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CONFORMAL SYMMETRY IN LAGRANGIAN FIELD THEORY

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ABSTRACT

Conformal symmetry in Lagrangian field theory is discussed for Lagrangians with derivatives upto first order. Conditions for 'invariance' and 'covariance' of the Lagrangian and for expressing the conformal currents as moments of 'improved' energy momentum tensor are discussed.

1. INTRODUCTION

The idea of approximate symmetry with respect to dilatation and special conformal transformation group of hadronic interactions has drawn renewed interest in recent years. This development arose out of the experimentally observe 'scaling' at high energies, which suggests the possibility of a dynamical limit where dimensional quantities become unimportant. The other important motivation has been the possibility of explaining, at least in part, the masses of the stable particles as arising from spontaneous break down of dilatation invariance.

We discuss here the symmetry of a Lagrangian field theory with respect to scale and special conformal transformations. Lagrantian is assumed to contain derivatives not higher than the first. Distinction is made between the cases in which the infinitesimal quantity $[\delta L]$ defined in eqn. (2.16)vanishes('invariance') and the case in which it is only a divergence ('covariance').

It is shown that in both cases the 'weak' conserved currents derived from Noether's theorem can be cast as moments of the 'improved' energy momentum tensor. We find also necessary and sufficient conditions for 'invariance' condition to hold and that a Poincaré invariant theory is invariant (covariant) simultaneouly, with respect to both scale and special conformal transformations if the conformal deficiency vector V^{λ} vanishes.

In section 2 we review the Lagrangian field theory and Noether's theorem. In section 3 we discuss the variation of field corresponding to infinitesimal conformal transformations.In section 4 conformal currents are constructed and the conditions of invariance and covariance of Lagrangian under infinitesimal transformations as well as the condition for expressing currents as moments of improved energy momentum tensor are discussed. The dilatation symmetry is discussed in some detail. In section 5 applications are made to spin 0,1/2 and 1 field theories and a short section 6 is devoted to the presence of fields with anomalous scale transformations.

2. REVIEW OF LAGRANGIAN FIELD THEORY AND NOETHER'S THEOREM[1]

a) Notation:

We will consider a classical field theory in four-dimensional space-time. The dynamical system is described by N field components $\phi_A(x)$, A=1,2,...N - the dependent variables - which are functions of independent variables(*) $x = (x^0, x^1, x^2, x^3)$. We assume that a Lagrangian density function L can be defined as a function of x^μ, $\phi_A(x)$ and derivatives of $\phi_A(x)$ only upto first order.

The action integral is given by

$$J[\phi_1, \ldots {}_N] = \int_a^b dx^0 \int_R d^3x \; L(x, \phi, \partial\phi) = \int_\Omega d^4x \; L(x, \phi, \partial\phi) \qquad (1)$$

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* We use the metric $g^{\mu\mu} = g_{\mu\mu} = (1, -1, -1, -1)$, $g^{\mu\nu} = g_{\mu\nu} = 0 \quad \mu \neq \nu$.

where R is a three dimensional region and Ω is a cylindrical space-time region (**).

The dynamical equations are then obtained from Hamilton's principle by requiring that the functional $J[\phi_1, \ldots \phi_N]$ be an extremum for all admissible variations $\delta\phi_A$, with region Ω kept fixed (e.g. $\delta x^\mu = 0$). By considering the particular case of $\delta\phi_A$ which vanish on the boundary of Ω we obtain Euler-Lagrange differential equations

$$[L]_A \equiv - \frac{\partial L}{\partial \phi_A} + \partial_\mu \left(\frac{\partial L}{\partial \, \partial_\mu \phi_A} \right) = 0 \tag{2}$$

Here $\partial_\mu F = \frac{\partial}{\partial x^\mu} F$ is the usual partial derivatives where coordinates other than x^μ are kept constant. We will use $\partial_\mu F/$ to indicate partial derivative w.r.t. x^μ which regards coordinates other than x^μ, ϕ_A and all $\partial_\lambda \phi_A$ as constants(+). For convinience in notation we introduce the vector $\phi = (\phi_1, \phi_2, .. \phi_N)$ and tensor $\nabla\phi$ with components $\partial_\mu \phi_A$ so that

$$J[\phi] = \int_\Omega L(x, \phi, \nabla\phi) dx \tag{3}$$

and

$$- \frac{\partial L}{\partial \phi} + \partial_\mu \pi^\mu = 0 \tag{4}$$

** Ω is cartesian product of R and the interval [a,b]

+ Note that:

$$\partial_\mu F(x, \phi, \partial\phi) = \partial_\mu F/ + \frac{\partial F}{\partial \phi} \partial_\mu \phi + \frac{\partial F}{\partial(\partial_\lambda \phi)} \partial_\mu \partial_\lambda \phi$$

where $\pi^{\mu}{}_{A} = \dfrac{\partial L}{\partial(\partial_{\mu}\phi_A)}$ and $\pi^{\mu}{}_{.} = (\pi^{\mu}_{1}, \ldots \pi^{\mu}_{N})$ (5)

We assume throughout that partial derivatives of L exist upto second order w.r.t all its argument and are continuous.

b) Noether's Theorem:

We now consider arbitrary infinitesimal transformations

$$x'^{\mu} = x^{\mu} + \delta x^{\mu}$$

$$\phi'_A(x) = \phi_A(x) + \bar{\delta}\phi_A + \ldots \tag{6}$$

or

$$\bar{\Delta}\phi_A(x) \equiv \phi'_A(x) - \phi_A(x) \simeq \bar{\delta}\phi_A + \ldots \tag{7}$$

where

$$\delta x^{\mu} = \sum_{k=1}^{r} \varepsilon_k \, C^{\mu}_{(k)}(x,\phi,\nabla\phi)$$

and

$$\bar{\delta}\phi_A = \sum_{k=1}^{r} \varepsilon_k \, \bar{B}^{(k)}_{A} \, (x,\phi,\nabla\phi) \tag{8}$$

are arbitrary functions of x, ϕ , $\nabla\phi$ and ε_k, (k=1,2...-.r) are the r essential parameters of the transformation. We introduce also

$$\Delta\phi_A(x) \equiv \phi'_A(x') - \phi_A(x) = \delta\phi_A + \ldots \tag{9}$$

with

$$\delta\phi_A = \sum_{k=1}^{r} \varepsilon_k \, B_A^{(k)} \, (x,\phi,\nabla\phi) \tag{10}$$

It is easily shown

$$\bar{\delta}(\partial_\mu \phi) = \partial_\mu (\bar{\delta}\phi)$$

$$\delta\phi = \bar{\delta}\phi + (\partial_\mu \phi)\, \delta x^\mu$$

$$\begin{aligned} \delta(\partial_\mu \phi) &= \bar{\delta}(\partial_\mu \phi) + (\partial_\nu \partial_\mu \phi)\delta x^\nu \\ &= \partial_\mu (\delta\phi) - (\partial_\nu \phi)\, \partial_\mu (\delta x^\nu) \end{aligned} \tag{11}$$

