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DIRAC FIELD IN THE GRK BACKGROUND

by

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Properties of Dirac equation on the GRK geometry are studied. Bound states solutions are obtained. Their physical significance is discussed.

The GRK (General Relativistic Kink) is a natural generalization, to the 4-dimensional space-time, of the 2-dimensional kink solution of nonlinear $\lambda\phi^4$ model as shown by Kodama[1]. It is a completely singularity-free, finite energy solution and has some interesting aspects. One of them is the topology which permits to associate conservative numbers with it. It was also shown to be stable, at least against radial perturbations[2].

The geometrical structure of the GRK resembles that of the Rosen-Einstein bridge[3] of the Schwarzschild geometry, but with an essential difference in that such a structure becomes similar to the Wheeler's wormhole[4], as was discussed in the references 1 and 2.

The GRK was thus introduced, as a natural consequence, in a scheme analogous to Wheeler's idea. This scheme consists essentially in considering that any kind of field flux can pass through the wormhole, and if it remains inside the geometry, the complete set (GRK+field) can be seen as a mass concentration due to the GRK, together with properties inherent to the fields. A solution was shown for the eletromagnetic case[5].

Another important property besides the charge is the spin one-half. We shall then investigate the Dirac's field, studying it in a GRK background.

In order to perform this task we shall treat the spinor field in a generalized space-time. We write it in the following way[6]:

$$\gamma^{\mu}(\partial_{\mu}-\Gamma_{\mu})\psi + im_{D}\psi = 0 \quad , \tag{1}$$

where γ^{μ} are Dirac matrices satisfying anti-commutation relations,

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}\,\mathbb{1} \quad ,$$

Γ_{μ} are Fock-Ivanenko coefficients, ψ is the spinorial field and m_D can be considered as a parameter.

The spherically symmetric line element is written as

$$ds^2 = e^{2\eta(r)}(dx^0)^2 - e^{2\alpha(r)}(dr)^2 - r^2\, d\Omega^2 \quad , \tag{2}$$

while the Fock-Ivanenko coefficients are given by

$$\Gamma_{\mu} = A_{\mu}\mathbb{1} - \frac{1}{8}\left[\gamma^{\alpha}(\partial_{\mu}\gamma_{\alpha}) - (\partial_{\mu}\gamma_{\alpha})\gamma^{\alpha} - \left\{{\rho \atop \alpha\mu}\right\}(\gamma^{\alpha}\gamma_{\rho} - \gamma_{\rho}\gamma^{\alpha})\right] \quad , \tag{3}$$

where A_{μ} is an arbitrary real vector field, and $\left\{{\rho \atop \alpha\mu}\right\}$ the usual

Christoffel symbol. We can take $A_\mu = 0$ without loss of generality[7].

Eq. (1) is reduced to

$$\Big\{ e^{-\eta}\tilde{\gamma}^0 \partial_0 + e^{-\alpha}(\mathbf{e}_r\cdot\tilde{\boldsymbol{\gamma}})\partial_r + \frac{1}{r}(\mathbf{e}_\theta\cdot\tilde{\boldsymbol{\gamma}})\partial_\theta + \frac{1}{r\,\mathrm{sen}\theta}(\mathbf{e}_\phi\cdot\tilde{\boldsymbol{\gamma}})\partial_\phi +$$

$$+ \left[\frac{1}{2}\eta' e^{-\alpha} + \frac{1}{r}(e^{-\alpha}-1)\right](\mathbf{e}_r\cdot\tilde{\boldsymbol{\gamma}}) + im_D \Big\}\psi = 0 \quad , \qquad (4)$$

where

$$(\mathbf{e}_r\cdot\tilde{\boldsymbol{\gamma}}) = \mathrm{sen}\theta\cos\phi\,\tilde{\boldsymbol{\gamma}}^1 + \mathrm{sen}\theta\,\mathrm{sen}\phi\,\tilde{\boldsymbol{\gamma}}^2 + \cos\theta\,\tilde{\boldsymbol{\gamma}}^3 \quad ,$$

$$(\mathbf{e}_\theta\cdot\tilde{\boldsymbol{\gamma}}) = \cos\theta\cos\phi\,\tilde{\boldsymbol{\gamma}}^1 + \cos\theta\,\mathrm{sen}\phi\,\tilde{\boldsymbol{\gamma}}^2 - \mathrm{sen}\theta\,\tilde{\boldsymbol{\gamma}}^3 \quad ,$$

$$(\mathbf{e}_\phi\cdot\tilde{\boldsymbol{\gamma}}) = -\mathrm{sen}\phi\,\tilde{\boldsymbol{\gamma}}^1 + \cos\phi\,\tilde{\boldsymbol{\gamma}}^2 \quad ,$$

and the $\tilde{\boldsymbol{\gamma}}$'s are the Dirac's matrices in flat space.

