A0005/79
Jan.1979

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ABSTRACT

Under the laws of Einstein's gravitational theory, a massless system consisting of the diffuse sources of two fields is discussed. One field is scalar, of long range, the other is a vector field of short range. A proportionality between the sources is assumed. Both fields are minimally coupled to gravitation, and contribute positive definitely to the time component of the energy momentum tensor. A class of static, spherically symmetric solutions of the equations is obtained, in the weak field limit. The solutions are regular everywhere, stable, and can represent large or small physical systems. The gravitational field presents a Schwarzschild-type asymptotic behavior. The dependence of the energy on the various parameters characterizing the system is discussed in some detail.

1. INTRODUCTION

It is an old belief that general relativity will find a leading place in the description of elementary physical structures[1]. Nonsingular solutions of the field equations are particularly looked for, in which the energy momentum tensor depends on a minimum number of simple physical quantities.

Massive static systems are more usually studied, where the attractive effects of self-gravitation are balanced by some kind of repulsive interaction. Such are the cases of the incompressible fluid sphere of Schwarzschild, in which the gradient of pressure is responsible for the equilibrium, and the charged sphere of Bonnor[2], where the electrostatic repulsion between the constituents prevents the collapse. Massive systems containing scalar sources have also been considered, in recent literature[3-7].

In an alternative line of research, static systems not containing matter explicitly can be considered, in which the energy momentum tensor is obtained from some covariant Lagrangean. Along this line, a system only containing the diffuse source of two different scalar fields was recently studied[8], and a nonsingular and stable solution was obtained. However, one of these scalar fields contributes negative definitely to the time component of the energy momentum, and is not considered a "reasonable" physical agent by some authors[9].

In the present paper, the study is made of a stable structure, source of a vector field of short range and of a scalar field of long range. Both fields are minimally coupled to gravitation, and contribute positive definitely to T_{00}. In Sec. 2, the covariant equations governing the system are obtained from a Lagrangean density, and the static, spherically symmetric equations are written in the weak field limit. In Sec. 3, the exact solutions for the vector and scalar fields are obtained, and are found to be regular everywhere. In Sec. 4, the expressions for the gravitational potentials are presented; these expressions also are regular everywhere, and show a Schwarzschild-type behavior at infinity. Finally, the three independent parameters which characterize the system are discussed in Sec. 5, and the influence each of them exerts on the total energy of the system is clearly explained. It is also shown that the solutions obtained may serve as basis to describe large or small actual physical systems.

2. THE EQUATIONS

One starts from the Lagrangean density

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_V + \mathcal{L}_S \quad , \tag{1}$$

$$K\,\mathcal{L}_G = \tfrac{1}{2}(-g)^{1/2}\, R \quad , \qquad K = 8\pi G/c^4 \quad , \tag{2}$$

$$K\,\mathcal{L}_V = (-g)^{1/2}\left[(V_{\mu,\nu} - V_{\nu,\mu})V_{\alpha,\beta}g^{\mu\beta} + \kappa^2\, V_\nu V_\alpha\right]g^{\nu\alpha} - 8\pi J_*^\alpha V_\alpha \quad , \tag{3}$$

$$K\,\mathcal{L}_S = (-g)^{1/2}\, S_{,\alpha}\, S_{,\beta}\, g^{\alpha\beta} - 8\pi\sigma_* S \quad . \tag{4}$$

In these equations, R is the scalar curvature, g is the determinent of the metric potentials $g_{\mu\nu}$, V_μ is a repulsive vector field of short range (κ^{-1}) and S is an attractive scalar field of long range. The vector quantity J_*^α and the scalar quantity σ_* are densities of weight +1, and represent the diffuse sources of V_α and S, respectively.