These relations lead to relation between the functions, B, $\bar{B}$ and C. The tranformation carries $J[\phi]$ to

$$\begin{aligned} J[\phi'] &= \int_{\Omega'} L(x',\phi'(x'), \nabla'\phi'(x'))dx' \\ &= \int_{\Omega} L\,(x',\phi',\nabla',\phi')\, \left| \frac{\partial x'}{\partial x} \right| dx = \int_{\Omega} L'(x,\phi,\nabla\phi,\nabla\nabla\phi)dx \end{aligned} \tag{12}$$

where Ω is mapped to new region Ω' and $L'(x,\phi,\nabla\phi,\nabla\nabla\phi) = L(x',\phi',\nabla'\phi')|\partial x'/\partial x|$ may contain second order derivatives. The variation of the action functional is thus

$$\begin{aligned} \Delta J &= J[\phi'] - J[\phi] \\ &= \int_{\Omega} [\Delta L]dx \end{aligned} \tag{13}$$

where

$$[\Delta L] = L(x',\phi', (x'), \nabla'\phi'(x')) \left| \frac{\partial x'}{\partial x} \right| - L(x,\phi(x), \nabla\phi(x))$$

$$= L'(x,\phi,\nabla\phi,\nabla\nabla\phi) - L(x,\phi,\nabla\phi) \tag{14}$$

$$= [\delta L]+\ldots$$

$$\Delta J = \delta J+\ldots \tag{15}$$

where δJ, $[\delta L]$ indicate the terms upto first order in infinitesimal parameters.

Clearly

$$\delta J = \int_\Omega [\delta L]dx \tag{16}$$

where

$$[\delta L] = [L(x',\phi',\nabla'\phi') - L(x,\phi,\nabla\phi)] + L(x,\phi,\nabla\phi)\partial_\mu(\delta x^\mu) \tag{17}$$

on using

$$\left|\frac{\partial x'}{\partial x}\right| = 1 + \partial_\mu(\delta x^\mu) \tag{18}$$

on making Taylor expansion

$$[\delta L] = \frac{\partial L}{\partial x^\mu}\Big|\delta x^\mu + \frac{\partial L}{\partial \phi}\delta\phi + \pi^\lambda\delta(\partial_\lambda\phi) + L\partial_\mu(\delta x^\mu)$$

$$= \frac{\partial L}{\partial x^\mu}\Big|\delta x^\mu + \frac{\partial L}{\partial \phi}\delta\phi + \pi^\lambda\{\partial_\lambda(\partial\phi) - (\partial_\nu\phi)\ \partial_\lambda(\delta x^\nu)\} + L\partial_\mu(\delta x^\mu) \tag{19}$$

This can be recast as [2]

$$[\delta L] = -[L]_A \,\bar{\delta}\phi_A + \partial_\mu(\pi^\mu_A \,\bar{\delta}\phi + L\delta x^\mu)$$

$$= -[L]_A \,\bar{\delta}\phi_A + \partial_\mu(\pi^\mu_A \delta\phi_A - \tau^{\mu\nu}\delta x_\nu) \qquad (20)$$

Here summation over components A= 1... N is understood and $\tau^{\mu\nu}$ is the canonical energy momentum tensor

$$\tau^{\mu\nu} = \pi^\mu_A \,\partial^\nu\phi_A - g^{\mu\nu}L$$

$$\tau^\mu{}_\mu = \pi^\mu\partial_\mu\phi - 4L \qquad (21)$$

It may be remarked that, due to arbitrariness in the region Ω, $\delta J=0$ implies $[\delta L]=0$ and vice versa.

If action is invariant under the infinitesimal transformations under consideration we find

$$\varepsilon\, \partial_\mu Z^\mu = [L]_A \,\bar{\delta}\phi_A \qquad (22)$$

where $\varepsilon Z^\mu = \pi^\mu_A \,\delta\phi_A - \tau^{\mu\nu}\delta x_\nu$. For constant parameter transformations this leads to "weak continuity" equation[2]

$$\varepsilon\, \partial_\mu Z^\mu \,\hat{=}\, 0 \qquad (23)$$

where $\hat{=}$ indicates the equality when the fields satisfy the Euler's equations of motion. For invariance under coordinate dependent parameter transformation, like gauge transformation we obtain identities. We will be concerned in this paper with the constant parameter transformations. The linear independence of the r parameters lead to r weak continuity equations.

It is clear also that weak continuity equation can be defined even in the case the actions is not invariant.

For the case(3)

$$[\delta L] = \varepsilon\partial_\mu \Lambda^\mu \tag{24}$$

We clearly have

$$\varepsilon Z^\mu = \pi^\mu_A \delta\dot{\phi}_A - \tau^{\mu\nu}\delta x_\nu - \varepsilon\Lambda^\mu \tag{25}$$

and

$$\varepsilon\partial_\mu Z^\mu = [L]_A \,\overline{\delta}\phi_A \overset{\circ}{=} 0 \tag{26}$$

This case is important since Euler's equations corresponding to $[\delta L]$ are then satisfied identically(4). This would then assure that the Euler's equation calculated from the transformed action $J[\phi']$ are the same as those derived from $J[\phi]$. In other words the equations of motion are form invariant w.r.t the infinitesimal transformations like in the case with $[\delta L]=0$, even though the invariance of action may be lost. For the case in discussion we call the theory 'covariant' while the former case will be called 'invariant' theory ($[\delta L] = 0$).

There is a still more general case(5), viz, $[\delta L] = \varepsilon\partial_\mu\Lambda^\mu - f$ with

$f \stackrel{\circ}{=} 0$ and $f \neq [L]_A \overline{\delta}\phi_A$ where we can write a weak continuity equation with Z^μ given in eqn. (2.25); the form invariance of the equations of motion may, however, also be lost.

3. CONFORMAL GROUP. TRANSFORMATION OF FIELDS.

a) Conformal Group[6]:

The connected conformal group containing the identity (called for simplicity conformal group) may be defined as the group of following transformations on the real space time coordinates x^μ of a vector in the four dimensional Minskowski space:

1. Translations

$$x'^\mu = x^\mu + a^\mu$$

2. Restricted Lorentz group of transformations

$$(\Lambda x)^\mu \equiv x'^\mu = \Lambda^\mu{}_\nu x^\nu; \quad g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\lambda = g_{\rho\lambda}, \; \Lambda^o{}_o > 1, \; \det \Lambda = 1$$

3. Scale or dilatation transformations: $(g_D x)^\mu \equiv x'^\mu = e^{-\rho} x^\mu$, ρ real

4. Special conformal transformations:

$$(g_c\, x) \equiv x'^\mu = (x^\mu - c^\mu x^2) \,/\, [1 - 2c.x + c^2 x^2]$$

These transformations constitute a 15 parameter group and the special transformations are non-linear. Each of these sets of transformations constitute a sub-group which is abelian except for the case of Lorentz transformations. Note that translations do not constitute an invariant subgroup.

