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ABSTRACT

In this paper we give the solution of the simplified unidirectional diffusion equations of nucleons and charged pions in the atmosphere when pions of the second generation are taken into account. A sufficiently smooth differential spectrum G(E)dE is assumed for the primary cosmic radiation in the top of the atmosphere.

INTRODUCTION

1. The development of the nuclear and pion components of the cosmic radiation in the atmosphere can be described in a first approximation by the one-dimensional diffusion equations[1,2,3,4,5].

$$\frac{\partial F_N(x,E)}{\partial x} = -\frac{1}{\lambda_N} F_N(x,E) + \frac{1}{\lambda_N(1-K_N)} F_N\left(x, \frac{E}{1-K_N}\right) \qquad (1)$$

$$\frac{\partial F_\pi(x,E)}{\partial x} = -\frac{1}{\lambda_\pi} F_\pi(x,E) + \frac{1}{\lambda_\pi(1-K_\pi)} F_\pi\left(x, \frac{E}{1-K_\pi}\right) + P_\pi^{NN}(x,E) + P_\pi^{\pi N}(x,E) \qquad (2)$$

$F_\alpha(x,E)$ is the flux per (cm^2.s.ster.) of a generic hadron α of energy E, E+dE at the atmospheric depth x (g/cm^2). We put $\alpha = N$ for nucleons and $\alpha = \pi$ for charged pions.

The functions F_N and F_π are supposed to satisfy the boundary conditions

$$F_N(0,E) = G(E) \geq 0 \qquad (3)$$

and

$$F_\pi(0,E) = 0 \qquad (4)$$

G(E)dE is the primary nucleon differential spectrum in the top of the atmosphere. The mathematical restrictions assumed for G(E)dE will be specified later. Here λ_α and K_α are respectively the interaction length (g/cm^2) and the inelasticity coefficient of a high energy interaction between a hadron α and an air nucleus;

$P_\pi^{\alpha N}(x,E)$ is the production rate of charged pions created by the interaction between a hadron α of energy E,E+dE and an air nucleus at the atmosphere depth x (g/cm^2).

In this approximation the production of Kaons is not taken into account. Here λ_α and K_α are supposed to be constants and independent of the hadron incident energy.

The $\pi \rightarrow \mu$ decay is disregarded because we consider only pions of energy equal or greater than 1 Tev and only consider the diffusion along the vertical direction.

In a preceding paper(4), we obtained the solution of these diffusion equations when the pions of the second generation are not taken into account, that is when $P_\pi^{\pi N}(x,E) = 0$. At the end of the paper, we said, also, that the special result obtained there could be used to construct the solution of the general case where $P_{\pi}^{\pi N}(x,E) \neq 0$. In this paper we shall consider this general case.

2. The solution of the equation (1) which describes the diffusion of the nucleon component needs no modification and is given by

$$F_N(x,E) = e^{-x/\lambda_N} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{x}{\lambda_N}\right)^n G\left(\frac{E}{(1-K_N)^n}\right) \frac{1}{(1-K_N)^n} \qquad (5)$$

Conditions for G(E)

Under the assumption that G(E)

A_1) is continuous, non negative and bounded in the interval $I = [a,\infty)$, $a > 0$, it was established(4) that the unique continuous solution of eq. (1) is given by (5).

A_2) The existence of the integral spectrum $\int_E^{\infty} G(E)dE$ was also required by the very nature of the physical problem.

3. In the same paper(4), we have given also the solution of the following inhomogeneous equation

$$\frac{\partial F(x,E)}{\partial x} = - \frac{1}{\lambda} F(x,E) + \frac{1}{\lambda(1-K)} F\left(x, \frac{E}{1-K}\right) + P(x,E) \qquad (6)$$

for $K < 1$, with the boundary condition

$$F(0,E) = H(E) \geq 0 \qquad (7)$$

where H(E) satisfies conditions (A_1) and P(x,E) is supposed to be a <u>continuous</u> function of (x,E) on every rectangle

$$T = \left[0 \leq x \leq X \quad , \quad a \leq E \leq b\right] \quad , a > 0,\ b > a,\ X > 0 .$$

It was shown there that for H(E) = 0 a solution of (6) continuous on T is given by

$$F(x,E) = \int_0^x e^{-\frac{1}{\lambda}(1-\sigma_K)(x-\eta)} P(\eta,E)d\eta \tag{8}$$

where the operators σ_K and $e^{\gamma\sigma_K}$ are defined as

$$\sigma_K \Gamma(x,E) = \frac{1}{1-K}\,\Gamma(x,\ \frac{E}{1-K}) \qquad K < 1 \tag{9}$$

and

$$e^{\gamma\sigma_K}\ \Gamma(x,E) = \sum_0^\infty \frac{\gamma^n}{n!}\ \sigma_K^{(n)}\ \Gamma(x,E) \tag{10}$$

where γ is a constant.

The <u>unicity</u> of the solution (8) results immediately from the fact that if $F_1(x,E)$ and $F_2(x,E)$ are both continuous solutions of $(x,E) \in T$ then the difference $F_1 - F_2$ will be <u>identically zero</u>, as the <u>unique</u> solution of the homogeneous equation

$$\frac{\partial F}{\partial x}(x,E) = -\frac{1}{\lambda}F(x,E) + \frac{1}{\lambda(1-K)}\ F(x,\ \frac{E}{1-K}) \tag{11}$$

with the condition

$$F(0,E) = \mathcal{H}(E) = 0 \tag{12}$$

4. For the integration of the equation (2) we must known explicitly the production rates $P_\pi^{\alpha N}(x,E)$, that are given by[5]

$$P_{\pi}^{\alpha N}(x,E) = \frac{1}{\lambda_{\alpha}} \int_{(E_{o\alpha})_{Min}}^{(E_{o\alpha})_{Max}} \phi_{\alpha}(E_o,E) F_{\alpha}(x,E) dE_o \qquad (13)$$

where $\phi_{\alpha}(E_o,E)dE$ is the number of charged pions with energy in the range $E, E+dE$ produced by the interaction between a hadron α of incident energy E_o, and an air nucleus. The explicit form of the functions ϕ_{α} depends on the specific model we adopt to describe the interactions.

Frequently $\phi_{\alpha}(E_o,E)$ are homogeneous functions of E and E_o, that is

$$\phi_{\alpha}(E_o,E)dE = f\left(\frac{E}{E_o}\right) \frac{dE}{E_o} \qquad (14)$$

and

$$(E_{o\alpha})_{Min} = E/B_{\alpha}$$

where B_{α} are constants.

If these conditions are verified and we put (to simplify) $(E_{o\alpha})_{Max} = \infty$, eq. (13) becomes

$$P_{\pi}^{\alpha N}(x,E) = \frac{1}{\lambda_{\alpha}} \int_{0}^{B_{\alpha}} f_{\alpha}(\eta)\, F_{\alpha}\left(x, \frac{E}{\eta}\right) \frac{d\eta}{\eta} \qquad (15)$$

where $\eta = E/E_o$.

The functions f_{α} are supposed to be known. Since $\eta = \frac{E}{E_o} \leq 1$ we have $B_{\alpha} \leq 1$.

Now introducing the operators

$$A\, F_{\pi}(x,E) = -\frac{1}{\lambda_{\pi}} F_{\pi}(x,E) + \frac{1}{\lambda_{\pi}(1-K_{\pi})} F_{\pi}\left(x, \frac{E}{1-K_{\pi}}\right) \qquad (16)$$

and

$$B\ F_\pi(x,E) = \frac{1}{\lambda_\pi} \int_0^{B_\pi} f_\pi(\eta)\ F_\pi(x, \frac{E}{\eta})\ \frac{d\eta}{\eta} \tag{17}$$

the equation (2) becomes

$$\frac{\partial F(x,E)}{\partial x} = A\ F(x,E) + P(x,E) + B\ F(x,E) \tag{18}$$

where we put to simplify $P(x,E) = P_\pi^{NN}(x,E)$ and $F_\pi(x,E) = F(x,E)$.

Equation (18) is an <u>operator equation</u> and we wish to seek a solution for it.

