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A0005/76
ABR,1976

PROPERTIES OF SEMI-INFINITE NUCLEI

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(Received August 22nd, 1975)

ABSTRACT

Several relations among density distributions and energies of semi-infinite and infinite nuclei are investigated in the framework of Wilets's statistical model.

The model is shown to be consistent with the theorem of surface tension given by Myers and Swiatecki. Some numerical results are shown by using an appropriate nuclear matter equation of state.

I - INTRODUCTION

As it is well known, nuclear masses are expressed mostly by a smooth function of A and Z (A, mass number and Z, atomic number), whose origin is usually understood through a liquid drop model of nuclei, and some small fluctuations arising from quantum effects of nuclei.

The liquid drop property is mainly due to the short rangeness and saturational nature of nuclear forces. Homogeneous nuclear matter is an extreme case of such a system idealized to study properties of the "liquid", neglecting surface effects.

For finite nuclei the surface effect is usually treated phenomenologically; a typical example may be the Weizsacker-Bethe mass formula.

There are several theoretical studies on surface effects of saturating systems. Wilets et al.[1] introduced a statistical model and investigated surface effects on semi-infinite nuclei. Their model was developed by several authors[2-7].

This kind of analysis about the behaviour of the liquid drop based on a theoretical model, is important especially for investigation of nuclei far from the β-stability line and super-heavy nuclei, since a large extrapolation of the theory is required.

Recently Myers and Swiatecki[8] introduced a refinement of the liquid drop model, called the droplet model, and discussed surface effects from a very general argument.

In this paper properties of semi-infinite nuclei are investigated taking Wilets' model as a base and trying to find a connection between the model and Myers and Swiatecki's argument to understand the physical mechanism of the system in the simplest way possible.

In Section II the energy-density formalism is reviewed using an expression for the total energy given by Wilets' statistical model. This energy is minimized with respect to the density distribution of nucleons, restricted to the condition that the total number of nucleons A be fixed, to find the ground state distribution.

We have studied some analytical properties of the solution of the differential equation for density distributions. It is shown that Wilets' model is consistent with the surface ten - sion theorem given by Myers and Swiatecki[8].

In Section III some numerical examples are presented taking a simple expression of nuclear matter energy.

II - ENERGY-DENSITY FORMALISM

In Wilets' statistical model[1] the total energy of a semi-infinite nucleus, neglecting Coulomb energy,is written as:

$$E = \int \{\varepsilon(\rho) + \frac{\zeta\hbar^2}{8M} \frac{(\nabla\rho)^2}{\rho} \}d\tau \qquad (1)$$

where ρ is the density of nucleons in the nucleus ($\rho = \rho_n + \rho_p$), $\varepsilon(\rho)$ represents the energy density of nuclear matter, and the second term is a correction to the kinetic energy due to the non-homogenity of the density distribution[1].

Eq. (1) is an energy density functional. In order to obtain the ground state distribution, the energy E should be minimized with respect to ρ, subject to the condition that the total number of particles $A = \int\rho d\tau$ be kept constant. This leads to the following differential equation[1].

$$- \frac{\hbar^2\zeta}{2M} \nabla^2 u + \frac{d\varepsilon}{d\rho} u = \mu u \qquad (2)$$

where $u = \sqrt{\rho}$ and μ is a Lagrangian multiplier.

Eq. (2) has the form of Schroedinger's equation for a particle of "mass" M/ζ in a "potential" $d\varepsilon/d\rho$ with energy μ.

For a spherically symmetrical case, we get:

$$\frac{\zeta\hbar^2}{2M}\left\{\frac{1}{r}\frac{d^2(ur)}{dr^2}\right\} + \frac{d\varepsilon}{d\rho}u = \mu u \tag{3}$$

or

$$-\frac{\zeta\hbar^2}{2M}\frac{d^2y}{dr^2} + \frac{d\varepsilon}{d\rho}y = \mu y \tag{4}$$

where $y = ru$.