The infinitesimal transformations are given by

Translations:

$$\delta x^{\mu} = \varepsilon^{\mu} = - i \varepsilon_{\nu} \overline{P^{\nu}} x^{\mu}$$

Lorentz transformations:

$$\delta x^{\mu} = \varepsilon^{\mu}{}_{\nu} x^{\mu} = \frac{i}{2} \varepsilon_{\rho\sigma} \overline{M}^{\rho\sigma} x^{\mu}$$

$$\varepsilon_{\mu\nu} = - \varepsilon_{\nu\mu}; \quad \left| \frac{\partial x'}{\partial x} \right| = 1$$

Dilatations:

$$\delta x^{\mu} = - \varepsilon x^{\mu} = i \varepsilon \overline{D} x^{\mu}$$

$$\left| \frac{\partial x'}{\partial x} \right| = (1 - 4\varepsilon)$$

Special transformations:

$$\delta x^{\mu} = \eta_{\nu} (2x^{\nu} x^{\mu} - g^{\mu\nu} x^2) = i\eta_{\nu} \overline{K^{\nu}} x^{\mu}$$

$$\left| \frac{\partial x'}{\partial x} \right| = (1 + 8\eta . x) \tag{1}$$

where $\overline{p^{\nu}} = i \partial^{\nu}$, $\overline{M}^{\rho\sigma} = i(x^{\rho} \partial^{\sigma} - x^{\sigma} \partial^{\rho})$, $\overline{D} = i(x.\partial)$ and

$\overline{K}^{\nu} = -i\,(2x^{\nu} x^{\lambda} - x^2 g^{\nu\lambda})\partial_{\lambda}$ are the fifteen infinitesimal generators. The Lie algebra of these generators also determines the Lie algebra of the abstract (connected) conformal group whose generator will be indicated by P^{μ}, $M^{\rho\sigma}$, D and K^{μ}. The Lie algebra is found to be:

$$[D, P_{\mu}] = -i\, P_{\mu} \qquad [D, K_{\mu}] = +i\, K_{\mu}$$

$$[D, M_{\mu\nu}] = 0 \qquad [K_{\mu}, K_{\nu}] = 0 \qquad [P_{\mu}, P_{\nu}] = 0$$

$$[P_{\sigma}, M_{\mu\nu}] = i(g_{\sigma\mu} P_{\nu} - g_{\sigma\nu} P_{\mu})$$

$$[K_{\sigma}, M_{\mu\nu}] = i(g_{\sigma\mu} K_{\nu} - g_{\sigma\nu} K_{\mu}) \tag{2}$$

$$[K_{\mu}, P_{\nu}] = -2i(g_{\mu\nu} D + M_{\mu\nu})$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(g_{\mu\sigma} M_{\nu\rho} - g_{\nu\sigma} M_{\mu\rho} - g_{\mu\rho} M_{\nu\sigma} + g_{\nu\sigma} M_{\mu\sigma})$$

Note that the commutation relations imply

$$e^{i\rho D} P_{\mu} e^{-i\rho D} = e^{\rho} P_{\mu}$$

$$e^{i\rho D} K_{\mu} e^{-i\rho D} = e^{-\rho} K_{\mu} \tag{3}$$

and that K_{μ} transforms as a four-vector. The exact dilatation symmetry (with an integrable generator D that takes one-particle states into one-particle states) implies that the mass spectrum is either continuons or all masses are zero.

Introducing J_{AB} (A,B = 0,1,2,3,5,6) where ($J_{AB} = - J_{BA}$):

$$J_{\mu\nu} = M_{\mu\nu} \qquad J_{65} = D \qquad J_{5\mu} = \frac{1}{2}(P_\mu - K_\mu)$$

$$J_{6\mu} = \frac{1}{2}(P_\mu + K_\mu) \tag{4}$$

one has

$$[J_{KL}, J_{MN}] = i(g_{KN} J_{LM} + g_{LM} J_{KN} - g_{KM} J_{LN} - g_{LN} J_{KM})$$

$$g_{AA} = (+ ---, - +) \tag{5}$$

$$g_{AB} = 0 \qquad ; A \neq B$$

which is the Lie algebra of SO (4,2). Thus conformal group is locally isomorphie to non-compact group SO(4,2) whose covering group is the spinor group SU(2,2). Three Casimir operators are then easily obtained:

$$J_{AB} J^{AB} = M_{\mu\nu} M^{\mu\nu} + 2P.K + 8iD - 2D^2,$$

$$\varepsilon_{ABCDEF} J^{AB} J^{CD} J^{EF} \quad \text{and} \quad J^{AB} J_{BC} J^{CD} J_{DA}$$

b) Transformation of Fields:

We postullate that there exist interpolating field (with a finite number N of components) to every particle which transforms according to a representation of the conformal algebra. Thus corresponding to a transformation

$$x^{\mu} = (gx)^{\mu} \qquad g \in \text{Conformal group, the field}$$

$\phi(x):(\phi_1, \ldots\ _N)$ transorms as

$$T(g)\ \phi(x') = \phi'(x') \equiv S(g,x')\ \phi(x) \tag{6}$$

where $\{T(g)\}$ constitute a N dimensional representation of the conformal group. For the infinitesimal transformations

$$T(g) \simeq I + i \sum_{k=1}^{15} \varepsilon_k I_k + \ldots \tag{7}$$

where the essential parameters are labelled as ε_k, k=(1, ..., 15) for convenience. We find

$$\bar{\delta}\phi(x) = i \sum_{k} \varepsilon_k I_k \phi(x) \tag{8}$$

The generators I_k satisfy the Lie algebra of conformal group.

When the fields are quantized field operators acting on the state vectors in Hilbert space which carries the representation according to

$$/\Psi> \ \rightarrow U(g)\ /\Psi> \tag{9}$$

whith U(g) a unitary operator we obtain the supplementary constraint[7]

$$\phi'(x') = U(g)^{+} \phi(x') U(g) \qquad (1b)$$

For infinitesimal transformations

$$U(g) \simeq \Pi + i \sum_k \varepsilon_k G_k \qquad (11)$$

it follows

$$\bar{\delta}\phi(x) = i \sum_k \varepsilon_k [\phi(x), G_k] \qquad (12)$$

where G_k satisfy the Lie algebra of conformal group. Since it is easier to calculate the commutators in q .f. theory where x^{μ} is simply a parameter we will frequently calculate the variation of ϕ regarding ϕ as quantized operators.

Homogeneity of space with respect to translations according to special relativity requires for any(+) field 0(x)

$$0'(x') = 0(x) = 0(t^{-1}x') \qquad (13)$$

when $(tx)^{\mu} = x'^{\mu} = x^{\mu} + \varepsilon^{\mu}$

thus $\delta_T 0(x) = 0$ (14)

and $$\bar{\delta}_T 0(x) = - \varepsilon_{\mu} \partial^{\mu} 0 = i\varepsilon_{\mu} p^{\mu} 0(x) \qquad (15)$$

(+) rather observable field.