To solve this equation we make the following successive approximations

$$\begin{cases} \dfrac{\partial F_o}{\partial x} = A\ F_o + P(x,E) \\ \\ \dfrac{\partial F_1}{\partial x} = A\ F_1 + P_1(x,E) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{\partial F_n}{\partial x} = A\ F_n + P_n(x,E) \end{cases} \tag{19}$$

where

$$\begin{cases} P_o(x,E) = P(x,E) = P_\pi^{NN}(x,E) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ P_n(x,E) = P(x,E) + B\ F_{n-1}(x,E) \end{cases} \tag{20}$$

and

$$F_n(0,E) = 0 \qquad (n = 0, 1, 2, \ldots)\ .$$

The physical idea that suggests the above approximations is the following.

First we seek the solution $F_o(x,E)$ of the equation (2) where the pions of the second generation are not taken into account, that is when $P_\pi^{NN}(x,E) = 0$; next we seek a first order approximation putting in eq. (18) B $F_o(x,E)$ in the place of the exact term B $F(x,E)$ that is unknown. Thus we have a first estimate of the second generation pions contributions to the flux $F_\pi(x,E)$. The successive approximations are given by the recurrence equations (19) and (20). As long as the successive production of pions is not too considerable a good convergence of the process is to be expected from the physical point of view.

To solve the system (19) we may use the result indicated before, in section 3, provided the continuity of the functions $P_n(x,E)$ is verified on every rectangle (T), defined in section 3.

It can be proved (see Appendix (A,8)) that the continuity of the function $P_n(x,E)$ and $F_n(x,E)$ on (T) is guaranteed when

4a) $G(E)$ satisfies condition A_1

4b) the $f_\alpha(\eta)$ are positive and continuous for $0 \leq \eta \leq B_\alpha$ and

4c) the integrals $\int_0^{B_\alpha} \frac{f_\alpha(s)}{s} ds$ exist.

Thus, if these conditions are verified, then the unique solution of the system (19), (20), according to eq. (8) is given by

$$F_n(x,E) = \Omega\, P_n(x,E) = \int_0^x e^{-\frac{1}{\lambda_\pi}(1-\sigma_{K\pi})(x-\eta)} P_n(\eta,E)\, d\eta \qquad (21)$$

$$(n=0,1,2,\ldots)$$

where the operator Ω is defined as

$$\Omega H(x,E) = \int_0^x e^{-\frac{1}{\lambda_\pi}(1-\sigma_{K\pi})(x-\eta)} H(\eta,E)d\eta \tag{22}$$

From eq. (20) and (21) we obtain successively

$$\begin{cases} F_0(x,E) = \Omega P(x,E) \\ F_1(x,E) = \Omega P_1(x,E) = \Omega\left[1 + B\Omega\right]P(x,E) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_n(x,E) = \Omega P_n(x,E) = \Omega\left[1 + B\Omega + \ldots + (B\Omega)^n\right]P(x,E) \end{cases} \tag{23}$$

The induction from n to n+1 is easily done.

In fact, suppose eq. (23) to be valid. Then according to eq. (20) and (21) we have

$$F_{n+1} = \Omega P_{n+1} = \Omega\left[P + BF_n\right] = \Omega P + \Omega B\ F_n =$$

$$= \Omega P + \Omega B\left[\Omega(1 + B\Omega + \ldots + (B\Omega)^n)\right]P =$$

$$= \Omega\left[1 + B\Omega + (B\Omega)^2 + \ldots + (B\Omega)^{n+1}\right]P$$

Q.E.D.

5. To prove that $F_\pi(x,E) = \lim_{n\to\infty} F_n(x,E)$ exists and is the unique continuous solution of (2) with the condition $F_\pi(0,E) = 0$, there are still some steps to be considered.

First we shall prove that $(B\Omega)^n\ P = B^n\Omega^n P$ (n=1,2,...)

so that eq. (23) can be written in the following more manageable form

$$F_n = \Omega\left[1 + B\Omega + \ldots + B^n\Omega^n\right] P \tag{24}$$

The proof is given in the Appendix A.

A straightforward calculation gives

$$\Omega^n\Gamma(x,E) = \int_0^x dt_n \int_0^{t_n} dt_{n-1} \ldots \int_0^{t_2} e^{-\frac{1}{\lambda_\pi}(1-\sigma_{K\pi})(x-t_1)} \Gamma(t_1,E)dt_1 \tag{25}$$

and

$$B^n\Gamma(x,E) = \frac{1}{(\lambda_\pi)^n} \int_0^{B_\pi} \ldots \int_0^{B_\pi} f_\pi(s_n)\ldots f_\pi(s_2)f_\pi(s_1) \cdot$$

$$\cdot \Gamma\left[x, \frac{E}{s_n \ldots s_2 \cdot s_1}\right] \frac{ds_n \ldots ds_2\ ds_1}{s_n \ldots s_2 \cdot s_1} \tag{26}$$

Putting $\Gamma(x,E) = \Omega^n P(x,E)$ in (25) we get

$$B^n\Omega^n P(x,E) = \frac{1}{(\lambda_\pi)^n} \int_0^{B_\pi} \ldots \int_0^{B_\pi} f_\pi(s_n)\ldots f_\pi(s_1) \frac{ds_n \ldots ds_1}{s_n \ldots s_1} \int_0^x dt_n \cdot$$

$$\cdot \int_0^{t_n} dt_{n-1} \ldots \int_0^{t_2} dt_1\ e^{-\frac{1}{\lambda_\pi}(1-\sigma_{K\pi})(x-t_1)} P(t_1, \frac{E}{s_n \ldots s_1}) = H(x,E) \tag{27}$$

and

$$\Omega B^n\Omega^n P(x,E) = \int_0^x e^{-\frac{1}{\lambda_\pi}(1-\sigma_{K\pi})(x-t_{n+1})} H(t_{n+1},E)dt_{n+1} =$$

$$= \frac{1}{\lambda_\pi^{\,n}} \int_0^{B_\pi} \cdots \int_0^{B_\pi} f_\pi(s_n)\cdots f_\pi(s_1)\, \frac{ds_n \cdots ds_1}{s_n \cdots s_1} \int_0^{X} dt_{n+1} \int_0^{t_{n+1}} dt_n \cdots$$

$$\cdots \int_0^{t_2} e^{-\frac{1}{\lambda_\pi}(1-\sigma_{K\pi})(n-t_1)}\, P\left(t_1, \frac{E}{s_1 \cdots s_{n+1}}\right) dt_1 \qquad (28)$$

6. <u>Convergence of the succession $F_n(x,E)$ for $(x,E)\varepsilon T$</u>.

Consider the following inequalities established in the Appendix (A, 5,12)

$$B^n \Omega^n P(x,E) \le \frac{Mc_1}{\lambda_N}\, e^{(\beta_N+\beta_\pi)X} \left[\frac{c_2 X}{\lambda_\pi}\right]^n \frac{1}{n!} = W_n(X) \qquad (29)$$

for $(x,E)\ \varepsilon\ T$.

Note that the series $\Sigma\, W_n(X)$ of positive terms which are independent of $(x,E)\ \varepsilon\ T$ converges to $Mc_1/\lambda_N\, e^{(\beta_N+\beta_\pi+c_2/\lambda_\pi)X}$.

Then $\Sigma B^n \Omega^n P(x,E)$ is absolutely and uniformly convergent in T. (Weierstrass test).

Since $B^n \Omega^n P(x,E)$ are continuous functions of (x,E) on T, (see Appendix A 5d) the sum $\Sigma B^n \Omega^n P$ is also a continuous function of $(x,E)\ \varepsilon\ T$. In using the same proceeding and the inequalities A(7,6) it can be easily seen that the series of positive terms $\Sigma \Omega B^n \Omega^n P(x,E)$ is absolutely and uniformly convergent for $(x,E)\ \varepsilon\ T$.

Since the $\Omega B^n \Omega^n P(x,E)$ are continuous functions of (x,E) on T (see Appendix (A,7)) then

$$F_\pi(x,E) = \lim_{n\to\infty} \sum_0^n \Omega B^n \Omega^n P(x,E) = \lim_{n\to\infty} F_n(x,E) \qquad (30)$$

is a continuous function of (x,E) on T and the proof is complete.