The Lagrangian multiplier, or chemical potential μ, which has the dimension of energy, can be interpreted as the separation energy of a particle of the system. It coincides with the mean energy per particle E/A in the limit $A \to \infty$, but for semi-infinite nuclei $\mu < E/A$.

This can be seen as follows: the mean energy per particle for a semi-infinite system is a monotonically decreasing function of A since the surface energy is defined as being positive (the gradient term in Eq. (1))and its effect on the mean energy per particle decreases as $A^{-1/3}$. Thus the total energy of a system with A nucleons can be expanded as:

$$E = E_0 A + \alpha A^{2/3} + \ldots$$

where the term $E_0 A$ corresponds to the volume energy in a liquid drop mass formula and E_0 is the energy per particle of nuclear matter at equilibrium density $\rho = \rho_{00}$. The second term is proportional to the area of the nuclear surface where α should be positive as seen from Eq. (1).

Then the energy per particle is

$$\frac{E}{A} = E_0 + \alpha A^{-1/3} + \ldots \tag{5}$$

showing that

$$\frac{E}{A} > E_o \qquad \text{since} \qquad \alpha > 0$$

As $\frac{d}{dA}(\frac{E}{A}) < 0$ for large A, the chemical potential (energy of the "last" particle) is

$$\mu \equiv \frac{dE}{dA} = E_o + \frac{2}{3}\alpha A^{-1/3} + \dots \qquad (6)$$

Thus for large A

$$E_o < \mu < E/A \qquad (7)$$

and for $A \to \infty$

$$E_o = \mu = E/A$$

It is shown that the physical solution for Eq. (4) should satisfy the following boundary condition[7]:

$$y \sim e^{-\sqrt{C'}\, r} \qquad (8)$$

where

$$C' \equiv \left(\frac{d\varepsilon}{d\rho} - \mu\right)\Big|_{\rho=0} \frac{2M}{\zeta\hbar^2} > 0$$

This boundary condition for Eq. (1) forms an eigenvalue problem for the chemical potential μ as a function of a given central density ρ_o.

The behaviour of μ with respect to ρ_o, in the neighbourhood of ρ_{oo}, (equilibrium density of nuclear matter), is studied as follows.

Since the energy density divided by ρ has a minimum at

$\rho_o = \rho_{oo}$, and the surface effect is always defined as being posi tive, we can deduce that the ground state distribution $\rho(r)$ for $A \to \infty$ is pratically constant until $r \simeq R$, where $R \propto A^{1/3}$, and falls off within a finite surface thickness Δ (fig. 1).

Introducing a set of new variables

$$\xi = Rx$$

$$\eta = Ry$$

the following equation is obtained

$$\frac{d^2\eta}{d\xi^2} = CR^2 \left[\frac{d\varepsilon}{d\rho} - \mu\right]\eta \qquad \text{where } C = \frac{2M}{\zeta\hbar^2} \tag{9}$$

when

$$\rho = \left(\frac{\eta}{\xi}\right)^2$$

In the limit $\rho_o \to \rho_{oo}$, we get $A \to \infty$, $R \to \infty$ and

$$\eta \simeq \sqrt{\rho_o}\ \ \xi\ \theta\ (1-\ \xi) \tag{10}$$

Integration of Eq. (9) with respect to ξ, from 0 to ∞ gi ves

$$\left.\frac{d\eta}{d\xi}\right|_{r\to\infty}^{o} - \left.\frac{d\eta}{d\xi}\right|_{r=0} \simeq CR^2 \int \left[\frac{d\varepsilon}{d\rho} - \mu\right]\eta \ \ d\xi$$