Regarding field as operator in Hilbert space ($U_T \simeq e^{i\varepsilon.P}$)

$$\bar{\delta}_T\, O = i\, \varepsilon_\mu [\, O(x), P^\mu \,]$$

Thus

$$[\, O(x), P^\mu \,] = i\partial^\mu\, O(x) \equiv p^\mu\, O(x) \tag{16}$$

from which it follows

$$O(x) = e^{ix.P}\, O(o)\, e^{-ix.P} \tag{17}$$

Homogeneity w.r.t. space-time rotations requires that the interpolating N component field ϕ transforms according to a (non-unitary and irreducible) representation of the homogeneous Lorentz group. viz.,

$$\phi'(x') = S(\Lambda)\, \phi(x) \tag{18}$$

with $S(\Lambda)$constituting a representation of the Lorentz group. For infinitesimal transformations we define

$$S(\Lambda) = I - \frac{i}{2}\, \varepsilon_{\rho\sigma}\, \Sigma^{\rho\sigma} \tag{19}$$

so that

$$\delta_L\, \phi = -\frac{i}{2}\, \varepsilon_{\rho\sigma}\, \Sigma^{\rho\sigma}\, \phi \tag{20}$$

and

$$\bar{\delta}_L\, \phi = -\frac{i}{2}\, \varepsilon_{\rho\sigma}\, m^{\rho\sigma}\, \phi \tag{21}$$

where

$$m^{\rho\sigma} = \Sigma^{\rho\sigma} + i(x^\rho \partial^\sigma - x^\sigma\, \partial^\rho) \tag{22}$$

Taking the field operator point of view

$$U_L \approx e^{-\frac{i}{2}\varepsilon_{\rho\sigma} M^{\rho\sigma}} \tag{23}$$

and

$$\bar{\delta}_L \, \phi = -\frac{i}{2}\varepsilon_{\rho\sigma} \left[\phi(x), M^{\rho\sigma} \right] \tag{24}$$

so that

$$\left[\phi(x), M^{\rho\sigma} \right] = m^{\rho\sigma} \, \phi(x) \tag{25}$$

Using the relation in eqn.(3.17) and the identity

$$\left[\phi(x), M^{\rho\sigma} \right] = e^{ix.P} \left[\phi(o), M^{\rho\sigma}(-x) \right] e^{-ix.P} \tag{26}$$

where $(M^{\rho\sigma} = M^{\rho\sigma}(o))$

$$M^{\rho\sigma}(-x) = M^{\rho\sigma} + (x^{\rho} P^{\sigma} - x^{\sigma} P^{\rho}) \tag{27}$$

we can show

$$\left[\phi(o), M^{\rho\sigma} \right] = \Sigma^{\rho\sigma} \, \phi(o) \tag{28}$$

Conversly, if we take this relation as definition of $\Sigma^{\rho\sigma}$ we can recover eqn. (3.20).

For dilatations we define

$$\left[\phi(o), D \right] = i \, L \, \phi(o) \tag{29}$$

Where L is a $N \times N$ matrix and $D \equiv D(o)$. We may now use the identity similar to eqn. (3.26) to obtain $[\phi(x), D]$. In the present case

$$D(-x) = D + x^{\lambda} P_{\lambda} \tag{30}$$

so that

$$[\,\phi(x),\, D\,] = i(L + x.\partial)\,\phi(x) \tag{31}$$

$$\equiv d\,\phi(x)$$

Then

$$\delta_D\,\phi(x) = -\,i\varepsilon\,[\,\phi(x),\, D\,] = -\,i\,\varepsilon\,d\,\phi(x) \tag{32}$$

where $U_D \simeq e^{-i\varepsilon D}$. It follows

$$\phi'(x) \simeq e^{\varepsilon L}\,\phi(e^{\varepsilon}\,x)$$

Comparing with $\phi'(x) = S(g,x)\,\phi(g^{-1}x)$ we see that under finite dilatations: $x'^{\mu} = e^{-\rho}\,x^{\mu}$

$$\phi'(x') = e^{\rho L}\,\phi(x) = e^{\rho L}\,\phi(e^{\rho}\,x') \tag{33}$$

and correspondingly $U_D = e^{-i\rho D}$, that is,

$$e^{i\rho D}\,\phi(x)\,e^{-i\rho D} = e^{\rho L}\,\phi(e^{\rho}\,x) \tag{34}$$

Also

$$\delta_D\phi = \varepsilon\,L\,\phi(x), \qquad \delta x^{\mu} = -\varepsilon x^{\mu} \tag{35}$$

For special conformal transformations we define for field operator $\phi(o)$ to satisfy[(3)]

$$[\,\phi(o),\, K_{\mu}\,] = \kappa_{\mu}\,\phi(o) \tag{36}$$

From

$$K_\mu(-x) \equiv e^{-ix.P} K_\mu e^{ix.P}$$

$$= K_\mu + 2(x_\mu D + x^\nu M_{\mu\nu}) + (2x_\mu x.P - x^2 P_\mu) \qquad (37)$$

and identity analogous to the used above

we find

$$[\phi(x), K^\mu] = k^\mu \phi(x) \qquad (38)$$

where

$$k_\mu = \kappa_\mu + i(2x_\mu x.\partial - x^2 \partial_\mu) + 2(x_\mu iL + x^\nu \Sigma_{\mu\nu}) \qquad (39)$$

and

$$\bar{\delta}_c \phi(x) = i \eta_\mu [\phi(x), K^\mu] = i \eta_\mu k^\mu \phi(x) \qquad (40)$$

or

$$\phi'(x) \simeq [I + i\eta^\mu \{2(x_\mu iL + x^\nu \Sigma_{\mu\nu}) + \kappa_\mu\}] \phi(g_c^{-1} x) \qquad (41)$$

or

$$S(g_c, x) \simeq I + i\eta^\mu \{2(x_\mu iL + x^\nu \Sigma_{\mu\nu}) + \kappa_\mu\} \qquad (42)$$

Thus

$$\delta_c\phi(x) = i\eta^\mu \{2(x_\mu iL + x^\nu \Sigma_{\mu\nu}) + \kappa_\mu\} \phi(x) \qquad (43)$$

It may be noted that in $\delta\phi$ (or δx^μ) no derivatives of the field appear. It follows that $[\delta L]$ contains derivatives only upto first order. In this case(4) Λ^μ is a function of x and ϕ alone. Note also that

p^μ, $m^{\rho\sigma}$, d, k^μ satisfy the commutation relations of the Lie algebra of conformal group' and that κ^μ makes transitions between fields with different Lorent transformation law; we will assume it to vanish in discussions to follow. Also it follows that $[L, \Sigma^{\rho\sigma}] = 0$ and, if the field ϕ constitute an irreducible representation of homogeneous Lorentz group, L is a multiple of identity matrix.

4. CONFORMAL CURRENTS AS MOMENTS OF A SYMMETRIC ENERGY MOMENTUM TENSOR:

We may now calculate $[\delta L]$ from eqn. (2.19):

$$[\delta L] \equiv \varepsilon_\mu I_T^\mu L + \frac{1}{2} \varepsilon_{\rho\sigma} I_L^{\rho\sigma} L + \varepsilon I_D L + \eta_\nu I_c^\nu L \tag{1}$$

where

$$I_T^\mu L = \frac{\partial L}{\partial x^\mu}\Bigg|$$

$$I_L^{\rho\sigma} L = (x^\sigma g^{\rho\mu} - x^\rho g^{\sigma\mu}) \partial_\mu L|$$

$$- i\left(\frac{\partial L}{\partial \phi} \Sigma^{\rho\sigma}\phi + \pi^\lambda \Sigma^{\rho\sigma} \partial_\lambda \phi\right) + (\pi^\rho \partial^\sigma - \pi^\sigma \partial^\rho)\phi$$

$$I_D L = - x^\mu \partial_\mu L / - 4L + \frac{\partial L}{\partial \phi} L\phi + \pi^\lambda (L + I)\partial_\lambda \phi$$

$$I_c^\nu L = (2x^\mu x^\nu - g^{\mu\nu} x^2) \partial_\mu L\Big| - 2x^\nu (x^\mu \partial_\mu L\Big| + I_D L)$$

$$+ 2x_\mu ([x^\mu g^{\nu\lambda} - x^\nu g^{\mu\lambda}] \partial_\lambda L\Big| - I_L^{\nu\mu} L)$$

$$+ V^\nu + i\left(\frac{\partial L}{\partial \phi} \kappa^\nu \phi + \pi^\lambda \kappa^\nu \partial_\lambda \phi\right) \tag{2}$$

where(*)

$$V^{\nu} = 2i\ \pi_{\lambda}(iLg^{\nu\lambda} + \textstyle\sum^{\nu\lambda})\phi \tag{3}$$

is conformal deficiency vector.

