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THE PENETRATION OF A WAVE INTO A POTENTIAL BARRIER

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THE PENETRATION OF A WAVE INTO A POTENTIAL BARRIER

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SUMMARY: - The distorsion of a wave pulse reflected by a potential step of finite height is determined and is given, in a simple case, in closed form. It shows, that part of the signal energy becomes temporarily stored inside the potential barrier, before it becomes reflected.

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The penetration of a wave into a potential barrier is a very general and well known phenomenon. The stationary treatment of a penetrating wave, however, shows only that the wave has a finite intensity inside the barrier, near its surface, and does not exhibit the characteristic features of the dynamics of the penetration phenomenon. We consider, therefore, a δ-shaped wave signal which incides on a potential step of finite height and determine the distorted signal which becomes reflected (fig. 1).

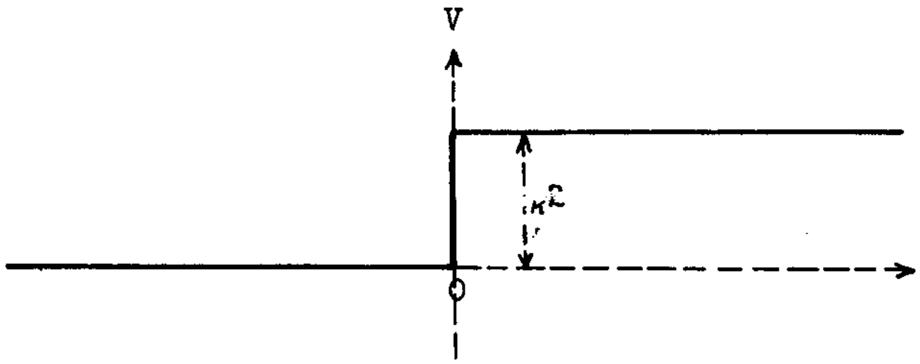

Fig. 1 Potential step function

As a characteristic example, we chose a light wave which, coming from vacuum, incides at $z = 0$ on the surface of a supraconductor of finite penetration depth $1/\kappa$. According to F. London the wave function obeys, then, the wave equations

$$z < 0: \qquad \frac{\partial^2\psi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0 \tag{1}$$

$$\psi_k = e^{ik(z-ct)} + R(k).e^{-ik(z+ct)}$$

$$z > 0 \qquad \frac{\partial^2\psi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \kappa^2\psi = 0 \tag{2}$$

$$\psi_k = T(k).e^{-\sqrt{\kappa^2-k^2}z - ikct}$$

The reflection and transmission coefficients R(k) and T(k) result from the continuity of the wave function and of its derivative at the boundary $z = 0$

$$R(k) = \frac{k - i\sqrt{\kappa^2 - k^2}}{k + i\sqrt{\kappa^2 - k^2}}\,, \qquad T(k) = \frac{2k}{k + i\sqrt{\kappa^2 - k^2}} \tag{3}$$

Both R(k) and T(k) have branch points at $k = \pm\kappa$. Their analytic behaviour in the complex k - plane is given by fig. 2.

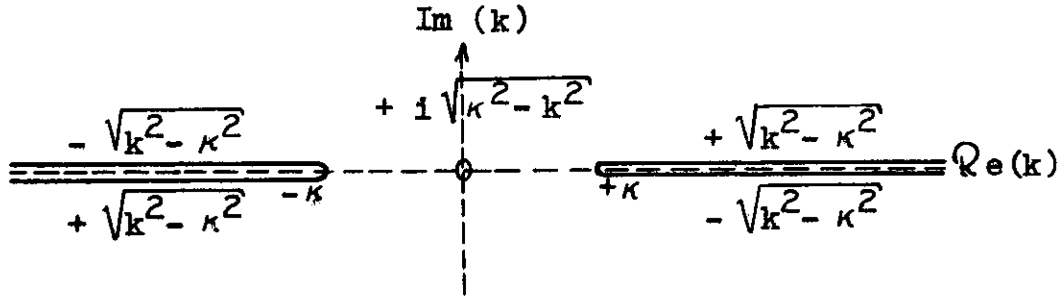

Fig. 2 The function $\sqrt{k^2 - \kappa^2}$ in the complex k - plane

We chose, now, the linear combination of the wave functions (1) and (2) which, for $t = 0$, represent a δ -shaped pulse at $z = -a$ moving towards the potential step. The incident wave is, then, given by

$$\psi_i = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ik(z+a-ct)}\,.dk = \delta(z+a-ct) \quad \text{for } ct < a,\ z < 0. \tag{4}$$

while the reflected and the transmitted signals become

$$\psi_R = \frac{1}{2\pi}\int_{-\infty+i\epsilon}^{+\infty+i\epsilon} \frac{k - i\sqrt{\kappa^2 - k^2}}{k + i\sqrt{\kappa^2 - k^2}}\; e^{-ik(z-a+ct)}\,dk \quad \text{for } z < 0 \tag{5}$$

$$\psi_T = \frac{1}{2\pi} \int_{-\infty+\epsilon}^{+\infty+\epsilon} \frac{2k}{k + i\sqrt{\kappa^2 - k^2}} e^{-\sqrt{\kappa^2-k^2}z+ik(a-ct)}\, dk \quad \text{for } z > 0 \tag{6}$$

and the integrals have to be taken on the positive imaginary border of the real axis of the complex k - plane.