For very large A, or $\rho_o \simeq \rho_{oo}$, we obtain

$$-\ 1 \simeq \frac{CR^2}{2}\left[\frac{d\varepsilon}{d\rho} - \mu\right]$$

or

$$\frac{d\varepsilon}{d\rho} - \mu \simeq -\ \frac{2}{CR^2} \tag{11}$$

Combining Eqs. (6) and (11) we see that

$$\mu = \frac{d\varepsilon}{d\rho_o} + \frac{2}{CR^2} = E_o + \frac{2}{3}\alpha A^{-1/3} + \ldots \qquad (12)$$

Since ρ_{oo} is the equilibrium density of nuclear matter

$$\frac{d\varepsilon}{d\rho} = E_o + \frac{K}{2}(\rho_o - \rho_{oo})^2 + K(\rho_o - \rho_{oo})\,\rho_o \qquad (13)$$

where

$$K = \frac{d^2}{d\rho^2}\left(\frac{\varepsilon}{\rho}\right)_{\rho = \rho_{oo}}$$

Substituting Eq. (13) into Eq. (12)

$$E_o + \frac{K}{2}(\rho_o - \rho_{oo})^2 + K(\rho_o - \rho_{oo})\rho_o + \frac{2}{CR^2} =$$

$$= E_o + \frac{2}{3}\alpha A^{-1/3} + \ldots$$

From this we can see that

$$K(\rho_o - \rho_{oo})\rho_o \sim \frac{2}{3}\alpha A^{-1/3}$$

$$(\rho_o - \rho_{oo})\, A^{1/3} \simeq \frac{2}{3K\rho_o} \qquad (14)$$

where α represents the surface coeficient.

Then

$$A^{-1/3} \simeq C'\,(\rho_o - \rho_{oo})$$

$$A = \frac{1}{(\rho_o - \rho_{oo})^3} + \text{higher order terms in } \left(\frac{1}{\rho_o - \rho_{oo}}\right) \qquad (15)$$

Knowing that the density is almost constant until $r = R$

and $A = \frac{4\pi R^3}{3}$ then

$$R \propto \frac{1}{(\rho_o - \rho_{oo})} \quad (16)$$

From Eqs. (16) and (11) it follows that

$$\frac{d\varepsilon}{d\rho} - \mu \propto - (\rho_o - \rho_{oo})^2$$

In Fig. 2, eigenvalue μ in function of ρ_o, in the neighbourhood of ρ_{oo}, shows that the central density ρ_o must be greater than ρ_{oo}, i.e., ρ_o tends to ρ_{oo} from the right side of ρ_{oo} where the system is infinitely large.

This can be proved in the following way: defining

$$E(\rho) = \frac{\varepsilon(\rho)}{\rho}$$

then

$$\frac{d\varepsilon}{d\rho_o} - E(\rho_o) = \frac{dE(\rho_o)}{d\rho_o} \rho_o$$

but

$$E(\rho_o) = E_o + \frac{K}{2} (\rho_o - \rho_{oo})^2 + \dots \quad (17)$$

so

$$\frac{d\varepsilon}{d\rho_o} - E(\rho_o) \propto K(\rho_o - \rho_{oo}) \quad \text{when} \quad \rho_o \sim \rho_{oo} \quad (18)$$

On the other hand

$$\frac{d\varepsilon}{d\rho_o} - \mu \propto -(\rho_o - \rho_{oo})^2 \quad (19)$$

Then, combining Eqs. (18) and (19) we get

$$\mu - E(\rho_0) \propto K(\rho_0 - \rho_{00}) + 0\left[(\rho_0 - \rho_{00})^2\right]$$

Using the Eq. (14) it follows that

$$\mu - E_0 \sim K(\rho_0 - \rho_{00}) + 0\left[(\rho_0 - \rho_{00})^2\right]$$

$$> 0 \quad \text{for} \quad \rho_0 - \rho_{00} > 0$$

$$< 0 \quad \text{for} \quad \rho_0 - \rho_{00} < 0$$

However from Eq. (7) $\mu > E_0$, so that we should have

$$\rho_0 - \rho_{00} > 0 \quad \text{for large } A \tag{20}$$

Myers and Swiatecki[8] investigated a semi-infinite system of compressible matter and deduced a general theorem for the property of surface tension.