The currents in conformally "covariant" theory satisfying the weak continuity equation are also easily found. Writing(**)

$$\varepsilon\ Z^{\lambda} = -\ \varepsilon_{\mu}\ J_{T}^{\ \lambda\mu} + \frac{1}{2}\ \varepsilon_{\rho\sigma}\ J_{L}^{\rho\sigma} + \varepsilon J^{\lambda} + \eta_{\nu}\ J_{C}^{\ \lambda\nu}$$

μ

$$\varepsilon\ \Lambda^{\lambda} = -\ \varepsilon\ \ \Lambda_{T}^{\lambda\mu} + \frac{1}{2}\ \varepsilon_{\rho\sigma}\ \Lambda_{L}^{\rho\sigma} + \varepsilon\ \Lambda_{D}^{\lambda} + \eta_{\mu}\ \Lambda_{C}^{\lambda\nu} \tag{4}$$

We have in Poincare invariant theory ($\Lambda_T = \Lambda_L = 0$)

$$J_{T}^{\ \lambda\mu} = \tau^{\lambda\mu}$$

$$J_{L}^{\lambda\rho\sigma} = -\ J_{L}^{\lambda\sigma\rho} = -\ i\ \pi^{\lambda}\ \textstyle\sum^{\rho\sigma}\phi + (x^{\rho}\tau^{\lambda\sigma} - x^{\sigma}\ \tau^{\lambda\rho})$$

$$J_{D}^{\ \lambda} = x_{\mu}\ \tau^{\lambda\mu} + \pi^{\lambda}\ L\phi - \Lambda_{D}^{\lambda}$$

$$J_{C}^{\lambda\nu} = -\ (2x^{\nu}\ x_{\mu} - g_{\mu}^{\nu}\ x^{2})\ \tau^{\lambda\mu} + 2ix_{\mu}\ \pi^{\lambda}(iLg^{\nu\mu} + \textstyle\sum^{\nu\mu})\ \phi$$

$$+\ i\pi^{\lambda}\ \kappa^{\nu}\phi - \Lambda_{C}^{\lambda\nu} \tag{5}$$

* Note that V^{ν} does not depend on $\partial L/\partial\phi$. Also we will assume κ^{ν}=0

** sign in front of J_T is for convinience.

where

$$\partial_\lambda J^{\lambda\cdots} \underset{=}{\circ} 0 \tag{6}$$

Poincare invariance leads to restrictions

$$\frac{\partial L}{\partial x^\mu}\Big| = 0 \tag{7}$$

e.g. L cannot depend explicitty on coordinates, and

$$i\left(\frac{\partial L}{\partial\phi}\Sigma^{\rho\sigma}\phi + \pi^\lambda\,\Sigma^{\rho\sigma}\,\partial_\lambda\phi\right) = (\pi^\rho\,\partial^\sigma - \pi^\sigma\,\partial^\rho)\phi \tag{8}$$

which may be used to determine the matrices $\Sigma^{\rho\sigma}$.

Exploiting the fact that J^λ and $J^\lambda + \partial_\mu\,\chi^{\lambda\mu}$ where $\chi^{\lambda\mu} = -\chi^{\mu\lambda}$ have the same divergence and charge (if χ^{io} vanishes, sufficiently rapidly at the surface at infinity) we can write the currents in simpler form. In terms of Belinfante tensor(8)

$$\bar{\Theta}^{\lambda\mu} = \tau^{\lambda\mu} + \frac{1}{2}\,\partial_\nu\,\chi^{\nu\lambda\mu} \tag{9}$$

where

$$\chi^{\lambda\mu\rho} = -\,i\left[\pi^\lambda\Sigma^{\mu\rho}\phi - \pi^\mu\,\Sigma^{\lambda\rho}\,\phi - \pi^\rho\,\Sigma^{\lambda\mu}\,\phi\right] \tag{10}$$

The currents take the form ($\kappa^\nu = 0$)

$$J_T^{\lambda\mu} = \bar{\Theta}^{\lambda\mu}$$

$$J_L^{\lambda\rho\sigma} = (x^\rho\,\bar{\Theta}^{\lambda\sigma} - x^\sigma\,\bar{\Theta}^{\lambda\rho})$$

$$J_D^{\lambda} = x_\mu \bar{\theta}^{\lambda\mu} - \frac{1}{2} V^\lambda - \Lambda_D^\lambda$$

$$J_C^{\lambda\nu} = -(2x^\nu x_\mu - g^\nu_\mu x^2)\bar{\theta}^{\lambda\mu} + x^\nu V^\lambda - \Lambda_C^{\lambda\nu} \qquad (11)$$

A further simplification can be achieved by introducing "improved" energy momentum tensor[9] $\theta^{\lambda\mu}$

$$\theta^{\mu\nu} = \bar{\theta}^{\mu\nu} + \frac{1}{2} \partial_\lambda \partial_\rho \chi^{\lambda\rho\mu\nu} \qquad (12)$$

Where $\partial_\lambda \partial_\rho \chi^{\lambda\rho\mu\nu}$ is symmetric and divergenceless on indices μ and ν and

$$\chi^{\lambda\rho\mu\nu} = g^{\lambda\rho} \sigma_+^{\mu\nu} + g^{\mu\nu} \sigma_+^{\lambda\rho} - g^{\lambda\mu} \sigma_+^{\nu\rho} - g^{\lambda\nu} \sigma_+^{\mu\rho}$$

$$+ \frac{1}{3} (g^{\mu\lambda} g^{\nu\rho} - g^{\mu\nu} g^{\lambda\rho})\sigma^{\alpha}_{+\alpha} \qquad (13)$$

with $\sigma^{\mu\nu}$ being any arbitrary tensor function of fields and $\sigma_\pm^{\mu\nu} = \frac{1}{2} [\sigma^{\mu\nu} \pm \sigma^{\nu\mu}]$.

The currents then become

$$J_T^{\lambda\mu} \doteq \theta^{\lambda\mu}$$

$$J_L^{\lambda\rho\sigma} \doteq (x^\rho \theta^{\lambda\sigma} - x^\sigma \theta^{\lambda\rho})$$

$$J_D^{\lambda} \doteq x_\mu \theta^{\lambda\mu} - \frac{1}{2} (V^\lambda + 2 \partial_\rho \sigma^{\lambda\rho}) - \Lambda_D^\lambda$$

$$J_C^{\lambda\nu} \doteq -(2x^\nu x_\mu - g^\nu_\mu x^2)\theta^{\lambda\mu} + x^\nu (V^\lambda + 2 \partial_\rho \sigma^{\lambda\rho}) - 2\sigma^{\nu\lambda} - \Lambda_C^{\lambda\nu} \qquad (14)$$

where the equality $\doteq$ means that we have dropped all the terms whose divergence w.r.t. index 'λ' identically vanishes.