In order to evaluate ψ_R we put

$$\xi = z - a + ct \tag{7}$$

and find

$$\psi_R(\xi) = \frac{1}{2\pi} \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} \frac{k - i\sqrt{\kappa^2 - k^2}}{k + i\sqrt{\kappa^2 - k^2}} e^{-ik\xi}\, dk$$

$$= \frac{1}{2\pi\kappa^2} \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} (2k^2 - \kappa^2 - 2ik\sqrt{\kappa^2 - k^2})\, e^{-ik\xi}\, dk$$

$$= -\left(1 + \frac{2}{\kappa^2} \frac{d^2}{d\xi^2}\right) \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ik\xi}\, dk + \frac{1}{\pi\kappa^2} \frac{d}{d\xi} \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} \sqrt{\kappa^2 - k^2}\, e^{-ik\xi}\, dk$$

$$= -\delta(\xi) - \frac{2}{\kappa^2}\delta''(\xi) + \frac{1}{\pi}\left(\frac{d}{d\xi} + \frac{1}{\kappa^2}\frac{d^3}{d\xi^3}\right) \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} \frac{e^{-ik\xi}}{\sqrt{\kappa^2-k^2}}\, dk$$

Remembering that [1]

$$\frac{2}{\pi} \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} \frac{e^{-ik\xi}}{\sqrt{\kappa^2 - k^2}}\, dk = \frac{2}{\pi} \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} \frac{e^{-i\omega\kappa\xi}}{\sqrt{1 - \omega^2}}\, d\omega = \begin{cases} 0 & \text{for } \xi < 0 \\ 4 . J_0(\kappa\xi) & \xi > 0 \end{cases}$$

$\psi_R(\xi)$ becomes $\psi_R(\xi) = 0$ for $\xi < 0$ and, for $\xi \geqslant 0$

$$\psi_R(\xi) = -\delta(\xi) - \frac{2}{\kappa^2}\delta''(\xi) - 2\kappa J_1(\kappa\xi) + 2.\delta(\xi)$$

$$- \frac{2}{\kappa}\frac{d^2}{d\xi^2}\left\{J_1(\kappa\xi)\right\} + \frac{2}{\kappa^2}\frac{d^2}{d\xi^2}\delta(\xi)$$

$$= \delta(\xi) - 2\kappa J_1(\kappa\xi) - 2\frac{d}{d\xi}\left\{\frac{d\ J_1(\kappa\xi)}{d(\kappa\xi)}\right\}$$

$$= \delta(\xi) - 2\kappa J_1(\kappa\xi) - \frac{d}{d\xi}\left\{J_o(\kappa\xi) - J_2(\kappa\xi)\right\}$$

$$= \delta(\xi) - \kappa J_1(\kappa\xi) + \frac{d}{d\xi}\left\{J_2(\kappa\xi)\right\} - \delta(\xi)$$

and, finally

$$\psi_R(\xi) = -\frac{\kappa}{2}\left\{J_1(\kappa\xi) + J_3(\kappa\xi)\right\} \quad \text{for } \xi \geqslant 0 \quad (8)$$

It is easily verified, that the reflected signal vanishes for $\kappa = 0$

$$\lim_{\kappa\to 0} \psi_R(\xi) = 0$$

and that, for $\to \infty$

$$\lim_{\kappa\to\infty} \psi_R(\xi) = -\delta(\xi)$$

as in the case of an ideal mirror.

In the neighbourhood of $\xi = 0$ ψ_R behaves like

$$\psi_R(\xi) \cong - \frac{\kappa^2}{4} \xi \tag{9}$$

while, for large values of ξ

$$\psi_R(\xi) \sim - \sqrt{\frac{8}{\pi}} \kappa \frac{\sin (\kappa \xi - \frac{3\pi}{4})}{(\kappa \xi)^{3/2}} \tag{10}$$

(8) shows, that part of the inciding pulse becomes accumulated within the potential step (supraconductor) and becomes reflected with a mean retardation time

$$\Delta t \sim \frac{1{,}9}{\kappa c} \tag{11}$$

but continues reemitting the accumulated energy after a long time with the intensity

$$\overline{\psi_R^2} \sim \frac{4}{\pi \kappa} \frac{1}{\xi^3} = \frac{4}{\pi \kappa (z - a + ct)^3} \tag{12}$$

The accumulation and the temporary storage of field energy inside a potential barrier is a very general feature of wave dynamics. It is essential for the understanding of phenomena like the tunnel effect. It occurs, however, whenever a wave meets a finite obstacle, even in free space in the neighbourhood of curved surfaces. In this case the potential barriers responsible for the accumulation are due to inertial forces, e.g. to centrifugal forces. The phenomenon of which we have treated above the mathematically simplest case, is of importance for the understanding of diffraction phenomena and it plays when an antenna or an atom becomes excited by an incoming wave pulse. In many important cases the accumulated energy decays according to an exponential law.[2] In the above treated example, however, the decay law is found to be a power law (12).

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1 - JAHNKE-EMDE, Tables of functions, Dover Publications, 1943, pp. 150 and ff.

2 - Compare G. Beck and H. M. Nussenzveig, N. Cim. 16, (1960), pp. 426 and ff.