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DIRAC'S EQUATION IN A WEYL SPACE

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ABSTRACT

Dirac's equation is written in a Weyl-space by means of a quaternionic formulation of the Weyl-geometry.

The so-called minimal-coupling of the electron with the electromagnetic field is an obvious consequence of such formalism.

Mass and charge can be inter-related by an investigation of the theory of mass-less charged particles imbedded in a Weyl-Space.

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1. DIRAC'S EQUATION

Let us represent by $\psi(x)$ a Dirac spinor field,

$$\left\{ i\gamma^{\mu} \frac{\partial}{\partial x^{\mu}} - m \right\} \psi(x) = 0 \tag{1}$$

in the well-known convenction, $\hbar$ and c are units of action and velocity respectively and the scalar product of two vectors is

$$\begin{aligned} a^{\mu} b_{\mu} &= a^{\mu} b^{\nu} g_{\mu\nu} \\ g_{\mu\nu} &= 0, \quad \text{for} \quad \mu \neq \nu \\ g_{00} &= - g_{11} = -g_{22} = - g_{33} = 1 \, . \end{aligned} \tag{2}$$

Besides this, the γ's satisfy the anti-commutation relations

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\, g^{\mu\nu}\, I \tag{3}$$

(I is the identity operator)

Equation (1) is written in a pseudo-euclidean space, the so-called Minkowski-space of the special theory of relativity. When one tries to generalize equation (1) for a more general type of space, such as the Riemannian-space, difficulties arise by the spinorial character of the ψ-field. Nevertheless, many authors have investigated an equation in a Riemannian-space that in many aspects resembles Dirac's equation. Generally one writes

$$\left\{ i\gamma^{\mu} \left[\frac{\partial}{\partial x^{\mu}} + \Gamma_{\mu} \right] - m \right\} \psi(x) = 0 \tag{4}$$

where now the relation (3) has to be understood as

$$\gamma^{\mu}(x)\gamma^{\nu}(x) + \gamma^{\nu}(x)\gamma^{\mu}(x) = 2\, g^{\mu\nu}_{(x)}\, I \tag{5}$$

The Γ_{μ}'s are 4×4 matrices that may be obtained by imposing a restriction on the structure of the space.

2. WEYL-SPACE

It is well-known that a Riemannian-space, besides having a non-null curvature R, has two properties of the metric tensor that characterize the space. One is that of symmetry and the other is that the covariant derivative of the metric tensor vanishes. So, we can write that in a Riemannian-space we must have

$$g^{\mu\nu}(x) = g^{\nu\mu}(x) \tag{6}$$

$$g^{\mu\nu}{}_{||\lambda}(x) = 0 \tag{7}$$

where the covariant derivative is defined by

$$g^{\mu\nu}{}_{||\lambda} = g^{\mu\nu}{}_{|\lambda} + \left\{ {\mu \atop \lambda\alpha} \right\} g^{\alpha\nu} + \left\{ {\nu \atop \lambda\alpha} \right\} g^{\mu\alpha} \tag{8}$$

$\left\{ {\mu \atop \lambda\alpha} \right\}$ are the Christoffell-symbols and $g^{\mu\nu}{}_{|\lambda}(x)$ means $\dfrac{\partial g^{\mu\nu}(x)}{\partial x^{\lambda}}$.

Weyl [1] has introduced a very interesting modification of this model by neglecting condition (7). He introduces a four-vector field $\varphi_\lambda(x)$, such that

$$g^{\mu\nu}{}_{||\lambda}(x) = \varphi_\lambda(x)\, g^{\mu\nu}(x) \; . \tag{9}$$

Weyl applied this kind of geometry in an unified theory of the classical electromagnetic and gravitational fields. In this paper we assume Weyl's physical interpretation of this kind of geometry but we intend to use the quaternionic formalism rather than the usual tensor relations. It will appear that this quaternionic treatment can make more understandable a few results of the theory of fields.

3. QUATERNIONIC FORMALISM

We will review some definitions and results of the theory of the quaternions. For a more extensive analysis the reader may consult the references [2].