7. Synthesis of the Solution

Now it is easy to show that $F(x,E) = F_\pi(x,E)$ given by eq. (30) satisfies equation (18).

In fact integrating both sides of eq. (18) and taking into account the condition $F(0,E) = 0$, we have the equivalent functional equation

$$F_\pi(x,E) = \int_0^x A\, F_\pi(\lambda,E)d\lambda + \int_0^x P(\lambda,E)d\lambda + \int_0^x B\, F_\pi(\lambda,E)d\lambda \qquad (31)$$

But from the system (19)(20) we have

$$\frac{\partial F_n}{\partial x} = A\, F_n + P_n(x,E) \qquad (32)$$

Integrating both members of this equation with the condition $F_n(0,E) = 0$ we get

$$F_n(x,E) = \int_0^x A\, F_n(\lambda,E)d\lambda + \int_0^x P_n(\lambda,E)d\lambda =$$

$$= \int_0^x A\, F_n(\lambda,E)d\lambda + \int_0^x \left[P(\lambda,E)+B\, F_{n-1}(\lambda,E)\right]d\lambda$$

Hence

$$F(x,E) = F_\pi(x,E) = \lim_{n\to\infty} F_n(x,E) = \int_0^x A\, F(\lambda,E)d\lambda +$$

$$+ \int_0^x P(\lambda,E)d\lambda + \int_0^x B\, F(\lambda,E)d\lambda \qquad (33)$$

Because the continuous functions $AF_n(\lambda,E)$ and $BF_{n-1}(\lambda,E)$ tend uniformly in T to the continuous functions $AF(\lambda,E)$ and $BF(\lambda,E)$ respectively.

The last equation (33) proves that $F(x,E)$ satisfies equation (31) and the proof is complete.

8. Uniqueness of the Solution

Suppose that there are two solutions F'_π and F''_π of the equation (2), both continuous on every rectangle

$$T = \left[0 \leq x \leq X \quad , \quad a \leq E \leq b\right] \quad , \; a > 0, \; b > a, \; X > 0 \quad ,$$

and both satisfying the same boundary conditions $F'_\pi(x,0) = 0$ and $F''_\pi(x,0) = 0$.

Then the difference $\phi(x,E) = F'_\pi(x,E) - F''_\pi(x,E)$ must satisfy the homogeneous equation

$$\phi(x,E) = \int_0^x (A+B)\phi(t,E)dt \tag{34}$$

Now substituting iteratively the function $\phi(t,E)$ under the sign of integration by means of (34) we have successively

$$\phi(x,E) = \int_0^x (A+B)\phi(t,E)dt = \int_0^x (A+B)dt_2 \int_0^{t_2} (A+B)\phi(t_1,E)dt_1 =$$

$$= \int_0^x dt_n \int_0^{t_n} dt_{n-1} \ldots \int_0^{t_2} (A+B)^n \phi(t_1,E)dt_1 \tag{35}$$

where

$$(A+B)\phi(t_1,E) = -\frac{1}{\lambda_\pi}\phi(t_1,E) + \frac{1}{\lambda_\pi(1-K_\pi)}\phi(t_1, \frac{E}{1-K_\pi}) +$$

$$+ \frac{1}{\lambda_\pi}\int_0^{B_\pi} f_\pi(s)\phi(t_1, \frac{E}{s}) \frac{ds}{s} \tag{36}$$

Since $F'_\pi(x,E)$ and $F''_\pi(x,E)$ are both continuous in every rectangle (T), the function $\phi(x,E)$ is also continuous in (T) and $\phi(x, \frac{E}{1-K})$ is continuous in the rectangle

$$T' = \left[0 \leq x \leq X, \quad \frac{a}{1-K_\pi} \leq E \leq \frac{b}{1-K_\pi}\right], \quad a > 0, \quad b > a$$

Let M_1 be the maximum of $\phi(x,E)$ in the rectangle

$$T'' = \left[0 \leq x \leq X, \quad a \leq E \leq \frac{b}{1-K_\pi}\right], \quad a > 0, \quad b > a$$

Thus we have

$$|A\phi| \leq \frac{2M_1}{\lambda_\pi(1-K_\pi)} \qquad (x,E) \in T'' \tag{37}$$

On the other side we note that $\psi(t_1,E) = B\phi(t_1,E)$ is continuous in every rectangle T. (The proof is the same as that given in the Appendix A, for P(x,E)). Let M_2 be the maximum of $\psi(t_1,E)$ in T".

Thus we have

$$|(A+B)\phi(t_1,E)| \leq |A\phi(t_1,E)| + |B\phi(t_1,E)| \leq M \tag{38}$$

where

$$M = \frac{2M_1}{\lambda_\pi(1-K_\pi)} + M_2, \qquad (t_1,E) \in T'' \tag{39}$$

then

$$(A+B)((A+B)\phi(t_1,E)) \le |A(M) + B(M)| \le$$

$$\le \frac{2M}{\lambda_\pi(1-K_\pi)} + \frac{M}{\lambda_\pi} \int_0^{B_\pi} f_\pi(s) \frac{ds}{s} =$$

$$= (\frac{2}{\lambda_\pi(1-K_\pi)} + \frac{c2}{\lambda_\pi})M$$

Therefore

$$(A+B)^{(2)}\phi(t_1,E) \le NM \quad , (t_1,E) \in T'' \tag{40}$$

where

$$N = \frac{2}{\lambda_\pi(1-K_\pi)} + \frac{c2}{\lambda_\pi} \tag{41}$$

By induction we get

$$(A+B)^{(n)}\phi(t_1,E) \le N^{n-1}M \quad , \quad (t_1,E) \in T'' \tag{42}$$

Now from eq. (35) and (42) we have

$$|\phi(x,E)| \le \int_0^X dt_n \int_0^{t_n} dt_{n-1} \ldots \int_0^{t_2} (A+B)^{(n)}\phi(t_1,E)dt_1$$

$$= MN^{n-1} \frac{X^{n-1}}{(n-1)!} \le M \frac{(NX)^{n-1}}{(n-1)!} \quad , \quad (x,E) \in T''$$

Letting $n \to \infty$ (with X fixed) it results that $|\phi(x,E)| \to 0$. Therefore $F'_\pi(x,E) = F''_\pi(x,E)$ for $(x,E) \in T''$. Since T" is arbitrary the proof is complete.

9. Taking Account the Equalities

$$(B\Omega)^n P = B^n \Omega^n P = \Omega^n B^n P \quad \text{for } n = 1, 2, \ldots$$

established in the Appendix A(6,1) and A(6,2) and the equality $F_o(x,E) = \Omega P(x,E)$, we can write

$$F_n(x,E) = \Omega\left[1+B\Omega + \ldots + B^n\Omega^n\right]P =$$

$$= \left[1+B\Omega + \ldots + B^n\Omega^n\right]F_o \qquad (43)$$

and therefore

$$F_\pi(x,E) = \lim_{n\to\infty} F_n(x,E) = \sum_0^\infty (B\Omega)^n F_o(x,E). \qquad (44)$$

Equation (44) shows how the solution $F_\pi(x,E)$ which take into account the pions of the second generation can be constructed starting from the special case in which only the pions produced by the interaction $N \to N$ are taken into consideration

This conclusion was stated in ref. (4) without proof.

Conclusion. The main result of this paper can be stated as follows.