From the invariance of the action integral upon variations of the metric potentials one obtains the Einstein's equations[10],

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = -\, 2(V_{\mu\alpha}V^{\alpha}{}_{\nu} + \kappa^2\, V_{\mu}V_{\nu} + S_{,\mu}S_{,\nu})$$
$$+ (\tfrac{1}{2}\, V^{\alpha}{}_{\beta}V^{\beta}{}_{\alpha} + \kappa^2 V^{\alpha}V_{\alpha} + S^{,\alpha}S_{,\alpha})g_{\mu\nu} \;, \tag{5}$$

$$V_{\mu\nu} \equiv V_{\nu,\mu} - V_{\mu,\nu} \;, \tag{6}$$

while variations of the vector and scalar potentials give

$$V^{\mu\nu}{}_{;\nu} - \kappa^2 V^{\mu} = -\, 4\pi J^{\mu} \quad , \quad J^{\mu} \equiv (-g)^{-1/2} J^{\mu}_{*} \quad , \tag{7}$$

$$S^{;\mu}_{;\mu} = -\, 4\pi\sigma \quad , \quad \sigma \equiv (-g)^{-1/2}\sigma_{*} \quad . \tag{8}$$

The semicolons mean covariant derivatives, and the quantities J^{μ} and σ have weight zero. From the Bianchi identities one obtains

$$J^{\alpha}\, V_{\alpha\nu} - \sigma S_{,\nu} = 0 \quad . \tag{9}$$

We now specialize these equations for the case of static, spherically symmetric systems. We write

$$ds^2 = e^{2\eta}(dx^0)^2 - e^{2\lambda}\, dr^2 - r^2\, d\theta^2 - r^2\, \sin^2\theta\, d\phi^2 \quad , \tag{10}$$

$$V_{\mu} = V\, \delta^0_{\mu} \quad , \quad J^{\mu} = J^0\, \delta^{\mu}_0 \quad , \tag{11}$$

and consider all quantities (η, λ, V, J^0, S, σ) functions of only r. We then obtain, as independent equations,

$$\eta' + \lambda' = r(\kappa^2 V^2\, e^{-2\eta+2\lambda} + S'^2) \quad , \tag{12}$$

$$\left[r(1-e^{-2\lambda})\right]' = r^2\left[(\kappa^2 V^2 + V'^2\, e^{-2\lambda})e^{-2\eta} + S'^2\, e^{-2\lambda}\right] \quad , \tag{13}$$

$$r^{-2}\, e^{-\eta-\lambda}(r^2 V'\, e^{-\eta-\lambda})' - \kappa^2 V\, e^{-2\eta} = -\, 4\pi J^0 \quad , \tag{14}$$

$$r^{-2}\, e^{-\eta-\lambda}(r^2\, S'\, e^{\eta-\lambda})' = 4\pi\sigma \quad , \tag{15}$$

$$J^0 V' + \sigma S' = 0 \quad , \tag{16}$$

where a dash means d/dr. Since in these five equations we have six functions, one constraint is necessary to obtain explicit solutions. We consider here the case where the sources $J^0(r)$ and $\sigma(r)$ bear a constant ratio,

$$J^0 = \omega\sigma \quad , \quad \omega = \text{const} \quad . \tag{17}$$

One finds difficulty in obtaining the exact integration of the field equations. We then try an approximate method: we expand the four fields (η, λ, V, S) and the two sources (J^0, σ) in integral powers of some dimensionless parameter ε . This parameter is identified later. We have been able to obtain the exact solution in the lowest order of approximation, in which J^0, σ, V, S are proportional to ε, while η and λ are proportional to ε^2. In this order of approximation the field equations become

$$\eta' + \lambda' = r(\kappa^2 V^2 + S'^2) \quad , \tag{18}$$

$$(r\lambda)' = \frac{1}{2} r^2 (\kappa^2 V^2 + V'^2 + S'^2) \quad , \tag{19}$$

$$V'' + 2V'/r - \kappa^2 V = -4\pi\omega\sigma \quad , \tag{20}$$

$$S'' + 2S'/r = 4\pi\sigma \quad , \tag{21}$$

$$(\omega V' + S')\sigma = 0 \quad , \tag{22}$$

where (17) has been used.

From the last three equations one obtains the fields V, S, and the source σ, then from (19) and (18) one gets the gravitational potentials λ and η , consecutively.