We can also show that our analysis is consistent with their study by the following:

It is known that the total energy of compressible nucleus with A nucleons is

$$E(\rho,A) = a_v(\rho)A + a_s(\rho)\, A^{2/3} + \ldots$$

where a_v and a_s are volume and surface energy coefficients generalized to a compressible liquid drop, and ρ represents average nuclear density.

Ground state density is specified as

$$\left.\frac{\partial E}{\partial \rho}\right|_{\rho=\rho_0} = 0 \text{ for } A \text{ fixed}$$

so

$$a_v'(\rho_0) + a_s'(\rho_0)\, A^{-1/3} = 0 \tag{21}$$

But from Eq. (17)

$$a_v = \frac{\varepsilon}{\rho} = E(\rho_0) - E_0 + \frac{K}{2}(\rho_0-\rho_{00})^2 \quad \text{for} \quad \rho_0 \simeq \rho_{00}$$

We get from Eq. (21)

$$a_s'(\rho_0) \simeq -K(\rho_0-\rho_{00})\, A^{1/3} \quad \text{for} \quad \rho_0 \simeq \rho_{00} \tag{22}$$

Substituting Eq. (14) into (22) we find

$$a_s'(\rho_{00}) = -\frac{2}{3}\frac{\alpha}{\rho_{00}}$$

Since from the definition of α and $a_s(\rho_0)$,

$$a_s(\rho_0) \to \alpha \qquad \text{for} \qquad \rho_0 \to \rho_{00}$$

Then

$$a_s'(\rho_{00}) = -\frac{2}{3}\frac{a_s(\rho_{00})}{\rho_{00}}$$

This relation coincides exactly with that of Myers and Swiatecki[8], i.e.

$$\frac{d\sigma}{d\rho_0} = 0$$

where σ is the surface tension[4] and is defined as

$$4\pi r_0^2\sigma = a_s(\rho) \quad \text{with} \quad r_0 = \left(\frac{4\pi}{3\rho}\right)^{1/3}$$

III - NUMERICAL EXAMPLES

Deduction of Eq. (9) is done assuming that there exists a solution of Eq. (4) for which nuclear density is approximately constant in the interior region and falls rapidly to zero at the surface.

In order to confirm this, we solved Eq. (4) numerically with boundary condition (Eq. (8)) using a conventional formula for energy density of nuclear matter. A simple expression for $a_v(\rho) = \varepsilon(\rho)/\rho$ was given by Kodama and Yamada from semi-empirical analysis[9,10] and we use their result here mostly for the sake of simplicity.

Numerical results show that in fact there exists a solution of Eq. (4) for a given A which satisfies the saturating properties of density distribution (see Fig. 5). Calculation is done for $\zeta = 2.10$ which is chosen by fitting calculated surface thickness to experimental values.

Surface thickness of a nucleus is plotted as a function of A in Fig. 4. We can see that it tends to a finite value for $A \to \infty$ which justifies the validity of Eq. (10).

In Fig. 5, total number A is plotted as a function of ρ_0 where equilibrium density of nuclear matter is also indicated. It is noted that the derivative of A with respect to ρ_0 for $A \to \infty$, is negative, in agreement with inequality (17).

To check the consistency of the results with the liquid drop expansion, the relation between A and $(\rho_0 - \rho_{00})$ is shown in Fig. 6. It was verified that the slope of $\ln A \times \ln(\rho_0 - \rho_{00})$ is $\simeq -3$, in accordance with Eq. (15).

IV - CONCLUDING REMARKS

Using Wilets' energy density formalism with a conventional energy density formula of nuclear matter, we have proved explicitly Myers and Swiatecki's theorem or the property of surface ten sion, showing that the energy density formalism is adequate to treat the semi-infinite saturating system.