Theorem. The diffusion equations

$$\frac{\partial F_N(x,E)}{\partial x} = -\frac{1}{\lambda_N} F_N(x,E) + \frac{1}{\lambda_N(1-K_N)} F_N(x, \frac{E}{1-K_N}) \qquad (1)$$

$$\frac{\partial F_\pi(x,E)}{\partial x} = -\frac{1}{\lambda_\pi} F_\pi(x,E) + \frac{1}{\lambda_\pi(1-K_\pi)} F_\pi(x, \frac{E}{1-K_\pi}) +$$

$$+ P_\pi^{NN}(x,E) + P_\pi^{\pi N}(x,E) \qquad (2)$$

where the unknown $F_N(x,E)$ and $F_\pi(x,E)$ are supposed to satisfy the boundary conditions

$$F_N(0,E) = G(E) \quad \text{and} \quad F_\pi(0,E) = 0$$

admit of the <u>unique</u> <u>continuous</u> solutions given by

$$F_N(x,E) = e^{-x/\lambda_N} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{x}{\lambda_N}\right)^n G\left(\frac{E}{(1-K_N)^n}\right) \frac{1}{(1-K_N)^n}$$

$$F_\pi(x,E) = \sum_{n=0}^{\infty} \Omega B^n \Omega^n \; P_\pi^{NN}(x,E)$$

provided that:

a) $G(E)$ is continuous, non negative and bounded in the interval $I = [a,\infty)$, $a > 0$;

b) the functions $P_\pi^{\alpha N}(x,E)$, $(\alpha = N,\pi)$, are given by the integrals

$$P_\pi^{\alpha N}(x,E) = \frac{1}{\lambda_\alpha} \int_0^{B_\alpha} f_\alpha(\eta) F_\alpha\left(x, \frac{E}{\eta}\right) \frac{d\eta}{\eta}$$

where $0 < B_\alpha \leq 1$;

c) the functions $f_\alpha(\eta)$ are positive and continuous on the interval $0 \leq \eta \leq B_\alpha$;

d) the integrals $\int_0^{B_\alpha} \frac{f_\alpha(s)}{s} ds$ exist.

The continuity of the $F_\alpha(x,E)$ is verified <u>on</u> every rectangle $T = [0 \leq x \leq X;\ a \leq E \leq b]$, where X and b can be arbitrarily choosen provided $X \geq 0$ and $b \geq a > 0$.

The terms $\Omega B^n \Omega^n P(x,E)$ are given by

$$\Omega B^n \Omega^n P(x,E) = \frac{1}{\lambda_\pi n} \int_0^{B_\pi} \cdots \int_0^{B_\pi} \frac{f_\pi(s_1)\ldots f_\pi(s_n)}{s_1 \ldots s_n} \int_0^{X} dt_{n+1} \cdots$$

$$\cdots \int_0^{t_2} dt_1 \; e^{-\frac{1}{\lambda_\pi}(1-\sigma_{K\pi})(x-t_1)} \; P\left(t_1, \frac{E}{s_1 \ldots s_n}\right) .$$

APPENDIX A

Design by T the rectangle

$$T = \left[0 \leq x \leq X \;,\; a \leq E \leq b\right] \;,\; a > 0 \;,\; X > 0 \;,\; b > a$$

A 1) - Continuity of $F_N(x,E)$, for $(x,E) \in T$.

Proof - We have from equation (5)

$$F_N(x,E) = \sum_0^\infty w_n(x,E) = \sum_0^\infty a_n(x)v_n(E) \quad . \qquad A(1.1)$$

where

$$a_n(x) = \frac{e^{-x/\lambda_N}}{n!} \left[\frac{x}{\lambda_N(1-K_N)}\right]^n \qquad A(1.2)$$

and

$$v_n(E) = G \left[\frac{E}{(1-K_N)^n}\right] \qquad K_N < 1 \qquad A(1.3)$$

The series of the right side of (1.1) is a series of positive terms.

Since $G(E) \leq M$ for $E \geq a > 0$, we have for any point (x,E) in T

$$w_n(x,E) \leq \frac{1}{n!} \left[\frac{X}{\lambda_N(1-K_N)^n}\right]^n \times M = M_n(X)$$

Note that the series of positive terms $M_n(X)$, which are independent of (x,E), converges to

$$\sum_0^\infty M_n(X) = M\, e^{\frac{X}{\lambda_N(1-K_N)}} = M\, e^{\beta_N X} \qquad \beta_N = \frac{1}{\lambda_N(1-K_N)} \qquad A(1.4)$$

Hence the series $\Sigma\, w_n(x,E)$ is uniformly and absolutely convergent in T. (Weierstrass test for uniform convergence).

Since G(E) is continuous for $E \geq a$, so also are $G\left(\frac{E}{(1-K_N)^n}\right)$ and $v_n(E)$. (Note that $K_N < 1$.)

Clearly the $a_n(x)$ are continuous functions of x, in T.

The product $w_n(x,E) = a_n(x)v_n(E)$ are continuous functions of (x,E) in T., as we can easily verify. Therefore the partial sum $S_N(x,E) = \sum_{n=o}^{N} w_n(x,E)$ converge to a function $F_N(x,E)$ continuous in T, that is

$$F_N(x,E) = \lim_{N\to\infty} S_N(x,E) = \sum_{n=1}^{\infty} w_n(x,E)$$

Q.E.D.

A,2) - Existence of $P(x,E) = P_{\pi}^{NN}(x,E)$

If $f(\eta)$ is a positive and continuous functions of η in the interval $0 \leq \eta \leq B_N$ and satisfies the condition

$$\int_{0}^{B_N} f_N(\eta)\,\frac{d\eta}{\eta} = C_1 \qquad A(2.1)$$

then P(x,E) exists.

Proof. In fact, for $\varepsilon > 0$ we consider the function

$$P_\varepsilon(x,E) = \frac{1}{\lambda_N}\int_{\varepsilon}^{B_N} \frac{f_N(\eta)}{\eta}\, F_N\left[x, \frac{E}{\eta}\right] d\eta$$

$$= \frac{1}{\lambda_N}\int_{\varepsilon}^{B_N} \frac{f_N(\eta)}{\eta} \sum_{0}^{\infty} a_n(x)\, G\left[\frac{E/\eta}{(1-K_N)^n}\right] d\eta \qquad A(2.2)$$

For any point (x,E) fixed in T, the integrand of the right side of A(2.2) is an <u>uniformly</u> and <u>absolutely</u> convergent series of continuous functions of η in $0 < \varepsilon \leq \eta \leq B_N$.

Thus we may write

$$P_\varepsilon(x,E) = \frac{1}{\lambda_N} \sum_0^\infty a_n(x) \int_\varepsilon^{B_N} \frac{f_N(\eta)}{\eta} \; G\left[\frac{E/\eta}{(1-K_N)^n}\right] d\eta \qquad A(2.3)$$

$$= \frac{1}{\lambda_N} \cdot \sum_0 a_n(x) \; d_n(E,\varepsilon) \qquad A(2.4)$$

But for every $0 < \varepsilon \leq B_N$ we have

$$b_n(E,\varepsilon) = \int_\varepsilon^{B_N} \frac{f_N(\eta)}{\eta} \, G\left[\frac{E/\eta}{(1-K_N)^n}\right] d\eta \leq M \int_0^{B_N} \frac{f_N(\eta)}{\eta} \, d\eta = MC_1 \qquad A(2.5)$$

Hence

$$P_\varepsilon(x,E) \leq \frac{MC_1}{\lambda_N} \sum_0^\infty a_n(x) \leq \frac{MC_1}{\lambda_N} \cdot e^{\frac{X}{\lambda_N(1-K_N)}} \qquad A(2.6)$$

for $(x,E) \varepsilon T$.

Now observe that, for (x,E) <u>fixed</u> in T, the function $\phi(\varepsilon) = P_\varepsilon(x,E)$ is <u>positive</u> <u>monotone</u> not <u>decreasing</u> and <u>bounded</u>, for $0 < \varepsilon \leq B_N$.

Hence exists the limit

$$\lim_{\varepsilon\to 0} P_\varepsilon(x,E) = \lim_{\varepsilon\to 0} \frac{1}{\lambda_N} \int_\varepsilon^{B_N} \frac{f_N(\eta)}{\eta} F_N(x, \frac{E}{\eta}) \, d\eta$$

$$= \frac{1}{\lambda_N} \int_0^{B_N} \frac{f_N(\eta)}{\eta} F_N(x, \frac{E}{\eta}) \, d\eta = P(x,E)$$

Q.E.D.

