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ON THE COVARIANCE OF EQUAL TIME COMMUTATORS AND SUM RULES *

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ABSTRACT

Conditions for covariance of equal time commutators are given. In momentum space they imply relations among the invariant amplitudes. It is shown that the most general expression for the sum rule, in any particular frame, is

$$\int d\tau \, n^{\mu} \, t_{\mu}(k^{\nu} + \tau n^{\nu}) = \text{form factor}$$

where n_{μ} is an arbitrary time-like vector. The covariance conditions are expressed by the fact that the integral is n-independent. In particular, Fubini's sum rule, which corresponds to a choice of n in the light cone, in spite of being covariant must be supplemented with a set of covariance conditions.

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1. INTRODUCTION

It is known that, assuming the validity of equal time canonical commutation relations, it is possible to deduce sum rules which have been tested in specific examples. At first sight, it is surprising to find in those examples that the final result is invariant in spite of having started from a time component of an equal time commutator. It is this point (and some similar ones) which we want to discuss in this paper.

Related problems have recently been considered by several authors [1, 2, 3].

For the deduction of sum rules, one can adopt FUBINI's method [4]. It consists essentially of the use of dispersion relations for the invariant amplitudes. The connexion of this method with the $\dot{P} \rightarrow \infty$ system is made clear by following AMATI's et al. procedure [5] starting from the method introduced in Ref. 6. We shall use the latter method for our discussion and shall comment in the Appendix on that of Fubini.

Let us first make a summary of the deduction of the sum rules. We shall take, for definiteness, a vector commutator. (The generalization to tensors can easily be done):

$$f_\mu(x) \equiv [j_\mu(x), j(0)] \qquad (1)$$

It is known from causality that $f_\mu(x)$ has to be zero outside the light cone. Inside the cone it is an unknown function, as we do not know how to compute the influence of the interaction. Nevertheless, following GELL-MANN [7], it is assumed that for equal times ($x_o = 0$) the commutator is given by the current algebra or by the commutation relations of free-field operators. With that assumption, if we multiply the time component of $f_\mu(x)$ by $\delta(x_o)$ we obtain (in any frame)

$$\delta(x_o)\, f_o(x) = \text{invariant factor} \times \delta^4(x) = \text{inv.} \qquad (2)$$

The Fourier transform of this expression is [6] (in any frame)

$$\int t_o . dk_o = \text{inv. form factor} \qquad (3)$$

where

$$t_\mu = \mathcal{F}\left[\langle 2|f_\mu|1\rangle\right] \;, \quad \mathcal{F} = \text{Fourier transform.}$$

Eqs. (2) and (3) show the striking result that the multiplication by $\delta(x_o)$ (or integration on k_o) of the fourth component of a vector, which is not an invariant operation, gives an invariant result. However, the left-hand side of eq. (2) can be written in the form

$$\frac{\partial \theta(x_o)}{\partial x_\mu} \cdot f_\mu(x) \;, \qquad \theta(x_o) = \text{step function} \;, \qquad (4)$$

which is formally invariant if $f_\mu(x)$ is zero outside the light cone. Nevertheless, $\theta(x_o)$ is not covariant and if it is substituted by $\theta(n.x)$ it remains to be proved that the result is independent of n. This will not be true in general but only for those f_μ fulfilling certain condition which we want to find.

We would also like to point out that for specific calculations, eq. (3) is usually written in a particular frame and, of course, in so doing one loses the information that $\int t_o\, dk_o$

in other frames gives the same result. To regain this information the covariance condition must be taken into account. One can express this in a different although equivalent way. If a vector n is introduced to give an invariant look to eq. (3), i.e.,

$$\int t_0 dk_0 = \int t.n_0 dk.n_0 \ , \quad \text{with} \quad n_0 = (1,0,0,0) \ , \qquad (3')$$

then, the r.h.s. of (3') must be n_0-independent and the conditions for this to be so are just the covariance relations. In particular, in the $P \to \infty$ system of Ref. 5 the corresponding sum rule (Fubini's) must be supplemented by the covariance conditions to have the full physical information of the original sum rule, (eq. 3). In Sec. 2 we find the condition for covariance in co-ordinate space. In Sec. 3 we consider the Fourier transform of the previous result. In this way, some explicit relations between the invariant amplitude are obtained. We write a compact expression for the sum rules, containing also the covariant conditions. In Sec. 4 we show that a form of those conditions has also to be taken into account even for non-covariant equal time commutators. In Sec. 5 we give a discussion of the previous results. In the Appendix we find the conditions for the covariance of Fubini's method and also of some more general sum rules.

2. COVARIANCE CONDITIONS

The problem is, then, given a four-vector $f_\mu(x)$, to find the conditions for the product $f_o(x)\,\delta(x_o)$ to be independent of the reference system. For any Lorentz transformation

$$x' = \alpha_\mu^\nu x_\nu \quad , \quad f'_\mu(x') = \alpha_\mu^\nu \; f_\nu(x) \tag{5}$$

we have

$$f'_\mu(x')\, \partial'^\mu \, \Theta(x'_o) = f_\mu(x)\partial^\mu \;\; \Theta(x'_o)$$

or

$$f_\mu(x)\partial^\mu \;\; \Theta(x_o) = f_\mu(x)\partial^\mu \;\; \Theta(x'_o) = f_\mu(x)\partial^\mu \;\; \Theta(n.x). \tag{6}$$