Let us assume that $\dot{\sigma}^{\mu}$ (μ = 0, 1, 2, 3) is a basis in the quaternionic-space. Then, any quaternion A may be decomposed as a sum

$$A = A_{\alpha} \dot{\sigma}^{\alpha} \quad (10)$$

or

$$A = A_0 \dot{\sigma}^0 + A_k \dot{\sigma}^k . \quad (10')$$

If we impose that

$$\dot{\sigma}^k \dot{\sigma}^{\ell} + \dot{\sigma}^{\ell} \dot{\sigma}^k = 0 \qquad \text{for } k \neq \ell$$

$$\dot{\sigma}^k \dot{\sigma}^k = I \qquad (\ell, k = 1, 2, 3) \quad (11)$$

then we see that there is an isomorphism between the basis $\dot{\sigma}^{\alpha}$ and Pauli-matrices plus the identity I_2 in the two-dimensional linear vector space. We can define a scalar product such that, if A and B are any two quaternions then the scalar product is defined as

$$A|B = \frac{1}{2} (A\bar{B} + B\bar{A}) \quad (12)$$

where $\bar{A}$ is the adjoint of A, that is

$$\bar{A} = A_0 \dot{\sigma}^0 - A_k \dot{\sigma}^k . \quad (13)$$

We introduce a metric $\dot{g}^{\alpha\beta}$ by the definition

$$\dot{g}^{\alpha\beta} = \dot{\sigma}^{\alpha} | \dot{\sigma}^{\beta} \quad (14)$$

and we note that this metric has the same form as the one introduced before by equation (2).

We can use this metric to lower and raise quaternionic-indices.

Let us generalise this approach by setting σ^{α} as function of X, by means of the 16-parameters $h^{\alpha}_{(\lambda)}(x)$:

$$\sigma^{\alpha}(x) = h^{\alpha}_{(\lambda)}(x)\, \dot{\sigma}^{(\lambda)} \, . \tag{15}$$

Then we obtain the metric tensor $g^{\alpha\beta}(x)$ as

$$g^{\alpha\beta}(x) = \sigma^{\alpha}(x)|\sigma^{\beta}(x) \tag{16}$$

or

$$g^{\alpha\beta}(x) = h^{\alpha}_{(\lambda)}(x)\, h^{\beta(\lambda)}(x) \, . \tag{16'}$$

4. Weyl's geometry in quaternionic formalism

A simple inspection of equation (16) shows that if we want to generalize equation (7) to equation (9), then we may impose a condition on the σ's rather than directly on the metric tensor. Indeed, we assume that

$$\sigma^{\mu}{}_{||\lambda} = \frac{1}{2} \mathrm{Tr}\,(A\bar{\sigma}_{\lambda})\, \sigma^{\mu}(x) \, . \tag{17}$$

Where the quaternion A may be decomposed as a sum (eq. 10) with the electromagnetic potential as the coefficients A_{α}. By Tr(B), where B is a quaternion, we understand the expression

$$\dot{\sigma}^{0}\, \mathrm{Tr}(B) = B + \bar{B} \, .$$

Now the covariant derivative has an expression a slightly modified

$$\sigma^{\mu}{}_{||\lambda}(x) = \sigma^{\mu}{}_{|\lambda} + \Gamma^{\mu}_{\lambda\nu}\, \sigma^{\nu} + \Gamma_{\lambda}\, \sigma^{\mu} + \sigma^{\mu}\, \Gamma_{\lambda}^{\dagger} \tag{18}$$

where the $\Gamma^{\mu}_{\lambda\nu}$ are no more the Christoffel symbols.

A strightforward calculation shows that we obtain by (16) and (17)

$$g^{\mu\nu}{}_{||\lambda}(x) = \mathrm{Tr}(A\bar{\sigma}_{\lambda})\, g^{\mu\nu}(x) \tag{19}$$

that is, the Weyl-field $\varphi_\lambda(x)$ is now $\mathrm{Tr}(A\bar{\sigma}_\lambda)$.

Using equations (17) and (18) we can calculate the Γ_λ's, and obtain

$$\Gamma_\lambda = -\frac{1}{4}\left\{\left(\sigma_{|\lambda}^{\mu} + \Gamma_{\lambda\nu}^{\mu}\right)\bar{\sigma}_\mu - 2\,\mathrm{Tr}(A\bar{\sigma}_\lambda)\right\}. \tag{20}$$

Note that when the electromagnetic potentials are null, the Γ_λ's take the usual expression, as would be expected.

Instead of using the two-dimensional formalism we can use the four-dimensional one and try to return to equation (4) generalized.