The same reasoning applies to the functions $b_n(E,\varepsilon)$ and proves the existence of the following limits

$$b_n(E) = \lim_{\varepsilon\to 0} b_n(E,\varepsilon) = \int_0^{B_N} \frac{f_N(\eta)}{\eta} G\left[\frac{E/\eta}{(1-K_N)^n}\right] \cdot d\eta \qquad A(2.7)$$

for n = 0,1,2... .

A,3) - Continuity of $P(x,E) = P_\pi^{NN}(x,E)$ for $(x,E) \varepsilon T$.

We have from relations (15), (1.1), (1.2) and (1.3)

$$P_\pi^{NN}(x,E) = \frac{1}{\lambda_N}\int_0^{B_N} f_N(s)\cdot F_N(x, \frac{E}{s}) \frac{ds}{s} =$$

$$= \frac{1}{\lambda_N}\int_0^{B_N} f_N(s)\ \Sigma a_n(x) v_n(\frac{E}{s}) \frac{ds}{s} \qquad A(3.1)$$

where

$$v_n(\frac{E}{s}) = G\left[\frac{E/s}{(1-K_N)^n}\right] \qquad K_N < 1$$

Since $K_N < 1$, $0 \leq B_N \leq 1$ and $G(E) \leq M$ for $E \geq a > 0$, we have for $0 \leq s \leq B_N$

$$\frac{E/s}{(1-K_N)^n} \geq a/B_N \geq a$$

$$v_n(\frac{E}{s}) \leq M$$

and

$$W_n(x, \frac{E}{s}) = a_n(x) v_n(\frac{E}{s}) \leq \frac{M}{n!}\left[\frac{X}{\lambda_N(1-K_N)}\right]^n = M_n(X)$$

But the series $\Sigma M_n(X)$ of positive terms $M_n(X)$, which are independent of (x,E) for (x,E) in T, converges.

Then $\Sigma W_n(x, \frac{E}{s})$ is absolutely and uniformly convergent on the region $0 < \varepsilon \leq s \leq B_N$, $E \geq a > 0$.

Now consider the integral

$$I_\varepsilon = \frac{1}{\lambda_N} \int_\varepsilon^{B_N} \frac{f_N(s)}{s} \Sigma a_n(x) v_n(\frac{E}{s})\, ds$$

Since a series of continuous and positive functions which sum is integrable may be integrated term by term, we have

$$I_\varepsilon = \frac{1}{\lambda_N} \Sigma a_n(x) \int_\varepsilon^{B_N} \frac{f_N(s)}{s} v_n(\frac{E}{s})\, ds$$

for all values of $0 < \varepsilon \leq B_N$.

But

$$\frac{f_N(s)}{s} a_n(x) v_n(\frac{E}{s}) \geq 0 \quad , \; s > 0$$

therefore

$$P(x,E) = \frac{1}{\lambda_N} \int_0^{B_N} \frac{f_N(s)}{s} \Sigma\, a_n(x) v_n(\frac{E}{s}) ds = \frac{1}{\lambda_N} \Sigma a_n(x) \int_0^{B_N} \frac{f_N(s)}{s} v_n(\frac{E}{s}) ds \qquad A(3.2)$$

<u>provided</u> that either side of (3.2) is convergent.

Since the first side of (3.2) exists there we have

$$P(x,E) = P_\pi^{NN}(x,E) = \frac{1}{\lambda_\pi} \Sigma\, a_n(x) \int_0^{B_N} \frac{f_N(s)}{s} G(\frac{E/s}{(1-K_N)^n}) ds =$$

$$= \frac{1}{\lambda_\pi} \sum_n a_n(x) b_n(E) \qquad A(3.3)$$

where

$$b_n(E) = \int_0^{B_N} \frac{f_N(s)}{s} G\left[\frac{E/s}{(1-K_N)^n}\right] ds$$

<u>Prop.</u> <u>$b_n(E)$ is a continuous functions of E, for $E \geq a > 0$</u> .

Proof. We can write

$$b_n(E) = \int_0^{B_N} \frac{f_N(s)}{s} \cdot G\left[\frac{E/s}{(1-K_N)^n}\right] ds = \int_0^{B_N} g_n(s,E)ds \qquad A(3.4)$$

Since $G(E) \leq M$ for $E \geq a > 0$, $0 \leq s \leq B_N \leq 1$, $\frac{1}{(1-K_N)^n} \geq 1$ for $K_N < 1$ and $n = 0,1,2,\ldots$, we have $g_n(s,E) \leq \mu(s) =$ $= M f_N(s)/s$. But the integral $\int_0^{B_N} \frac{f_N(s)}{s} ds$ was assumed to exist, then the integral $b_n(E)$ converges uniformly, whatever be $E \geq a > 0$.

Note that $g_n(s,E)$ is continuous for $0 < s \leq B_N$ and $E \geq a > 0$.

Thus, for $0 < X \leq B_N$, we have

$$\left|b_n(E+h) - b_n(E)\right| \leq \left|\int_X^{B_N} \left\{g_n(s,E+h) - g_n(s,E)\right\} ds\right| +$$

$$+ \left|\int_0^X g_n(s,E+h)ds\right| + \left|\int_0^X g_n(s,E)ds\right| \qquad A(3.5)$$

Now, given $\varepsilon > 0$, we can choose $0 < X < B_N$ such that $\int_0^X \frac{f_N(s)}{s} ds <$ $< \frac{\varepsilon}{2M}$, and thus the sum of the last two terms of (3.5) will be less than $\frac{\varepsilon}{2}$.

Moreover, for fixed X, since $g_n(s,E)$ is continuous in T, we can choose h such that $|g(s,E+h)-g(s,E)| < \frac{\varepsilon}{2(B_N-X)}$, so that $|b_n(E+h)-b_n(E)| \leq \varepsilon$.

This proves the continuity of $b_n(E)$.

To proves the continuity of $P(x,E)$ in (T) it suffices to note that

a) The functions $a_n(x)$ and $b_n(E)$ are positive and continuous functions of its arguments, for $x \geq 0$, $E \geq a$;

b) The product $a_n(x)$, $b_n(E)$ is continuous in (T);

c) The series of positive terms $\frac{1}{\lambda_\pi} \Sigma a_n(x) b_n(E)$ is uniformly convergent in (T) because we have

$$|a_n(x)b_n(E)| \leq \frac{1}{n!}\left[\frac{X}{\lambda_N(1-K_N)}\right]^n \cdot M \int_0^{B_N} \frac{f_N(s)}{s} ds$$

$$= \frac{MC_1}{n!}\left[\frac{X}{\lambda_N(1-K_N)}\right]^n = M'_n(X) \qquad A(3.6)$$

and the series of positive terms M'_n which are independent of $(x,E) \varepsilon T$ converges in T. to $MC_1 e^{\beta_N X}$, where $\beta_N = \frac{1}{\lambda_N(1-K_N)}$.

A,4) - <u>Continuity of $\Omega^n P(x,E)$ for $(x,E) \varepsilon T$</u>

<u>Proof</u>. The function P(x,E) is given by the series

$$P(x,E) = \frac{1}{\lambda_N} \Sigma_\nu a_\nu(x)\, b_\nu(E) \qquad A(4.1)$$

This series of positive functions continuous on (T) is uniformly convergent in (T).

A simple calculation gives

$$\Omega^n P(x,E) = \int_0^x dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} e^{-\frac{1}{\lambda_\pi}(1-\sigma_{K\pi})(x-t_1)} P(t_1,E)dt_1 =$$

$$= \int_0^x dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} e^{-\frac{1}{\lambda_\pi}(1-\sigma_{K\pi})t} P(x-t,E)dt =$$

$$= \int_0^x dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} e^{-\frac{t}{\lambda_\pi}} \sum_m \frac{1}{m!} \left[\frac{t}{\lambda_\pi(1-K_\pi)}\right]^m P(x-t, \frac{E}{(1-K_\pi)^m})dt =$$

$$= \frac{1}{\lambda_N} \sum_{m\nu} b_\nu \left(\frac{E}{(1-K_\pi)^m}\right) \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} e^{-\frac{t}{\lambda_\pi}} \frac{1}{m!} \left[\frac{t}{\lambda_\pi(1-K_\pi)}\right]^m a_\nu(x-t)dt =$$

$$= \frac{1}{\lambda_N} \sum_{m\nu} a_{n,m\nu}(x) b_\nu\left(\frac{E}{(1-K_\pi)^m}\right) = \frac{1}{\lambda_N} \sum_{m\nu} a_{n,m\nu}(x) b_{m\nu}(E) \qquad A(4.2)$$

where

$$a_{n,m\nu}(x) = \int_0^x e^{-\frac{t}{\lambda_\pi}} \frac{(x-t)^{n-1}}{(n-1)!} \frac{1}{m!} \left[\frac{t}{\lambda_\pi(1-K_\pi)}\right]^m a_\nu(x-t)dt \qquad A(4.3)$$

and

$$b_{m\nu}(E) = b_\nu\left(\frac{E}{(1-K_\pi)^m}\right) \qquad A(4.4)$$

We beginn to prove the uniform convergence of the series A(4.2) in (T).