3. VECTOR AND SCALAR FIELDS

One initially considers the region $r \leq a$, where the diffuse source σ exists. From (20) to (22) one then obtains the solutions, regular in the origin,

$$V_i(r) = \alpha\, j_0(\nu r) \quad , \quad \nu \equiv \kappa(\omega^2-1)^{-1/2} \quad , \tag{23}$$

$$4\pi\sigma(r) = \alpha\omega\nu^2\, j_0(\nu r) \quad , \tag{24}$$

$$S_i(r) = -\alpha\omega\left[j_0(\nu r) + \beta\right] \quad , \tag{25}$$

where $j_0(x) = x^{-1} \sin x$ is the spherical Bessel function of order zero, and α, β are constants of integration. The subscript i means internal. One finds that the parameter ω necessarilly satisfies $\omega^2 > 1$, otherwise the mathematical solutions obtained are physically unsatisfactory; this subject is further discussed in Sec. 5.

In the region $r > a$, where the source $\sigma = 0$, one obtains from (20)

$$V_e(r) = \alpha j_0(\nu a)(a/r)\, e^{-\kappa(r-a)} \quad , \tag{26}$$

where the continuity of the vector field through $r = a$ was imposed. The subscript e means external. One observes the rapid decay of the short range field, for increasing distance from the origin. One also imposes the continuity of the radial derivative of vector field, and obtains

$$\nu a j_1(\nu a) = (1 + \kappa a)\, j_0(\nu a) \quad , \tag{27}$$

where $j_1(x) = -\, dj_0(x)/dx$ is the spherical Bessel function of order one. This relation represents a constraint for the radius a, for a given set of parameters κ and ω. Since variations of sign in the diffuse sources of fields induce instability in the system, one finds from (24) that only the smallest positive value of νa satisfying (27) is of physical interest, namely

$$\pi/2 < \nu a < \pi \quad . \tag{28}$$

The external scalar field is obtained from (21), with $\sigma = 0$:

$$S_e(r) = -\alpha\omega\left[j_0(\nu a) + \beta\right](a/r) \quad , \quad \beta = (1-\omega^{-2})^{1/2} \quad , \tag{29}$$

where the continuity of the field and of its radial derivative were again imposed. In order to obtain the value of β, use was made of the relation (27). One remarks the hyperbolic behavior (r^{-1}) of the scalar field, in the regions outside the sources.

4. GRAVITATIONAL FIELDS

In the internal region ($r \leq a$) one obtains, using (19), (23) and (25),

$$\lambda_i(r) = \frac{1}{2}\alpha^2\left[\omega^2 + j_0(2\nu r) - (\omega^2+1)\, j_0{}^2(\nu r)\right] \quad , \tag{30}$$

while from (18) one obtains

$$\eta_i(r) = \eta(0) + \alpha^2\left[(2\omega^2-1)\textstyle\sum(\nu r) + (\omega^2 - \frac{1}{2})\, j_0(2\nu r) - \omega^2 + \frac{1}{2}\, j_0{}^2(\nu r)\right] . \tag{31}$$

For convenience, we introduced the constant

$$\eta(0) = -\alpha^2\left[(2\omega^2-1)\textstyle\sum(\nu a) - (1-\omega^{-2})e^{2\kappa a}\,\mathrm{Ei}(-2\kappa a) - \frac{1}{2}\right] \quad , \tag{32}$$

where the function $\sum(x)$ and the exponential integral $\mathrm{Ei}(-x)$

are defined by

$$\sum(x) = \int_0^x t\left[j_0(t)\right]^2 dt \quad , \quad Ei(-x) = -\int_x^\infty t^{-1} e^{-t} dt \quad , \quad x > 0 . \tag{33}$$

An easy inspection of (30) shows that $\lambda(0) = 0$; less trivially, one finds that $\eta(0) < 0$, and that both η_i and λ_i increase monotonically outwards. All these general features are also encountered in the weak field limit of the internal Schwarzschild solution.