Now, we can separate the radial and angular parts of eq. (4) through standard procedures[8] and then

$$\frac{dG}{dr} = e^\alpha \left[-\frac{\kappa}{r}G + (Ee^{-\eta} + m_D)F\right] \qquad (5)$$

and

$$\frac{dF}{dr} = e^\alpha \left[\frac{\kappa}{r}F - (Ee^{-\eta} - m_D)G\right] \quad , \qquad (6)$$

where F, G are radial functions and κ is the eigenvalue of $(\boldsymbol{\sigma}.\boldsymbol{\ell} + 1)$. In order to obtain eqs. (5-6) we used the ansatz

$$\psi = \frac{1}{r} e^{-\frac{1}{2}\eta(r)} \Phi(r)\, e^{-iEt} \quad .$$

Let us make the coordinate transformation

$$(dr/dR)^2 = e^{-2\alpha} \quad ,$$

for which the coordinate system itself is singularity-free[2]. Eqs. (5-6) become

$$\frac{dG}{du} = \left[-\frac{\kappa}{x} G + (\bar{E}e^{-\eta} + \bar{m}_D) F \right] \tag{7}$$

and

$$\frac{dF}{du} = \left[\frac{\kappa}{x} F - (\bar{E}e^{-\eta} - \bar{m}_D) G \right] \quad , \tag{8}$$

where $\bar{m}_D = m_D/\mu$, $\bar{E} = E/\mu$ and $u = \mu R$, $x = \mu r$ are dimensionless variables.

Eqs. (7-8) are quite similar to the usual Dirac radial equation in a central field.

The gravitational field generated by the GRK localizes the Dirac field around the bridge, if there is a non-singular solution satisfying the following boundary condition:

$$|G| \text{ and } |F| \to 0 \quad \text{for } |u| \to \infty \quad . \tag{9}$$

In this way, eqs. (7-8) with the boundary condition eq. (9) constitute an eigenvalue problem for $\bar{E}$.

To solve this problem, it is useful to analize some properties of the solutions. We know that the GRK static solution $x(u)$ and $\eta(u)$ are even functions of u, and their asymptotic properties are

$$x \to |u| \quad \text{and} \quad e^{-\eta} \to 1 \quad \text{for} \quad |u| \to \infty \quad .$$

It is easy to verify that the asymptotic forms of the solution are given by

$$G,F \simeq e^{\pm\sqrt{\bar{m}_D^2 - \bar{E}^2}\; u} \quad , \quad |u| \to \infty \quad . \tag{10}$$

Therefore, for $\bar{E} > \bar{m}_D$, the boundary condition [eq.(9)] is always satisfied and the spectrum of $\bar{E}$ is continuous. However, for $\bar{E} < \bar{m}_D$, we must find a value of $\bar{E}$ for which the wave functions take the asymptotic form:

$$G,F \sim e^{-\sqrt{\bar{m}_D^2 - \bar{E}^2}\; u} \quad \text{for} \quad u \to +\infty$$

and

$$G,F \sim e^{+\sqrt{\bar{m}_D^2 - \bar{E}^2}\; u} \quad \text{for} \quad u \to -\infty \quad .$$

In such a case the spinor field is localized around $u \simeq 0$.

For simplicity, we restric ourselves to ground state solutions, $\kappa = 1$ case. The problem is completely similar for the $\kappa < 0$ case, because the substitution $\kappa \to -\kappa$ is equivalent to the transformation

$$u \to -u \quad , \quad G \to -G \quad , \quad F \to F \quad , \tag{11}$$

which obviously does not change the spectrum of $\bar{E}$.

Since eqs. (7-8) are linear, we can fix, without loss of generality, one of initial values, say $G(0)$, to any finite constant. We put then

$$G(0) = 1 \quad . \tag{12}$$

Any other choice changes only the normalization of F and G.

Hence, we have to determine the two parameters $\bar{E}$ and $F(0)$, for the ground state eigenfunction.

From eqs. (7-8) it is easy to show that, if G tends to zero for $u \to +\infty$, then F tends also to zero. Likewise, if G diverges when $u \to +\infty$, F also diverges. Therefore, there is no possibility of one of them converge and the other diverge for $u \to +\infty$. The same happens when $u \to -\infty$. Then, there exists a particular value of $F(0)$ for which G and F tend simultaneously to zero when $u \to +\infty$. Obviously such $F(0)$ is a function of $\bar{E}$. Call this value $F_+^0(\bar{E})$. However, this does not necessarily imply that G and F tend to zero for $u \to -\infty$.