Our particular numerical examples show the influence of higher terms of the liquid drop expansion in $A^{1/3}$ over the nuclei with $A < 40$ (Fig. 3). In the region of the nuclei for $A > 40$ we observed that the central density tends to nuclear matter density when the nucleus becomes infinitely large.

For large but finite nuclei, surface tension compresses the central region causing an increase of density above nuclear matter equilibrium density by an amount proportional to surface-volume ratio ($\sim A^{-1/3}$). For small A, the central density decreases with mass number; for lighter nuclei the statistical model does not work and the liquid drop expansion in $A^{-1/3}$ is not valid.

ACKNOWLEDGEMENT

Numerical calculations are done on the Eletronic Computer IBM/370 at Centro Brasileiro de Pesquisas Físicas.

The authors would like to express sincere thanks to the members of the computer center and also to D.C. Binns for reading the manuscript.

One of the authors (L. El-Jaick) is indebted to Drs. S. M. Calzavara and C.M. do Amaral for their interest and encouragement.

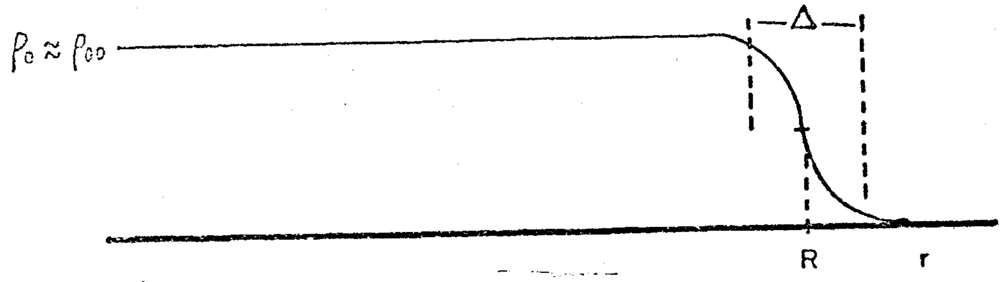

Figure 1 - Density distribution near the surface for a semi-infinite saturating system.