It is easy to see that if we put

$$\gamma^\mu(x) = \begin{pmatrix} 0 & \sigma^\mu(x) \\ \bar{\sigma}^\mu(x) & 0 \end{pmatrix} \tag{21}$$

then, by equation (16), we obtain equation (5).

The covariant derivative of the γ's are easily obtained:

$$\gamma^\mu_{||\lambda}(x) = \gamma^\mu_{|\lambda} + \Gamma^\mu_{\lambda\nu}\gamma^\nu + \gamma^\mu\tau_\lambda - \tau_\lambda\gamma^\mu \tag{22}$$

where

$$\tau_\lambda = \begin{pmatrix} -\Gamma_\lambda & 0 \\ 0 & \Gamma_\lambda^+ \end{pmatrix} \tag{23}$$

the Γ_λ's are defined by equation (20): Γ_λ^+ means the transposed conjugate of Γ_λ.

A straightforward calculation gives us another expression of τ_λ in terms of the γ's,

$$\tau_\lambda = \frac{1}{8}\left\{\gamma^\mu_{|\lambda}\gamma_\mu - \gamma_\mu\gamma^\mu_{|\lambda} + \Gamma^\mu_{\lambda\nu}(\gamma_\mu\gamma^\nu - \gamma^\nu\gamma_\mu)\right\}.$$

This expression reduces to the well-know expression of the Υ_λ of equation (4) when free of the electromagnetic field.

4. Dirac's equation in interaction with the electromagnetic field

A natural extension of the Dirac equation (4) written in a Riemannian space to a Weyl space, may be obtained by letting the Υ_μ go into $\Upsilon_\mu - \frac{3}{4}\,\mathrm{Tr}(A\bar{\sigma}_\mu)$ or, explicitly,

$$\left[i\gamma^\mu \left\{ \frac{\partial}{\partial x^\mu} + \Upsilon_\mu - \frac{3}{4}\,\mathrm{Tr}(A\bar{\sigma}_\mu) \right\} - m \right] \psi(x) = 0 \, . \qquad (24)$$

Expression (24) shows why the so called minimal coupling may be interpreted as the natural interaction of the electron with the electromagnetic field. By Weyl's interpretation, the electromagnetic field is responsible for the aditional factor $-\frac{3}{4}\mathrm{Tr}(A\bar{\sigma}_\mu)$ in equation (24) and this may be interpreted as changing the derivative - operator $\frac{\partial}{\partial x^\mu}$ into the translated - operator $\frac{\partial}{\partial x^\mu} - \frac{3}{4}\,\mathrm{Tr}(A\bar{\sigma}_\mu)$, giving origin to the minimal coupling. It is interesting to note that the electromagnetic-field, as seen by the particle represented by the ψ-field, contains the charge-factor inside the potentials. So, we see that distinct particles see distinct space-structure by means of its proper charge-value.

Another observation can be stated by analyzing a ψ-particle that does not interact with the electromagnetic-field. If this particle does not see the Weyl-structure of the space the only responsible for that is, possibly, the gauge of the mass [3].

Indeed, if we put

$$m' = m - i\,\frac{3}{4}\,\gamma^{\mu}\ \mathrm{Tr}(A\bar{\sigma}_{\mu}) \tag{25}$$

then equation (24) reduces to

$$\left\{ i\,\gamma^{\mu}\left(\frac{\partial}{\partial x^{\mu}} + \Gamma_{\mu}\right) - m' \right\} \psi(x) = 0 \tag{26}$$

which is the equation of a particle with mass m', considered as an operator given by equation (25), in a space free of the electromagnetic field.

The equation of a mass-less charged particle in a Weyl-space is

$$\left\{ i\,\gamma^{\mu}\left(\frac{\partial}{\partial x^{\mu}} + \Gamma_{\mu} - \frac{3}{4}\ \mathrm{Tr}(A\bar{\sigma}_{\mu})\right)\right\} \psi(x) = 0 \tag{27}$$

which may be interpreted as the equation of a particle with mass $m' = +\frac{i3}{4}\gamma^{\mu}\,\mathrm{Tr}(A\bar{\sigma}_{\mu})$ in a Riemannian - space. Thus, a mass -less charged particle in a Weyl-space (that is, interacting with the electromagnetic-field) behaves as a particle with variables mass in a Riemannian-space (without electromagnetic-field).

So, giving up the absolute character of the proper-mass may give origin to the charge-concept.

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