The arbitrariness in the choice of $\sigma^{\mu\nu}$ may allow is to write

$$J_D^{\lambda} \doteq x_\mu \, \theta^{\lambda\mu}$$

$$J_c^{\lambda\nu} = -(2x^\nu x_\mu - g_\mu^\nu x^2)\theta^{\lambda\mu} \tag{15}$$

Since, in a Poincaré invariant theory, $\theta^{\lambda\mu}$ and $\bar{\theta}^{\lambda\mu}$ can be shown to be symmetric tensors it is easily shown then

$$\partial_\lambda J_D^{\lambda} \overset{\circ}{=} \theta^\mu{}_\mu$$

$$\partial_\lambda J_c^{\lambda\mu} \overset{\circ}{=} -2x^\nu \partial_\lambda J_D^\lambda \overset{\circ}{=} -2x^\nu \theta^\mu{}_\mu \tag{16}$$

or

$$\frac{d}{dt}\int J_D^o \, d^3x = \frac{d}{dt}\int x_\mu \, \theta^{o\mu} \, d^3x \overset{\circ}{=} \int \theta^\mu{}_\mu \, d^3x$$

$$-\frac{d}{dt}\int J_c^{o\nu} \, d^3x = 2\int x^\nu \, \theta^\mu{}_\mu \, d^3x \tag{17}$$

In such a theory the trace $\theta^\mu{}_\mu$ determines whether the dilatation and conformal charges are conserved or not. It may be remarked that $\theta^\mu{}_\mu$ is much 'softer' than the trace of canonical energy momentum tensor in the sense that it involves less derivatives of the field.

The conditions necessary in Poincaré invariant theory, to obtain eqns. (4.16) and (4.17) are

$$\partial_\lambda \left[\frac{1}{2} (V^\lambda + 2\partial_\rho \sigma^{\lambda\rho}) + \Lambda_D{}^\lambda \right] = 0$$

$$\partial_\lambda \left[x^\nu (V^\lambda + 2\partial_\rho \partial^{\lambda\rho}) - 2\sigma^{\nu\lambda} - \Lambda_c{}^{\lambda\nu} \right] = 0 \qquad (18)$$

while the conditions that theory be conformal 'covariant' are from eqns. (4.2) and (4.4), $(I_T L = I_L L = 0$:

$$I_D L = \partial_\lambda \Lambda_D^\lambda$$

$$I_c{}^\nu L = -2x^\nu I_D L + V^\nu = \partial_\lambda \Lambda_c{}^{\lambda\nu} \qquad (19)$$

where we have assumed* $\kappa^\nu = 0$. It is clear from eqns. (4.19) that[9] scale invariant theory is also invariant w.r.t. the special conformal transformations if and only if

$$V^\nu = 0 \qquad (20)$$

In this case we may choose $\sigma^{\mu\nu} = 0$ to satisfy eqns. (4.18). In case eqn. (4.20) is not satisfied the scale invariance leads only to special conformal 'covariance' (c-covariance) and $V^\nu = \partial_\lambda \Lambda_c{}^{\lambda\nu}$. Eqns. (4.18) can then be satisfied by the choice

$$\sigma^{\lambda\rho} = -\frac{1}{2} \Lambda_c{}^{\rho\nu} \qquad (21)$$

This is the case, for example with massless scalar ϕ^4 theory and the

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* Note κ^ν makes transitions between fields with different L.T. law.

improved tensor $\theta^{\lambda\mu}$ involves a contribution from scalar fields but not for example from a massless spin 1/2 field for which $V^{\nu} \equiv 0$.

If the theory is c-invariant we have $V^{\nu} = 2x^{\nu} I_D L$ so that c-invariance implies a scale invariant theory if and only if $V^{\nu}=0$. If this is not the case only 'covariance' w.r.t. scale transformations is obtained. In this case, we can satisfy eqns. (4.18) by choosing

$$\sigma^{\lambda\rho} = - x^{\lambda} \Lambda^{\rho}_{D} \tag{22}$$

For a theory with only 'covariance' w.r.t. scale and c-transformations, we have

$$V^{\lambda} = \partial_{\rho} \Lambda_{c}^{\rho\lambda} + 2x^{\lambda} \partial_{\rho} \Lambda^{\rho}_{D} \tag{23}$$

and the choice for $\sigma^{\lambda\rho}$ is

$$\sigma^{\lambda\rho} = - \left(\frac{1}{2} \Lambda^{\rho\lambda}_{c} + x^{\lambda} \Lambda^{\rho}_{D} \right) \tag{24}$$

Thus if theory has symmetry w.r.t. conformal transformations and is Poincaré invariant it is always possible to write the currents in the form of eqn. (4.15) and the conservation of dilatation and special conformal currents implies then

$$\theta^{\mu}{}_{\mu} \triangleq 0 \tag{25}$$

We note also that Poincaré invariant theory, has symmetry w.r.t conformal group only if we may write the conformal deficiency vector V^{ν} of the Lagrangian in the form given by eqn. (4.23) from which Λ_c and Λ_D can be identified and the improved traceless tensor $\theta^{\mu\nu}$ then defined with a choice of $\sigma^{\mu\nu}$ given by eqn. (4.24). For the case of conformal invariance the tensor $\theta^{\lambda\mu}$ may be identified with the Belinfante tensor $\bar{\theta}^{\lambda\mu}$ whose trace must vanish. The lack of vanishing of $\bar{\theta}^{\lambda\mu}$ thus provides a measure of lack of (exact) conformal invariance in a Poincaré invariant theory but it does not exclude conformal 'covariance', for which $\theta^{\mu}_{\ \mu}$ is required to vanish. Eqn. (4.19) shows that if $V^{\nu} \equiv 0$ the theory with conformal symmetry is either invariant or 'covariant' w.r.t both the scale and special conformal transformations.

A remark on the scale invariance condition may be interesting. Working with natural units $\hbar = c = 1$ all quantities in the Lagrangian have dimensions of length. Let us denote them by

$$[m] = L^{-1} \quad [\phi_A] = L^{\ell_A} \qquad [\partial_\mu \phi_A] = L^{\ell_A - 1} \quad [f] = L^{\ell_f} \tag{26}$$

where f are the coupling constants appearing in the Lagrangian. Since in Pincaré invariant theories $[L] = L^{-4}$ we obtain on applying Euler's theorem for homogeneous functions

$$- 4L = \frac{\partial L}{\partial \phi} \ell \phi + \pi^{\lambda} (\ell - I) \partial_\lambda \phi$$

$$- \sum m \frac{\partial L}{\partial m} + \sum \ell_f f \frac{\partial L}{\partial f} \tag{27}$$

where $\ell \equiv (\ell_A I_{AB})$ is a diagonal matrix. Then

$$I_D L = \frac{\partial L}{\partial \phi} (L+\ell)\phi + \pi^{\lambda} (L+\ell)\partial_\lambda \phi$$

$$- \sum m \frac{\partial L}{\partial m} + \sum \ell_f f \frac{\partial L}{\partial f} \tag{28}$$

We may write $L = \sum g_\gamma L_\gamma$ where g_γ are coupling constants constructed from the masses and couplings f. The last two terms can then be written as $\sum g_\gamma \alpha_\gamma L_\gamma$ where dimension of g_γ is L^{α_γ}. Then

$$I_D L = \sum \{ \frac{\partial L_\gamma}{\partial \phi} (L + \ell)\phi + \pi_\gamma^\lambda (L + \ell)\partial_\lambda \phi + \alpha_\gamma L_\gamma \} g_\gamma \tag{29}$$

Scale invariance condition then implies that for each dimension less coupling we must have

$$\frac{\partial L_\gamma}{\partial \phi} (L + \ell)\phi + \pi_\gamma{}^\lambda (L + \ell)\partial_\lambda \phi = 0 \tag{30}$$

and for each dimensional coupling the parenthesis { } must vanish. If we assume* $L = - \ell = - (\ell_A \delta_{AB})$ no dimensional couplings may be present if scale invariance holds. For interacting field theory, it is clear that not all the masses need to vanish in the scale invariant limit.