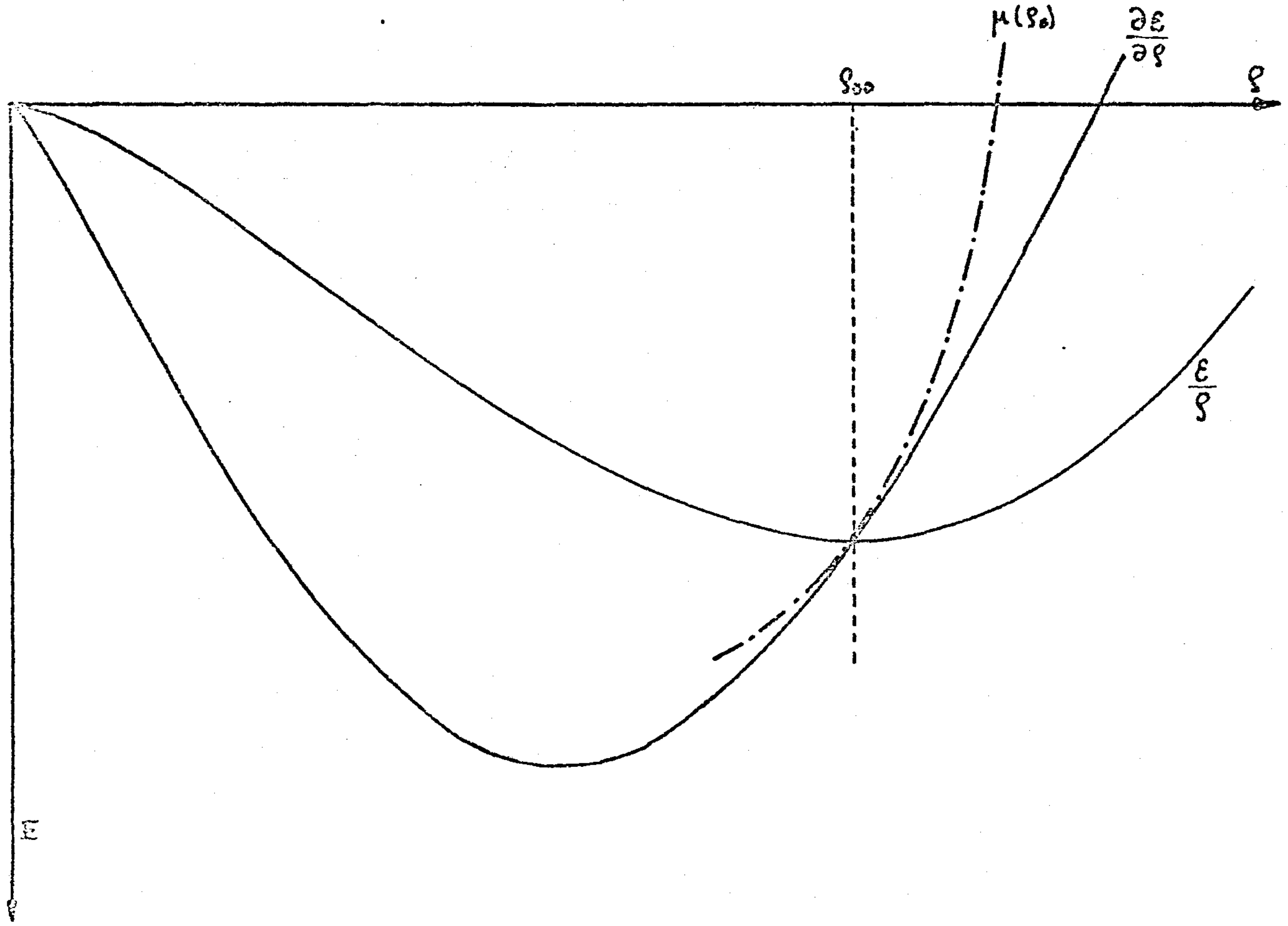

Figure 2 - Schematic graphs of $\partial\varepsilon/\partial\rho$, $E(\rho)$ and $\mu(\rho_0)$. μ contacts the curve $\partial\varepsilon/\partial\rho$ quadratically at $\rho_0 = \rho_{00}$.