On the other hand, there exists another value of $F(0)$, say $F_-^0(\bar{E})$, so that G and F tend to zero when $u \to -\infty$. In this way, the boundary condition Eq. (9) is expressed as the following equation for $\bar{E}$:

$$F_+^0(\bar{E}) = F_-^0(\bar{E}) \quad . \tag{13}$$

Therefore eigenvalues of $\bar{E}$ are given by the intersections points of two curves $y_1 = F_+^0(\bar{E})$ and $y_2 = F_-^0(\bar{E})$.

In fig. 1 we show F_+^0 and F_-^0 plotted against $\bar{E}/\bar{m}_D$, for the cases $\bar{m}_D = 1$ and 10 (with $|\kappa| = 1$), where the GRK geometry corresponding to $f = 1.25$ is taken[1].

The y_1 and y_2 curves can intersect several times. The first intersection provides the ground state energy eigenvalues, while the others, up to the value $\bar{E} = \bar{m}_D$, provide the excited bound states.

In fig. 2 we show the wavefunctions corresponding to the ground state (for $\kappa = 1$) of the solution shown in fig. 1 at the $\bar{m}_D = 1$ case. Such functions are shown in fig. 3, with $\kappa = -1$. In this figure the transformations given by eq. (11) become evident.

It is verified[9] that there is no bound states, for the case of $f = 1.25$, if $\bar{m}_D$ is smaller than 0.65.

Hence, we see that the GRK can localize the Dirac field around the bridge, for $\bar{m}_D > 0.65$. Therefore, a distant observer interpretes such a configuration as a massive particle of spin one-half: the mass is given by GRK geometry, while the spin is given by the localized Dirac field.

Such an approach, together with the electromagnetic field, would provide a pure theoretical-field, singularity-free model of elementary particles including anti-particles, at least at the classical level.

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FIGURE CAPTIONS

FIG. 1 - F_{+}^{0} and F_{-}^{0} plotted against $\bar{E}/\bar{m}_D$.

FIG. 2 - G(u) and F(u) functions corresponding to the ground state with: $\bar{m}_D = 1$ and $\kappa = 1$.

FIG. 3 - G(u) and F(u) functions corresponding to the ground state with: $\bar{m}_D = 1$ and $\kappa = -1$.