We have from A(4.3)

$$a_{n,m\nu}(x) = \int_0^x e^{-\left(\frac{x-t}{\lambda_\pi}\right)} \frac{t^{n-1}}{(n-1)!} \frac{1}{m!} \left[\frac{x-t}{\lambda_\pi(1-K_\pi)}\right]^m a_\nu(t)dt$$

From A(1.2) and A(4.3) we get

$$|a_{n,m\nu}(x)| \le \int_0^X \frac{t^{n-1}}{(n-1)!} \frac{1}{m!} \left[\frac{X}{\lambda_\pi(1-K_\pi)}\right]^m \frac{1}{\nu!} \left[\frac{X}{\lambda_N(1-K_N)}\right]^\nu dt =$$

$$= \frac{X^n}{n!} \frac{1}{m!} \left[\frac{X}{\lambda_\pi(1-K_\pi)}\right]^m \frac{1}{\nu!} \left[\frac{X}{\lambda_N(1-K_N)}\right]^\nu = \frac{X^n}{n!} R_{m\nu}$$

for $0 \leq x \leq X$.

From A(4.4) and eq. (2.5) we have

$$|b_{m\nu}(E)| = |b_\nu(\frac{E}{(1-K_\pi)^m})| \leq MC_1 \quad \text{for} \quad E \geq a > 0 \qquad A(4.4)$$

Hence

$$\frac{1}{\lambda_N} |a_{n,m\nu}(x) b_{m\nu}(E)| \leq \frac{MC_1}{\lambda_N} \frac{X^n}{n!} R_{m\nu}$$

But the series $\sum_{m\nu} R_{m\nu}$ of positive terms which are independent of (x,E) in (T) converges in (T) to $e^{(\beta_\pi+\beta_N)X}$, where $\beta_\alpha = \frac{1}{\lambda_\alpha(1-K_\alpha)}$ $\alpha = (\pi,N)$.

Therefore the series A(4.2) is uniformly and absolutely convergent in (T).

Since its terms are easily seen to be continuous in (T) so is its sum $\Omega^n P(x,E)$.

A,5) - Continuity of $B^n\Omega^n P(x,E)$ in (T)

a) Consider the integral

$$I_{\varepsilon_1,\ldots,\varepsilon_n}(x,E) = \frac{1}{\lambda_\pi^{\,n}} \int_{\varepsilon_n}^{B_\pi} \cdots \int_{\varepsilon_1}^{B_\pi} \frac{f_\pi(s_n)}{s_n} \cdots \frac{f_\pi(s_1)}{s_1} ds_n \ldots ds_1 \Big[\Omega^n P(x, \frac{E}{s_n \ldots s_1}) \Big] = \frac{1}{\lambda_N} \frac{1}{\lambda_\pi^{\,n}} \int_{\varepsilon_n}^{B_\pi} \cdots \int_{\varepsilon_1}^{B_\pi} \frac{f_\pi(s_n)}{s_n} \cdots \frac{f_\pi(s_1)}{s_1} \cdot$$

$$\cdot \sum_{m\nu} a_{n,m\nu}(x)\, d_{m\nu}(\frac{E}{s_n \ldots s_1})\, ds_n \ldots ds_1 \qquad A(5.1)$$

where the ε_j are positive numbers, arbitrarily choosen in the intervals $0 \leq \varepsilon_j \leq B_\pi \leq 1$.

Since $b_n(E)$ is positive and continuous for $E \geq a > 0$ so is

$$b_{m\nu}\left(\frac{E}{s_1 \ldots s_n}\right) = b_\nu\left(\frac{E}{s_1 \ldots s_n (1-K_\pi)^n}\right) ,$$

because

$$\frac{E}{s_1 \ldots s_n (1-K_\pi)^n} \geq \frac{a}{s_1 \ldots s_n (1-K_\pi)^n} \geq a > 0 .$$

The function $\frac{f_\pi(s)}{s}$ is positive and continuous for $\varepsilon_i \leq s \leq B_\pi$, and the functions $a_{n,m\nu}(x)$ are also positive and continuous for $0 \leq x \leq X$. The series that appears in the integrand of A(5.1) is easily seen to be absolutely and uniformly convergent in (T).

The proof is the same as that given before because we have

$$|a_{n,m\nu}(x)| \leq \frac{X^n}{n!} R_{m\nu} \qquad A(5.2)$$

and

$$\left|b_{m\nu}\left(\frac{E}{s_n \ldots s_1}\right)\right| = \left|b_\nu\left(\frac{E}{s_1 \ldots s_n (1-K_\pi)^n}\right)\right| \leq MC_1 \text{ for } E \geq a > 0 \qquad A(5.3)$$

Thus we can perform term-by-term the integrations that figure in A(5.1) and we get

$$I_{\varepsilon_1, \ldots, \varepsilon_n}(x,E) = \frac{1}{\lambda_N} \frac{1}{(\lambda_\pi)^n} \sum_{m\nu} a_{n,m\nu}(x)\, b_{n,m\nu,\varepsilon_1 \ldots \varepsilon_n}(E) \qquad A(5.4)$$

where

$$b_{n,m\nu,\varepsilon_1\ldots\varepsilon_n}(E) = \int_{\varepsilon_n}^{B_\pi}\ldots\int_{\varepsilon_1}^{B_\pi} \prod_1^n \frac{f_\pi(s_j)}{s_j}\, b_\nu\left(\frac{E}{s_1\ldots s_n(1-K_\pi)^m}\right)ds_1\ldots ds_n \qquad A(5.5)$$

b) Now we shall prove that, if the $\varepsilon_j \to 0$, then $b_{n,m\nu,\varepsilon_1\ldots\varepsilon_n}(E)$ tends uniformly to

$$b_{n,m\nu}(E) = \int_0^{B_\pi}\ldots\int_0^{B_\pi} \prod_1^n \frac{f_\pi(s_j)}{s_j}\, b_\nu\left(\frac{E}{s_1\ldots s_n(1-K_\pi)^n}\right) ds_1\ldots ds_n \qquad A(5.6)$$

In fact according to eq. (2.5) we have $b_\nu(E) \leq MC_1$ for $E \geq a > 0$. Moreover the positive integrand of eq. A(5.5) is less or equal to the positive function

$$MC_1 \prod_1^n \frac{f_\pi(s_i)}{s_i} = \phi(s_1\ldots s_n)$$

The integral

$$\int_0^{B_\pi} \frac{f_\pi(s)}{s}\, ds = c_2$$

was supposed to exist according to the condition c of the section 3.

Therefore the integral

$$\int_0^{B_\pi}\ldots\int_0^{B_\pi} \phi(s_1,\ldots,s_n)ds_1\ldots ds_n = Mc_1c_2^n$$

exists.

This is sufficient to guarantee the absolute and uniform convergence of the integral A(5.6).

