression eq.(16) for $D \leq 4$. As a first observation we note that all the path inegrals involved in eq.(16) can be exactly evaluated by making a power series in λ_0. The resulting proper-times ζ integrals will in general be divergents. By using a regularization (such as a cut off for small proper-times) one can show that the divergencies can be absorbed by a renormalization of the bare mass μ_0 and the action path term in eq.(16) (or equivalently, a wave-function and λ_0-coupling renormalization in the field formulation eq.(1)).

At this point of our remarks and comments, it is worth to point out that there is no simple relation between our random loop space approach for QCD where the loop defining the string world-sheet boundary is a non-differentiable path and representing rigorously the functional determinant associated to the matter content E [this means that there is no pure Yang-Mills quantum theory without matters source in our approach (no rings of Gluons!)]; and others approach based on suitable supersymmetric σ-models formulations for conformal superstrings moving in non quantum back-grounds (see E.S. Fradkin and A.A. Tseytlin, PLB ISS, 316, (1981) and M. Maldacena, Phys. Rev. Lett 80, 4859, (1988)). Note that in this case there is still no true derivation of this string/gauge field duality from first principles.

In our string representation for Bosonic QCD as we have proposed in this note, one can see that the Hausdorff dimension of the continuous manifolds sampled by the (euclidean) quantum string vector position is four (a very rough Brownion Bosonic Surface filling up any four-volume in R^4). However, it is expected that the Hausdorff dimension of the manifold sampled by the $2D$-Fermion Field should be minus two. Combining these results one can see that the effective Hausdorff dimension of the QCD string world-sheet is two, so allowing one applies all concepts of classical smooth Differential Topology and Geometry. If all these results turn out to be rigorous, one can see that our self-avoiding fermionic string representation gives a "proof" that $QCD(U(N_c))$ should be expected to be a trivial quantum field theory (with on infrared cut off!) for space-times dimension greater than four.

Finally, we should remark that our proposal for string representations in QCD has no apparent overlap with those proposals relying heavily in the existence of the string Liouville field theory as a bonafide $2D$ Field Theory as proposed in S.S. Gibson, I.R. Klebanov and A.M. Polyakov "Gauge Theory Correlators from Non-Critical String Theory - arXiv: hep-th/980210902; even if they can be interpetred as an extra (unphysical) five dimension coordinate after some conformal impositions in the non-critical string theory.

It appears interesting to remark that these Kaluza-Klein string representatoins for $N = 4$

Supersymmetric QCD may be considered as “modern/geometrical/topological” version of the old beautiful result in String Theory that Strings with $U(N)$ Chan-Paton factors leads formally to Massless and Massive Yang-Mills scattering amplitude in its low energy limit of vanishing Regge Sloppe limit (Luiz C.L. Botelho - PRD35,4,1515/1518 (1987).