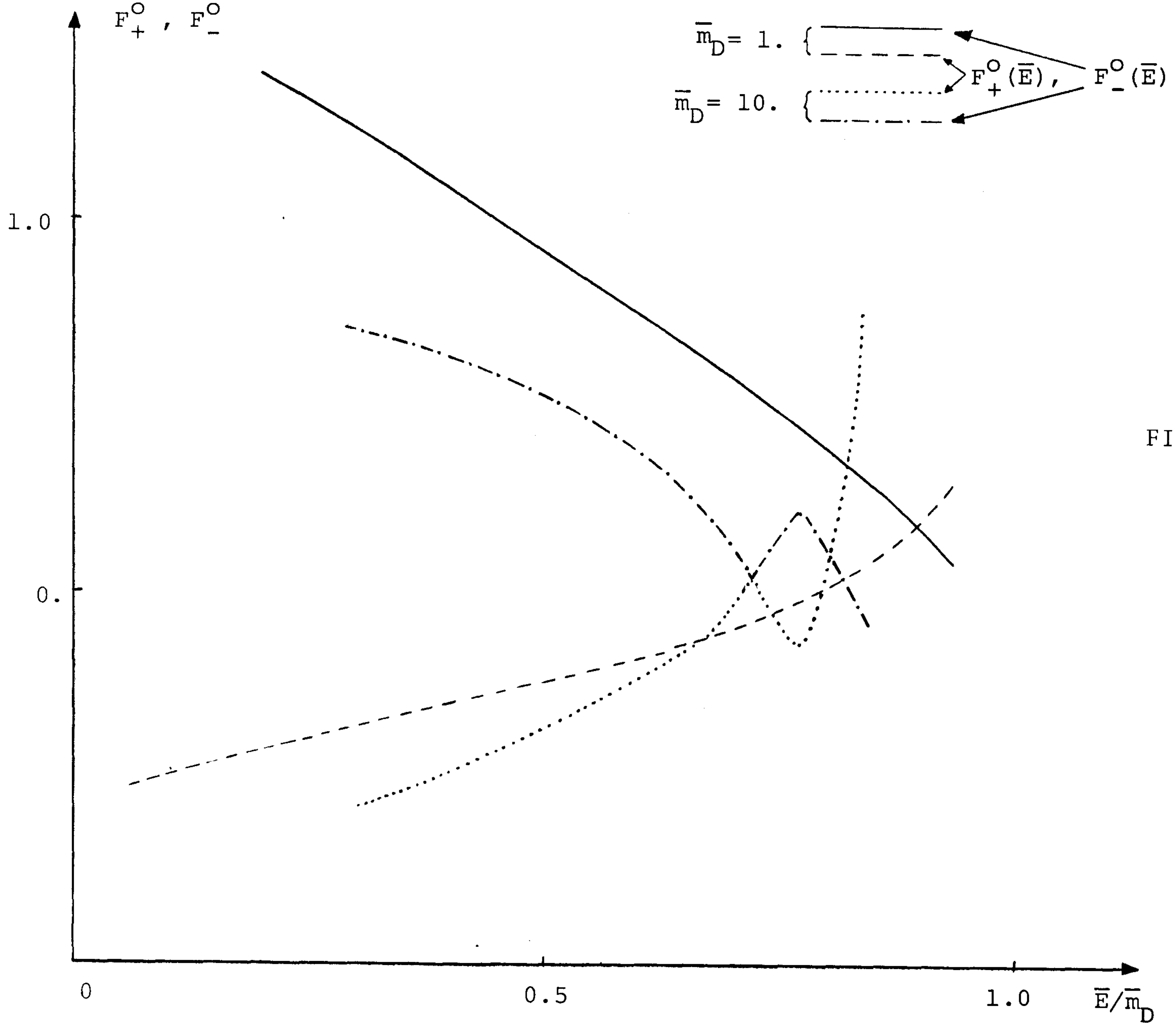

FIG. 1

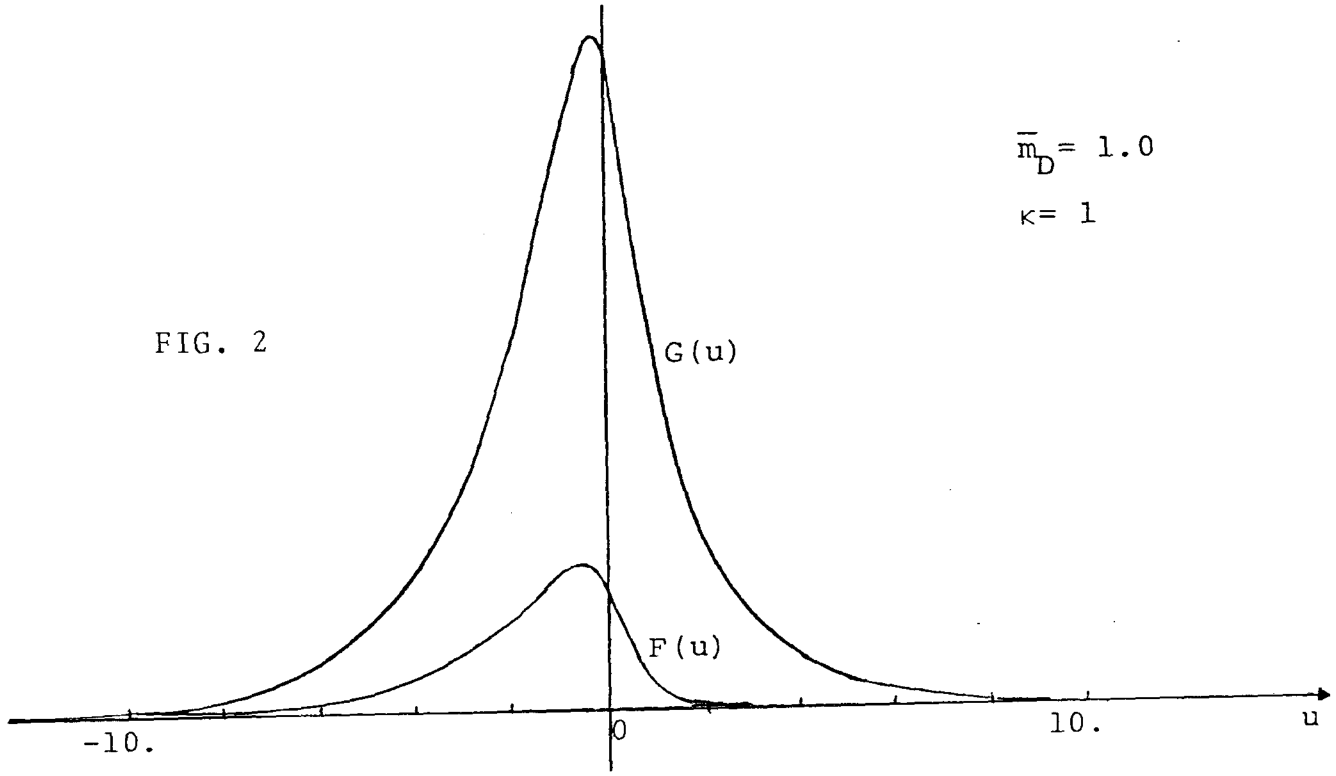