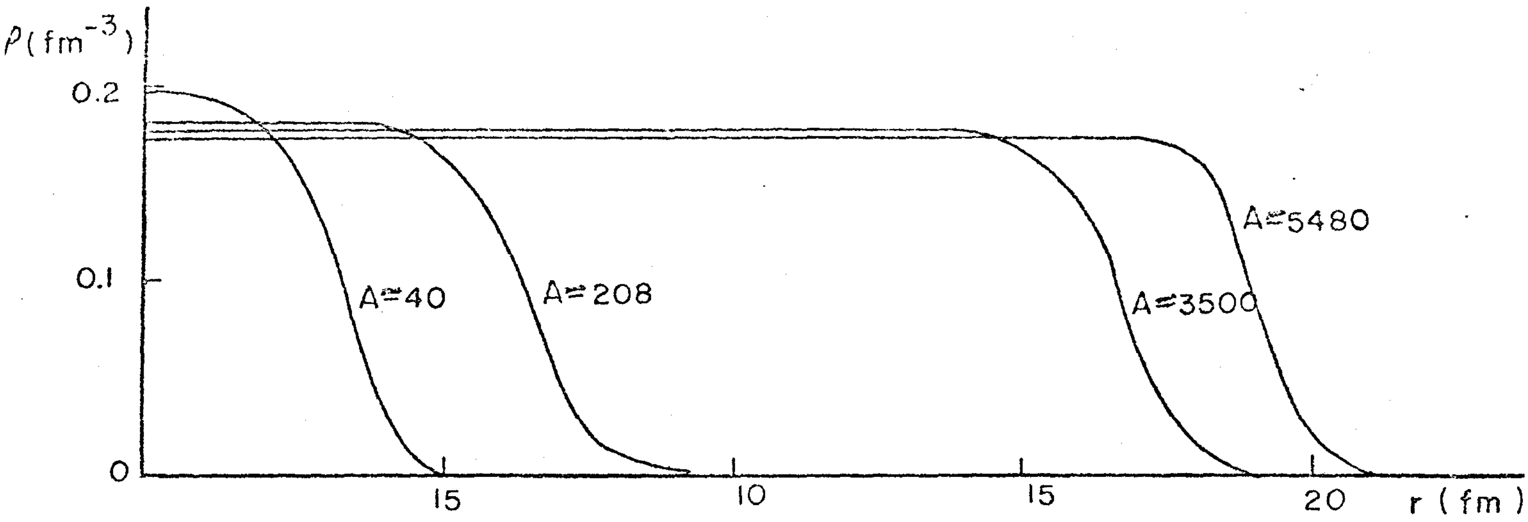

Figure 3 - Examples of density distribution, calculated as a function of radial coordinate r.

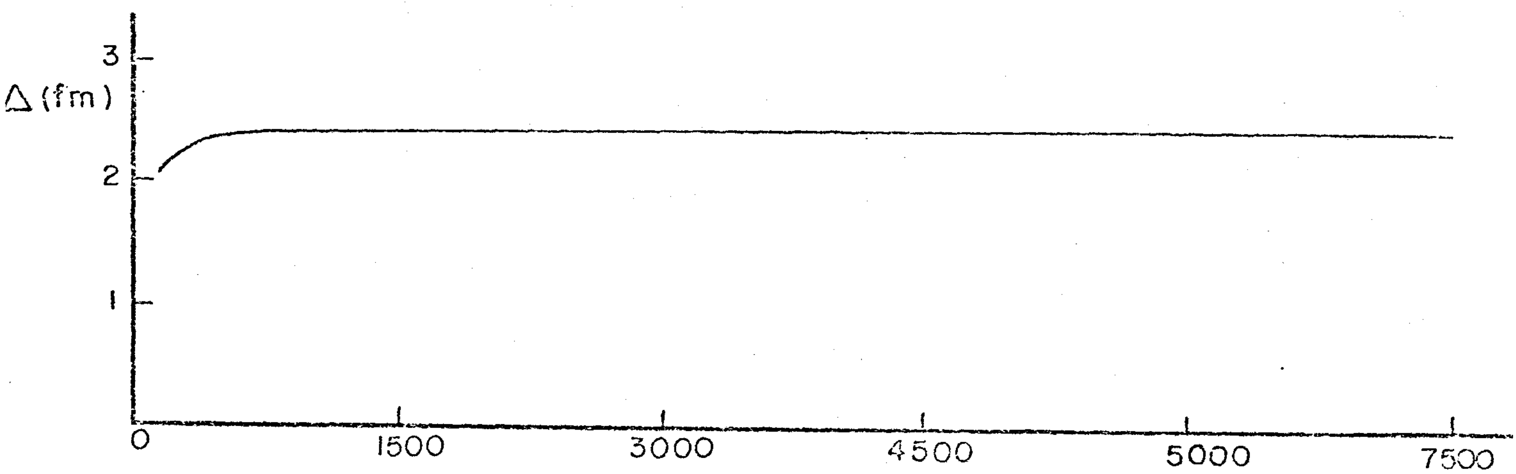

Figure 4 - Surface thickness (defined as $\Delta \equiv r_{\rho=0.1\rho_0} - r_{\rho=0.9\rho_0}$, plotted as a function of A.