Taking into account this last result and eq. A(5.4) we have

$$\beta^n \Omega^b P(x,E) = \lim_{\varepsilon_j \to 0} I_{\varepsilon_1 \ldots \varepsilon_n}(x,E) = \frac{1}{\lambda_N} \frac{1}{(\lambda_\pi)^n} \sum_{m\nu} a_{n,m\nu}(x)\, b_{n,m\nu}(E) \qquad A(5.7)$$

where $a_{n,m\nu}(x)$ and $b_{n,m\nu}(E)$ are given by eq. A(4.3) and A(5.6) respectively.

c) The functions $b_{n,m\nu}(E)$ are continuous in T

The proof is an abvious extension to a multiple integral of the proof given in Section 3, of this Appendix. In fact, the integral

$$b_{n,m\nu}(E) = \int_0^{B_\pi} \cdots\cdots \int_0^{B_\pi} g_{n,m\nu}(s_i,E)\, ds_1 \ldots ds_n$$

where

$$g_{n,m\nu}(s_j,E) = \prod_1^n \frac{f_\pi(s_j)}{s_j}\, b_\nu \left(\frac{E}{s_1 \ldots s_n (1-K_\pi)^m}\right)$$

is uniformly convergent for $E \geq a > 0$, and $g_{n,m\nu}(s_i,E)$ is continuous for $0 < S_j \leq B$, $E \geq a > 0$.

Thus, for $0 < X_j \leq B_\pi$ we have

$$\left| b_{n,m\nu}(E+h) - b_{n,m\nu}(E) \right| \leq \int_{X_1}^{B_\pi} \cdots \int_{X_n}^{B_\pi} \left\{ g_{n,m\nu}(s_j,E+h) - g_n(s_j,E) \right\} ds_1 \ldots ds_n + \left| \int_0^{X_1} \cdots \int_0^{X_n} g_{n,m\nu}(s_j,E+h) ds_1 \ldots ds_n \right| + \left| \int_0^{X_1} \cdots \int_0^{X_n} g_{n,m\nu}(s_j,E) ds_1 \ldots ds_j \right| \qquad A(5.8)$$

Taking into account eq. A(4.4) we have

$$\left|g_{n,m\nu}(s_j,E)\right| \leq M\ c_1 c_2^n \quad \text{for } 0 < s_j \leq B_\pi \ ;\ E \geq a > 0 \ .$$

Now we can choose the X_j so that the second and third integrals of the right side of the inequality A(5.8) are less than $\frac{\varepsilon}{3Mc_1c_2^n}$. Moreover for fixed X_j since $g_{n,m\nu}(s_j,E)$ is continuous in T we can choose h > 0 such

$$\left|g_{n,m\nu}(s_j,E+h) - g_{n,m\nu}(s_j,E)\right| < \frac{\varepsilon}{3 \prod_1^n (B_\pi - X_i)}$$

This proves the continuity of $b_{n,m}(E)$ in T.

Since $a_{n,m\nu}(x)$ and $b_{n,m\nu}(E)$ are continuous functions of its arguments for $(x,E) \in T$ it is easily seen that the product $a_{n,m\nu}(x) . b_{n,m\nu}(E)$ is continuous as a function of $(x,E) \in T$.

d) Absolute and uniform convergence (in T) of the series A(5.7)

From A(5.2) we have

$$\left|a_{n,m\nu}(x)\right| \leq \frac{X^n}{n!} R_{m\nu} \qquad A(5.9)$$

and from A(5.3) and A(5.6) we have

$$\left|b_{n,m\nu}(E)\right| \leq M\ c_1 c_2^n \qquad A(5.10)$$

From A(5.9) and A(5.10) we see that the series A(5.7) of positive terms which represents $B^n \Omega^n P(x,E)$ converges absolutely and uniformly in T, because we have

$$\frac{1}{\lambda_N} \frac{1}{(\lambda_\pi)^n}\ a_{n,m\nu}(x)\ b_{n,m\nu}(E) \leq \frac{M\ c_1\ c_2^n}{\lambda_N (\lambda_\pi)^n} \frac{X^n}{n!} R_{m\nu} \qquad A(5.11)$$

Hence

$$B^n \Omega^n P(x,E) \leq \frac{Mc_1}{\lambda_N} \left(\frac{c_2 X}{\lambda_\pi}\right)^n \frac{1}{n!} \sum_{m\nu} R_{m\nu} = \frac{Mc_1}{\lambda_N} e^{(\beta_\pi + \beta_N)X} . \left(\frac{c_2 X}{\lambda_\pi}\right)^n \frac{1}{n!}$$

A(5.12)

Because the series $\sum_{m\nu} R_{m\nu}$ of positive terms which are independent of (x,E) in T converges to $e^{(\beta_\pi+\beta_N)X}$, where $\beta_\alpha = \frac{1}{\lambda_\alpha(1-K_\alpha)}$ $\alpha = (\pi,N)$.

Since the products $a_{n;m\nu}(x)\cdot b_{n,m\nu}(E)$ are <u>continuous</u> in T, so is $B^n\Omega^n P(x,E)$ as the sum of an <u>uniformly</u> and <u>absolutely</u> series of continuous functions in T.

A,6) - <u>$B^n\Omega^n P(x,E) = \Omega^n B^n P(x,E)$ for $n = 1,2,\ldots$</u>

<u>Proof</u>. Note that in eq. A(5.1) the order of integrations can be changed so that

$$I_{\varepsilon_1,\ldots,\varepsilon_n}(x,E) = \frac{1}{\lambda_\pi^n}\,\Omega^n \int_{\varepsilon_n}^{B_\pi}\cdots\int_{\varepsilon_1}^{B_\pi} \frac{f_\pi(s_n)}{s_1}\cdots\frac{f_\pi(s_1)}{s_1}\,P\left(x,\frac{E}{s_n\ldots s_1}\right)ds_1\ldots ds_n \qquad A(6.1)$$

The equality A(6.1) is verified whatever be the positive numbers $0 < \varepsilon_j \leq B_\pi$. Therefore both sides of eq. A(6.1) which are positive tend to the same limit when $\varepsilon_j \to 0$, provided either limit exists. But the limit of the lift side of A(6.1) which is $B^n\Omega^n(x,E)$ exists as was shown before.

Hence the limit of the right side of A(6.1) exists also. But this limit is $\Omega^n B^n P(x,E)$. The proof is complete.

The inequality

$$B^n\Omega^n P(x,E) = (B\Omega)^n\, P(x,E) \qquad A(6.2)$$

is easily verified, taking into account that the order of integration in A(6.1) is irrelevant.

A,7) - Proof an inequality

We have.

$$\Omega B^n\Omega^n P(x,E) = \Omega B\cdot B^{n-1}\Omega^{n-1}\Omega P(x,E) \qquad A(7.1)$$

But according to section 6 of this Appendix

$$B^m\Omega^m P(x,E) = \Omega^m B^m P(x,E) \qquad A(7.2)$$

From A(7.1) and A(7.2) we get

$$\Omega B^n\Omega^n P(x,E) = B^n\Omega^n\cdot\Omega P(x,E) = B^n\Omega^{n+1}P(x,E) \qquad A(7.3)$$

But from A(4.3) and A(4.4) we have

$$\Omega^{n+1}P(x,E) = \frac{1}{\lambda_N}\sum_{m\nu} a_{n+1,m\nu}(x)\; b_\nu\left(\frac{E}{(1-K_\pi)^m}\right)$$

$$= \frac{1}{\lambda_N}\sum_{m\nu} a_{n+1,m\nu}(x)\; b_{m\nu}(E) \qquad A(7.4)$$

The same proceeding used in Section A,5 to obtain A(5.7) gives

$$B^n\Omega^{n+1}P(x,E) = \frac{1}{\lambda_N}\frac{1}{(\lambda_\pi)^n}\sum_{n\nu} a_{n+1,m\nu}(x)\; b_{n,m\nu}(E) \qquad A(7.5)$$

$$\left|a_{n+1,m\nu}\right| \leq \frac{X^{n+1}}{(n+1)!} R_{m\nu}$$

$$\left|b_{n,m\nu}\right| \leq Mc_1c_2^n .$$

Hence we have

$$\left|\Omega B^n\Omega^n P(x,E)\right| \leq \frac{Mc_1c_2^n}{\lambda_\pi\lambda_\pi^n}\frac{X^{n+1}}{(n+1)!} e^{(\beta_\pi+\beta_N)X} < \frac{Mc_1}{\lambda_N}\left(\frac{c_2X}{\lambda_\pi}\right)^n\frac{X}{n!} e^{(\beta_\pi+\beta_N)X}$$

A(7.6)

Hence the series of the right side of A(7.5) is absolute and uniformly convergent on T. This ensures the continuity of $B^n\Omega^{n+1}P(x,E)$ on T, because the functions $a_{n+1,m\nu}(x)b_{n,m\nu}(E)$ are continuous on T.