FIG. 2

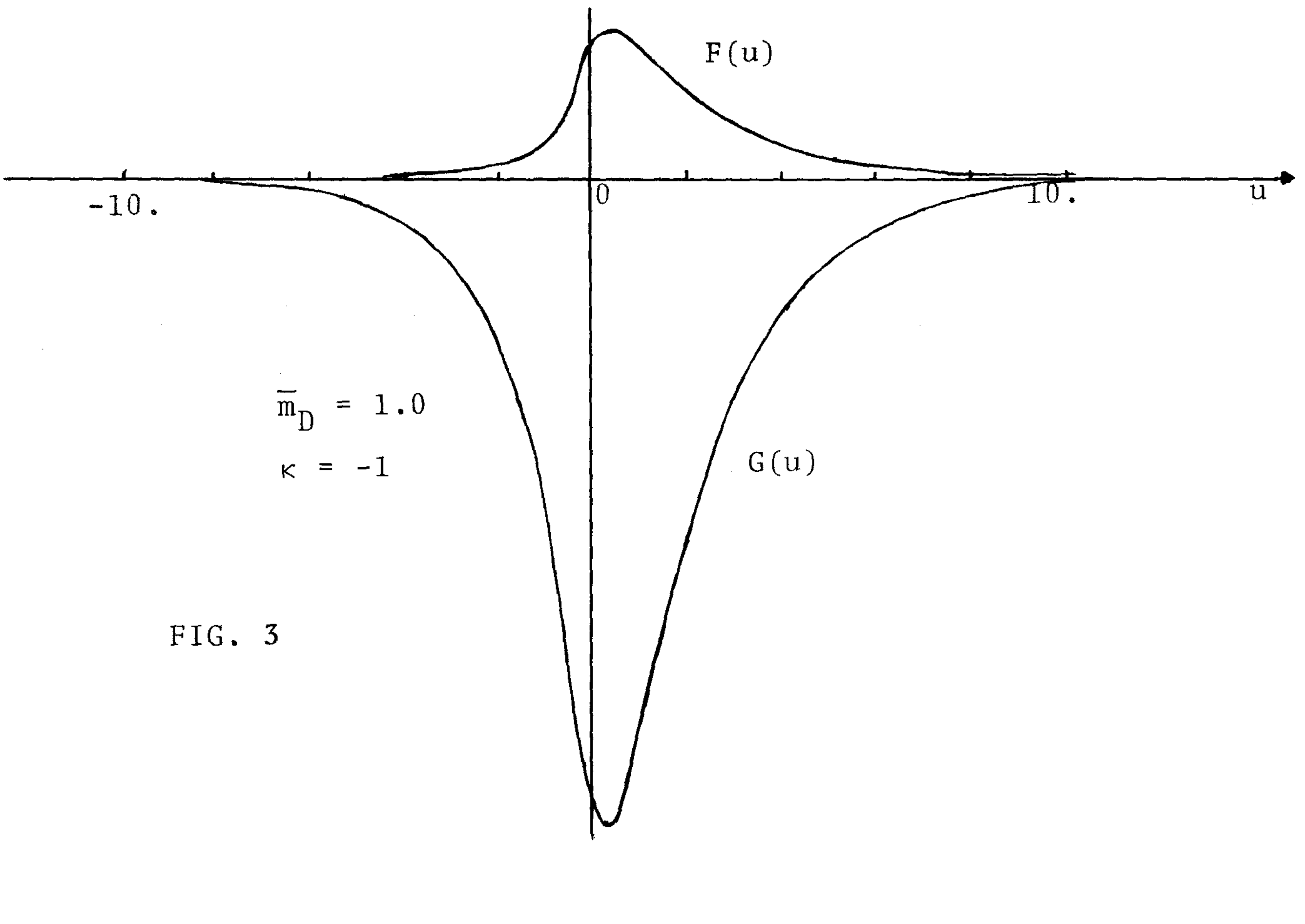

FIG. 3