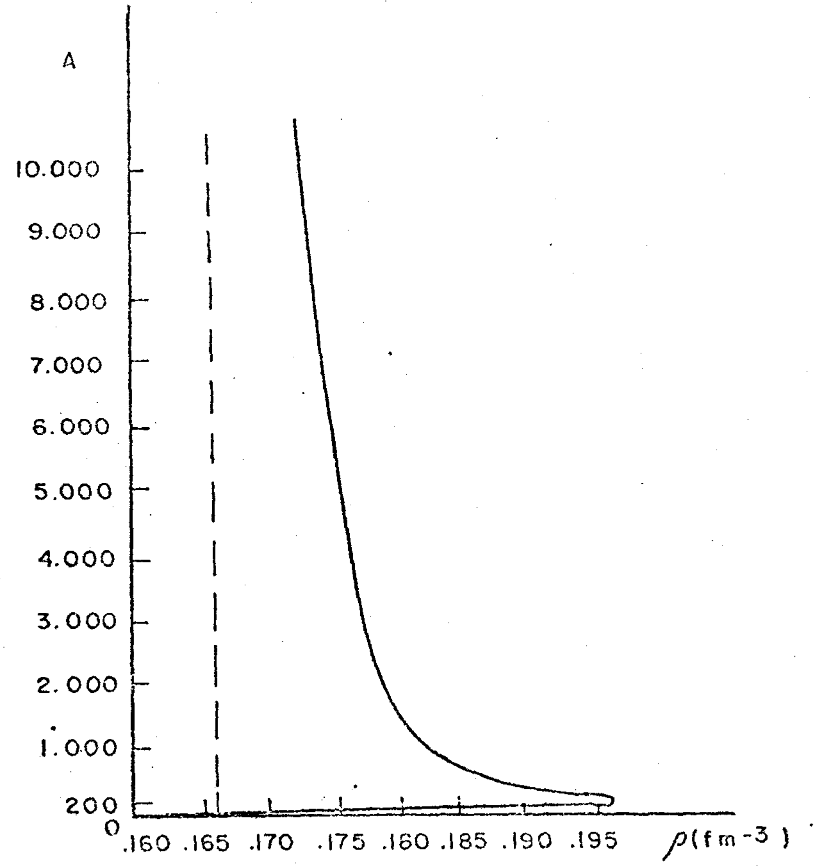

Figure 5 - Total number of nucleons calculated as a function of the central density ρ_0. The position of the equilibrium density of nuclear matter ρ_{00} is shown by a dotted line.

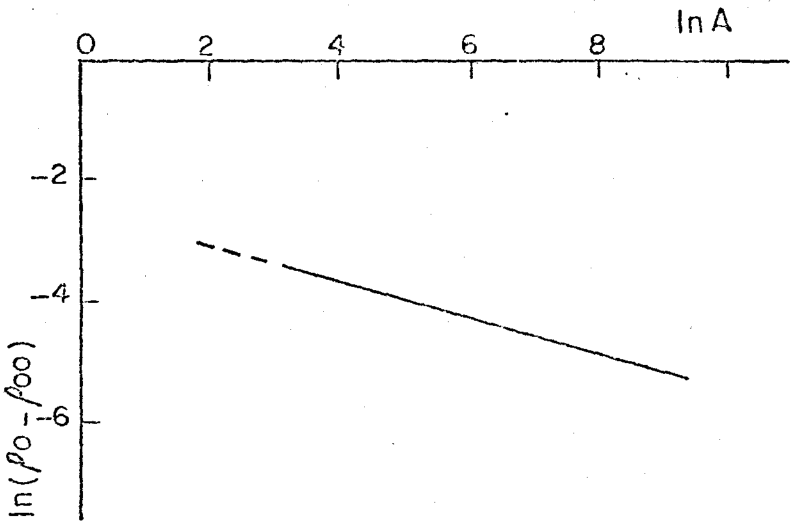

Figure 6 - Logarithm of the total number of nucleons A plotted against the logarithm of the difference $(\rho_0 - \rho_{00})$. The inclination is approximately $\simeq 3.0$.

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