In the external region ($r > a$), one obtains from (19), (26) and (29),

$$\lambda_e(r) = Gm/rc^2 - \frac{1}{2}(1 + \kappa a) V^2(r) - \frac{1}{2} S^2(r) \quad , \tag{34}$$

where m is the mass parameter, given by

$$Gm/ac^2 = \alpha^2\left[\omega^2 - \frac{1}{2} - \omega^2 j_0(2\nu a)\right] \quad . \tag{35}$$

Finally, from (18) one gets

$$\eta_e(r) = -Gm/rc^2 + \frac{1}{2}(1 + \kappa r) V^2(r) + \left[\kappa a V(a) e^{\kappa a}\right]^2 Ei(-2\kappa r). \tag{36}$$

One remarks, in (34) and (36), the usual Schwarzschild gravitational behavior in the asymptotic regions,

$$\eta(r) = -\lambda(r) = -Gm/rc^2 \quad , \qquad r \to \infty . \tag{37}$$

The continuity properties of the gravitational potentials are easily seen from (18) and (19). As in the cases of massive spheres, one finds that η, λ and η' are continuous through $r = a$. In addition, one finds that also λ' and η'' are continuous, in our system.

5. DISCUSSIONS

Three independent parameters characterize our physical system: κ, α and ω. The inverse length parameter κ is mainly responsible for the size of the system; indeed, one finds from (27) and (23) that the radius a is inversely proportional to κ.

The parameter α is dimensionless. In Sec. 3, we found that all vector and scalar field quantities are proportional to α, while in Sec. 4 we found that the gravitational potentials η and λ are proportional to α^2. This suggests to identify α as the small, dimensionless parameter ε in terms of which the series expansion of Sec. 2 were made. As can be seen in (35), the smallness of α^2 implies $m/a \ll c^2/G$, a condition usually met both in large physical systems (stars, galaxies) and small ones (atomic nuclei).

Finally, we found that the dimensionless parameter $\omega = J^0/\sigma$ must satisfy $\omega^2 > 1$. This has a simple physical interpretation. The collapse of the system is only prevented when there is sufficient source J^0 of repulsive, short range vector field to balance the attractive effects of the long range scalar field on the corresponding source σ.

From (5), (10) and (11) one finds that the time component of the energy momentum tensor is

$$(8\pi G/c^4)\ T_{00} = \kappa^2 V^2 + (V'^2 + S'^2\, e^{2\eta})e^{-2\lambda} \quad ; \qquad (38)$$

this is an exact result, and shows that both fields V and S contribute positive definitely to T_{00}.

An alternative expression for the mass m is obtained from (35) and (27):

$$Gm/c^2 = (\alpha^2/\kappa)\left[(\omega^2 - \tfrac{1}{2})(\omega^2 - 1)^{1/2} \text{ arc sin}|\omega|^{-1} + \omega^2 - 1\right] . \quad (39)$$

This expression exhibits more clearly the dependence of m on the parameters α, κ and ω . The inverse sine is taken between $\pi/2$ and π, in order to satisfy (28). A direct computation shows that the energy mc^2 of the system monotonically increases with $|\omega|$. The following two extreme behaviors are obtained, for small and large $|\omega|$:

$$Gm/c^2 \simeq \pi\ (\alpha^2/\kappa)(\delta/8)^{1/2} \qquad \text{for } |\omega| = 1 + \delta\ ,\ 0 < \delta \ll 1\ ; \quad (40)$$

$$Gm/c^2 \simeq \pi\ (\alpha^2/\kappa)\ |\omega|^3 \qquad \text{for } |\omega| \gg 1\ . \quad (41)$$

We did not attempt to rigorously demonstrate the stability of our system. However, a nonrelativistic form of ressoning is appropriate[6] in the present case, where the Newtonian concept of force can be used: Starting from an equilibrium configuration, admit a small perturbation which produces some local compression of the diffuse sources. Since $\omega^2 > 1$, the additional repulsive, short range forces will exceed the additional forces of the long range, attractive field. As a consequence, a tendency to local rarefaction is manifested. In the reverse situation of a local small expansion, the same final tendency to restore the equilibrium configuration is observed.

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