5. ILLUSTRATIONS FOR SOME FIELD THEORIES

a) *Scalar Field Theory:*

To illustrate our discussion we study the following Lagrangian for a scalar field ϕ:

$$L = \frac{1}{2} [(\partial^\mu \phi)(\partial_\mu \phi) - m^2 \phi^2] + \frac{g}{3} \phi^3 + \frac{\lambda}{4} \phi^4 \tag{1}$$

$$\pi^\mu = \partial^\mu \phi \qquad \frac{\partial L}{\partial \phi} = - m^2\phi + g\phi^2 + \lambda\phi^3 \tag{2}$$

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* See remarks at the end of section 3.

Euler's eqns. are ($\Box \equiv \partial^\mu \partial_\mu$)

$$(\Box + m^2)\phi = g\phi^2 + \lambda\phi^3 \tag{3}$$

Lorentz invariant condition is verified to be satisfied with $\sum^{\rho\sigma} = 0$. The energy momentum tensor is

$$\tau^{\mu\nu} = \tilde{\theta}^{\mu\nu} = (\partial^\mu \phi)(\partial^\nu \phi) - g^{\mu\nu}L \tag{4}$$

$$\tilde{\theta}^\mu{}_\mu = -(\partial^\mu \phi)(\partial_\mu \phi) + 2m^2 \phi^2 - \frac{4}{3} g\phi^3 - \lambda\phi^4 \neq 0 \tag{5}$$

Theory, therefore, can at best be conformal covariant. This may also be seen from conformal deficiency vector

$$V^\lambda = -2(\partial^\lambda \phi)L\phi = -L\,\partial^\lambda\phi^2 = -L\,\partial_\rho(g^{\rho\lambda}\phi^2) \tag{6}$$

which does not vanish due to the kinetic energy term[10]. It also shows that w.r.t special conformal transformations we may at best obtain 'covariance', while scale invariance is not excluded. Since ϕ and g have length dimension (-1) the scale invariance condition* is

$$(L-1)(\partial^\mu\phi)\partial_\mu\phi) + m^2(2-L)\phi^2 + g(L-\tfrac{4}{3})\phi^3 + \lambda(L-1)\phi^4 = 0 \tag{7}$$

---

* It is interesting to note that if we apply scale invariance transformation $\phi'(x') = \rho^{-1}\phi(x)$; $x' = \rho x$ to the Euler's equation it is left invariant also with the choice $L = 2$, $m = 0$, $\lambda = 0$.

For kinetic energy term $(L-1)\ (\partial^{\mu}\phi)\ (\partial_{\mu}\phi)$ to vanish identically $L = 1$; it then follows $m=0$ and $g=0$.

Massless scalar theory with

$$L = \frac{1}{2}\,(\partial_{\mu}\phi)\ (\partial^{\mu}\phi) + \frac{\lambda}{4}\,\phi^4 \qquad (8)$$

is thus scale invariant. We find

$$I_c{}^{\lambda} L = V^{\lambda} = -\ \partial\rho(g^{\rho\lambda}\ \phi^2) \qquad (9)$$

so that $\Lambda_c{}^{\lambda\rho} = -\ g^{\rho\lambda}\ \phi^2$ and $\sigma^{\lambda\rho} = \frac{1}{2}\,g^{\rho\lambda}\ \phi^2$

The improved energy momentum tensor of eqn. (4.12) can be calculated easily

$$\theta^{\lambda\rho} = \bar{\theta}^{\lambda\rho} - \frac{1}{6}\,(\partial^{\lambda}\ \partial^{\rho} - g^{\lambda\rho}\,\Box\,)\phi^2 \qquad (10)$$

and

$$\theta^{\mu}{}_{\mu} = \phi\ [-\ \lambda\phi^3 + \Box\ \phi] + 2m^2\ \phi^2 - \frac{4}{3}\,g\phi^3 \qquad (11)$$

$$\underline{\underline{o}}\ \ m^2\ \phi^2 - \frac{1}{3}\,g\phi^3$$

Thus m and g are responsible for breaking scale invariance. All the currents can be written as moments of the tensor $\theta^{\lambda\mu}$ according to eqn. (4.15).

b) *Dirac Field Theory:*

$$L = \frac{1}{2}\{ \bar{\Psi}(i\gamma.\partial - m)\Psi + \bar{\Psi}(-i\gamma.\overleftarrow{\partial} - m)\Psi \} \qquad (12)$$

$$\bar{\Psi} = \psi^{\dagger}\gamma^{0}; \qquad \pi^{\mu} = \frac{i}{2}\bar{\Psi}\gamma^{\mu}; \qquad \frac{\partial L}{\partial \Psi} = \frac{i}{2}\bar{\Psi}\gamma.\overleftarrow{\partial} - m\bar{\Psi}$$

$$\pi^{*\mu} = -\frac{i}{2}(\gamma^{0}\gamma^{\mu}\Psi); \qquad \frac{\partial L}{\partial \Psi^{*}} = \gamma^{0}\left(\frac{i}{2}\gamma.\partial\Psi - m\Psi\right) \qquad (13)$$

Euler's eqns. are

$$(-i\gamma.\partial + m)\Psi = 0$$

$$\bar{\Psi}(i\gamma.\partial + m) = 0 \qquad (14)$$

Lorentz invariance condition is identically satisfied* for $\Sigma^{\rho\sigma} = \frac{i}{4}[\gamma^{\rho}.\gamma^{\sigma}]$. For free field with canonical dimension $\ell = -3/2\ I$, the Lagrangian L has

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* Note $\delta_L \Psi = -\frac{i}{2}\varepsilon_{\rho\sigma}\Sigma^{\rho\sigma}$ $\quad \delta_L \Psi^{*} = \frac{i}{2}\varepsilon_{\rho\sigma}\overset{*}{\Sigma}^{\rho\sigma}\Psi^{*}$ and

$\delta_D \Psi = \varepsilon L\Psi$, $\delta_D \Psi^{*} = \varepsilon L^{*}\Psi$. We also use $\gamma_0 L^{\dagger}\gamma_0 = L$.

length dimension -4. The scale invariance is obtained for massless theory with L = 3/2 I. The conformal deficiency vector vanishes identically even for massive case so that scale invariance also implies special conformal invariance, and $\theta^{\mu\nu}$ has no contribution from massless spin 1/2 field.