A,8) - Continuity of $F_n(x,E)$ and $P_n(x,E)$ on T

We have eq. (23) and eq. A(6.2)

$$F_n(x,E) = \Omega\left[1 + B\Omega + \ldots + (B\Omega)^n\right] P(x,E)$$

$$= \Omega\left[1 + B\Omega + \ldots + B^n\Omega^n\right] P(x,E)$$

But according to (A.7) the $\Omega B^n\Omega^n P(x,E)$ are continuous on T, so is $F_n(x,E)$.

Concerning to $P_n(x,E)$ we can write

$$P_n(x,E) = P(x,E) + BF_{n-1}(x,E) =$$

$$= P(x,E) + B\Omega\left[1 + B\Omega + \ldots(B\Omega)^{n-1}\right]P(x,E) =$$

$$= \left[1 + B\Omega + \ldots + (B\Omega)^n\right]P(x,E)$$

But we have seen in (A,5d) that the $(B\Omega)^n P(x,E)$ are continuous on T, so is $P_n(x,E)$.

APPENDIX B

If the primary nuclear differential spectrum in the top of the atmosphere is represented approximately by a power function

$$F_N(0,E) = G(E) = N_o E^{-(\gamma+1)} \quad , \quad E \geq a > 0$$

the solution (5) of equation (1) reduces to

$$F_N(x,E) = F_N(0,E)\, e^{-x/L_a} \qquad B(1)$$

where

$$L_a = \frac{\lambda_N}{1 - (1-K_N)^{\gamma}} \qquad B(2)$$

L_a is the absorption length of the nuclear component (N) in the atmosphere.

In this special case eq. (2) can be immediatly integra_ted by the method of separation of variables. The result thus obtained can be used to check our previous calculations. Thus we have

$$P(x,E) = P_{\pi}^{NN}(x,E) = \frac{1}{\lambda_{\pi}} \int_0^{B_{\pi}} f_N(\eta)\, F_N(0, \frac{E}{\eta})\, e^{-x/L_a}\, d\eta/\eta$$

$$= A(x)\, E^{-(\gamma+1)} \qquad B(3)$$

where

$$C_N = \int_0^{B_N} f(\eta)\eta^{\gamma}\, d\eta \qquad B(4)$$

and

$$A(x) = \frac{C_N N_o}{\lambda_N}\, e^{-x/L_a} \qquad B(5)$$

$$\Omega^n F_o(x,E) = \Omega^{n+1} P(x,E) =$$

$$= \int_0^x dt_{n+1} \int_0^{t_{n+1}} dt_n \dots \int_0^{t_2} dt_1 \; e^{\frac{1}{\lambda_\pi}(1-\sigma_{K\pi})(x-t_1)} \; P(t_1,E) \; . \qquad B(6)$$

A straightforward calculation gives

$$e^{-\frac{1}{\lambda_\pi}(1-\sigma_{K\pi})(x-t_1)} P(t_1,E) \doteq A(t_1)\, e^{-\frac{1}{\lambda_\pi}(1-\sigma_{K\pi})(x-t_1)} E^{-(\gamma+1)} =$$

$$= E^{-(\gamma+1)} A(t_1)\, e^{-\frac{1}{L_\pi}(x-t_1)} =$$

$$= \left(\frac{C_N N_o}{\lambda_N}\right) E^{-(\gamma+1)} E^{-\frac{x}{L_\pi}} e^{-\left(\frac{1}{L_a} - \frac{1}{L_\pi}\right)t_1}$$

B(7)

where we put

$$\frac{1}{L_\pi} = \frac{1-(1-K_\pi)^\gamma}{\lambda_\pi} \qquad B(8)$$

Hence

$$\Omega^{n+1} P(x,E) = \left(\frac{C_N N_o}{\lambda_N}\right) E^{-(\gamma+1)} \; e^{-\frac{x}{L_\pi}} \int_0^x dt_{n+1} \dots \int_0^{t_2} dt_1 \; e^{-\left(\frac{1}{L_a} - \frac{1}{L_\pi}\right)t_1} \; dt_1$$

$$= \left(\frac{C_N N_o}{\lambda_N}\right) E^{-(\gamma+1)} \; e^{-\frac{x}{L_\pi}} \frac{1}{n!} \int_0^x e^{-\left(\frac{1}{L_a} - \frac{1}{L_\pi}\right)} (x-t_1)^n dt_1 =$$

$$= \left(\frac{C_N N_o}{\lambda_N}\right) E^{-(\gamma+1)} \; e^{-\frac{x}{L_\pi}} \frac{1}{n!} \int_0^x e^{-\left(\frac{1}{L_a} - \frac{1}{L_\pi}\right)(x-t_1)} t_1^n dt_1 =$$

$$= \left(\frac{C_N N_o}{\lambda_N}\right) E^{-(\gamma+1)} \; e^{-\frac{x}{L_a}} \frac{1}{n!} \int_0^x e^{-\left(\frac{1}{L_a} - \frac{1}{L_\pi}\right)t} t^n dt \qquad B(9)$$

Moreover

$$B^n\Omega^{n+1}P(x,E) = \left(\frac{C_N N_o}{\lambda_N}\right) e^{-\frac{x}{L_a}} \frac{1}{n!}\int_0^x e^{\left(\frac{1}{L_a}-\frac{1}{L_\pi}\right)t} t^n dt .\ \frac{1}{\lambda_\pi}\int_0^{B_\pi}\frac{ds_1}{s_1} f_\pi(s_1)\ldots$$

$$\ldots\int_0^{B_\pi}\frac{ds_n}{s_n} f_\pi(s_n)\left(\frac{E}{s_1\ s_2\ldots s_n}\right)^{-(\gamma+1)} =$$

$$= \left(\frac{C_N N_o}{\lambda_N}\right) e^{-\frac{x}{L_a}} \frac{1}{n!}\int_0^x e^{\left(\frac{1}{L_a}-\frac{1}{L_\pi}\right)t} t^n dt .\ E^{-(\gamma+1)}\left(\frac{C_\pi}{\lambda_\pi}\right)^n \qquad B(9)$$

where

$$C_\pi = \int_0^{B_\pi} f_\pi(\eta)\ \eta^\gamma d\eta \qquad B(10)$$

Hence

$$F_\pi = \sum_o^\infty B^n\Omega^n F_o(x,E) = \sum_o^\infty B^n\Omega^{n+1}P(x,E) =$$

$$= \frac{C_N}{\lambda_N} . N_o . E^{-(\gamma+1)} e^{-\frac{x}{L_a}} \int_0^x e^{\left(\frac{1}{L_a}-\frac{1}{L_\pi}\right)t} \sum_o^\infty \frac{t^n\ C_\pi^n}{n!\ \lambda_\pi^n} dt \qquad B(11)$$

Putting

$$\frac{1}{L_a} - \frac{1}{L_\pi} + \frac{C_\pi}{\lambda_\pi} = \frac{1}{L_a} - \left[\frac{1}{L_\pi} - \frac{C_\pi}{\lambda_\pi}\right] = \frac{1}{L_a} - \Lambda_\pi = \frac{1}{L_a} - \frac{1}{\mathcal{L}_\pi} \qquad B(12)$$

we obtain finally

$$F_\pi(x,E) = \frac{C_N}{\lambda_N}\ N_o\ E^{-(\gamma+1)} \frac{e^{-\frac{x}{\mathcal{L}_\pi}} - e^{-\frac{x}{L_a}}}{\frac{1}{L_a} - \frac{1}{\mathcal{L}_\pi}} \qquad B(13)$$

which coincides with the result fiven by eq.(14)and (15) of ref. (5).

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