c) *Vector Field Theory*

$$L = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu$$

where† $$F^{\mu\nu} = - F^{\nu\mu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{15}$$

$$\frac{\partial L}{\partial A_\lambda} = - m^2 A^\lambda, \qquad \frac{\partial L}{\partial(\partial_\rho A_\lambda)} = F^{\rho\lambda} \tag{16}$$

Euler's equation are

$$\partial_\rho F^{\rho\lambda} = - m^2 A^\lambda$$

$$\partial_\lambda A^\lambda = - \frac{1}{m^2} \partial_\lambda \partial_\rho F^{\rho\lambda} \tag{17}$$

For $m \neq 0$ then

$$(\Box + m^2) A^\lambda = 0 \tag{18}$$

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† $\partial F_{\mu\nu}/\partial(\partial_\rho A_\lambda) = (g^\rho_\mu g^\lambda_\nu - g^\rho_\nu g^\lambda_\mu)$, $\quad \dfrac{\partial(F^{\mu\nu} F_{\mu\nu})}{\partial(\partial_\rho A_\lambda)} = 2F^{\mu\nu} \partial F_{\mu\nu}/\partial(\partial_\rho A_\lambda)$

Applying Lorentz invariance $\Sigma^{\rho\sigma}$ may be easily found to be

$$(\Sigma^{\mu\nu})_{\lambda\sigma} = i(g^{\mu}_{\lambda}\, g^{\nu}_{\sigma} - g^{\mu}_{\sigma}\, g^{\nu}_{\lambda}) \tag{19}$$

The commutation relations of $\Sigma^{\mu\nu}$ can be verified to be analogous to the equn. (2.28). Conformal deficiency vector is

$$\begin{aligned} V^{\nu} &= 2i\, \{iF^{\nu\lambda}\, L_{\lambda\sigma}\, A^{\sigma} + F_{\mu\lambda}(\Sigma^{\nu\mu})^{\lambda\sigma}\, A_{\sigma}\, \} \\ &= -\, 2(L_{\lambda\sigma} - g_{\lambda\sigma})\, F^{\nu\lambda}\, A^{\sigma} \end{aligned} \tag{20}$$

It vanishes if

$$L_{\lambda\sigma} = g_{\lambda\sigma} \qquad \text{or} \qquad L^{\lambda}{}_{\sigma} = g^{\lambda}_{\sigma} \tag{21}$$

We also note that $[\, A^{\lambda}\,] = L^{-1}$. For scale invariance, if $L^{\lambda}{}_{\sigma} = g^{\lambda}_{\sigma}$, theory must be massless which is well known and theory is then conformal invariant. There is no contribution to $\theta^{\mu\nu}$ from massless vector field. We also note

$$\tau^{\mu\nu} = F^{\mu\nu}\, \partial^{\nu} A_{\lambda} - g^{\mu\nu}\, L \tag{22}$$

$$\frac{1}{2}\chi^{\lambda\mu\rho} = F^{\lambda\mu}\, A^{\rho} \tag{23}$$

$$\begin{aligned} \bar{\theta}^{\nu\mu} = \bar{\theta}^{\mu\nu} &= g^{\rho\nu}\, F^{\mu\lambda}\, F_{\rho\lambda} + (\partial_{\lambda}\, F^{\lambda\mu})A^{\nu} - g^{\mu\nu}\, L \\ &\hat{=}\; g^{\rho\nu}\, F^{\mu\lambda}\, F_{\rho\lambda} - m^{2}\, A^{\mu}\, A^{\nu} - g^{\mu\nu}\, L \end{aligned} \tag{24}$$

and

$$\theta^{\mu}_{\ \mu} = (\partial_{\lambda} F^{\lambda\mu})A_{\mu} + 2m^2 A_{\mu} A^{\mu} \stackrel{\circ}{=} m^2 A_{\mu} A^{\mu} \tag{25}$$

6. FIELDS WITH ANOMALOUS SCALE TRANSFORMATIONS

To illustrate the consequences of modifield scale invariance condition in case some of the fields do not have the normal scale transformation we consider a field theory with the fields $\{\phi_A\} \equiv \phi$ with normal transformation and a single scalar field $\sigma(x)$ with the scale transformation given by

$$\delta x^{\mu} = -\varepsilon x^{\mu} \qquad \delta\sigma(x) = \varepsilon T\sigma_o \qquad \partial_{\lambda}(\delta\sigma) = 0 \tag{1}$$

where σ_o is a constant field with dimensions of mass i.e. $[\sigma] = [\sigma_o] = L^{-1}$. It is convinient to work with dimensionless field $p(x) = \sigma(x)/M$, $p_o = \sigma_o/M$ where M is some mass. We have $[p(x)] = [p_o] = L^o$ but $[\partial_{\mu}p(x)] = L^{-1}$

$\delta p(x) = \varepsilon T p_o$. The invariance condition is

$$I_D L = -4L + \frac{\partial L}{\partial\phi} L\phi + \pi^{\lambda}(L + I)\partial_{\lambda}\phi + \frac{\partial L}{\partial p} Tp_o + \frac{\partial L}{\partial(\partial_{\lambda} p)} (\partial_{\lambda} p) \tag{2}$$

where from Euler's theorem

$$-4L = \frac{\partial L}{\partial \phi} \ell\phi + \pi^{\lambda} \quad (\ell - I)\partial_\lambda \phi - \frac{\partial L}{\partial\,(\partial_\lambda\, p)} (\partial_\lambda\, p)$$

$$- \sum m \frac{\partial L}{\partial m} + \sum \quad f\, \ell_f \frac{\partial L}{\partial f} \tag{3}$$

Hence assuming the fields with normal transformation have the canonical dimension viz $(L + \ell) = 0$ the scale invariance requires

$$- \frac{\partial L}{\partial p} T p_o = - \sum m \frac{\partial L}{\partial m} + \sum f \ell_f \frac{\partial L}{\partial f} \tag{4}$$

Writing the Lagrangian $L = \sum g_\gamma L_\gamma$, where g_γ are quantities constructed out of m and f and have dimension α_γ we obtain

$$\frac{\partial L_\gamma}{\partial p} T p_o = - \alpha_\gamma L_\gamma \tag{5}$$

or

$$L_\gamma = L_\gamma^{(o)} \quad \exp(- \frac{p(x)}{T p_o} \alpha_\gamma) \tag{6}$$

where $L_\gamma^{(o)}$ is independent of $\sigma(x)$ but may depend on $(\partial_\lambda \sigma)$. Thus $\sigma(x)$ appears in Lagrangian in a very specific form. Consider, for example, the kinetic energy term of field; it is of the form $(\partial_\mu \sigma)(\partial^\mu \sigma) A(\sigma) = M^2 (\partial^\mu p)(\partial_\mu p) A(p)$ where $A(\sigma)$ is a dimensionless function. Then

$$L_{KE} = L^{(o)} \quad \exp\,(2 \frac{\sigma(x)}{T\sigma_o}) = \frac{1}{2} (\partial^\mu \sigma)^2 \exp\,(2 \frac{\sigma(x)}{T\sigma_o}) \tag{7}$$

with appropriate normalization factors.

Another type of anomalous scale transformation is

$$I_D \sigma = \varepsilon T(\sigma(x) - \sigma_o)$$

or (8)

$$(\sigma'(x') - \sigma_o) \simeq (I + \varepsilon T)(\sigma(x) - \sigma_o)$$

The invariance condition may be discussed as above.

* * *

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