(We see that the covariance condition implies that, in a given reference frame, the last term of (6) be independent of the time-like vector n). Using now (see Ref. 8)):

$$\Theta(x'_o) = \Theta(a_o^o x_o + a_o^i\, x_i) = \Theta(x_o + \bar{a}_o^i\, x_i)$$

$$= \sum_{p=0}^{\infty} \frac{1}{p!} \frac{d^p \; \Theta(x_o)}{dx_o^p} \; \bar{a}_o^{\,i_1} \ldots \bar{a}_o^{\,i_p} \, ,$$

(where $\bar{a}_o^{\,i} = \dfrac{a_o^i}{a_o}$ and $\Theta(\alpha x) = \Theta(x)$ for $\alpha > 0$, have been used) and, replacing in (6), we find the following conditions for covariance:

$$f_\mu(x)\,\partial^\mu \left[\delta^{(p-1)}(x_o) x_{i_1} \ldots x_{i_p}\right] = 0 \text{ (no summation over p)} . \tag{7}$$

This is the set of infinite equations which must be satisfied in a particular system in order that in any other frame of reference the expression $\delta(x'_o)\; f'_o(x')$ has the same value as in the original one [9]. Of course eqs. (7) are the conditions for the expression $f_\mu(x)\partial^\mu\, \Theta(n.x)$ to be n-independent.

If one remains in a single reference system, the infinite set of eqs. (7) give all the information we can obtain from the covariance of sum rule (3).

Of course, as soon as we assume that the eqs. (7) for $p = p_0$ are valid in <u>any</u> system, then all the equations with $p > p_0$ follow as consequences and are thus unnecessary. For example, if we can show that, in any frame, a particular commutator f_μ satisfies the eqs. (7) for p = 1, namely,

$$f_\mu(x)\partial^\mu\left[\delta(x_0)x_i\right] = 0 \qquad \text{or} \qquad \delta(x_0)f_i + \delta'(x_0)x_i f_0 = 0 \ , \qquad (8)$$

then the covariance of $f_\mu \, \partial^\mu \theta(x_0)$ is guaranteed. This is the case of the commutator of two-vector currents

$$j_\mu^\alpha = \bar{\Psi}\gamma_\mu \lambda^a \Psi$$

calculated in the free-field model [10]. It can be seen that

$$f_{\mu\nu}^c = \varepsilon_{abc}\left[j_\mu^a(x), \ j_\nu^b(0)\right]$$

satisfies eqs. (7) for p = 1 in any frame, i.e.:

$$\delta(x_0)f_{i\nu} + \delta'(x_0)x_i f_{0\nu} = 0$$

so that $\delta(x_0) \, f_{0\nu}$ is a vector, a fact that is easy to check directly.

Also the free-field commutator of the pseudoscalar current $j = \bar{\Psi}\,\gamma_5\Psi$ and a tensor current $j_{\mu\nu} = \bar{\Psi}[\gamma_\mu,\gamma_\nu]\Psi$ satisfies condition (8).

On the other hand, if we consider a C-number Schwinger term

$$f_{\mu\nu} = \partial_\mu\partial_\nu\Delta \qquad (\Delta = \text{Pauli-Jordan invariant function})$$

or the free-field commutator of a scalar current $j = \bar{\psi}\psi$ and a tensor current $j_{\mu\nu} = \bar{\psi}[\gamma_\mu, \gamma_\nu]\psi$, we can verify that none of them fulfils eq. (8), showing non-covariance of $\delta(x_o)f_{o\nu}$.

3. COVARIANCE CONDITIONS IN MOMENTUM SPACE

We shall now discuss covariance conditions in momentum space as there they will be more appropriate for physical applications.

Taking the Fourier transform of eq. (6) we obtain

$$\int d\tau\, n_o^\mu\, t_\mu(k^\nu + \tau n_o^\nu) = \int d\tau\, n^\mu t_\mu(k^\nu + \tau n^\nu) \; . \qquad (9)$$

Thus the result of the integration must be n-independent if it is to be invariant. If we now develop the integrand as a series in $n^i - n_o^i$, we find (to simplify we have introduced k_o as a new integration variable)

$$\int dk_o \Big[k_o^p \frac{\partial}{\partial k^{i_1}} \cdots \frac{\partial}{\partial k^{i_p}}\, t_o(k_o, \vec{k}) +$$

$$+ k_o^{p-1} \sum_{s=1}^{p} \frac{\partial}{\partial k^{i_1}} \cdots \frac{\partial}{k^{i_{s-1}}} \frac{\partial}{\partial k^{i_{s+1}}} \cdots \frac{\partial}{\partial k^{i_p}}\, t_{i_s}(k_o, \vec{k}) \Big] \qquad (10)$$

which are the Fourier transform of eqs. (7). So, the original sum rule has to be supplemented with all the set (10) in order to guarantee invariance.

Conditions (10) give additional relations between the invariants entering into t_μ. For instance, let us take p = 1.

$$\int dk_o \left(k_o \frac{\partial t_o}{\partial k^i} + t_i \right) = 0 \qquad (11)$$

For the case of spinless one-particle initial and final states, we have

$$t_\mu = \alpha P_\mu + bk_\mu + c\Delta_\mu = \sum_i a_i V^i_\mu . \qquad (12)$$

The three invariants α_i are functions of the scalars

$$u^1 = P.k \ , \quad u^2 = \frac{1}{2} k^2 \ , \quad u^3 = \Delta.k$$

and

$$\frac{\partial u^i}{\partial k^\mu} = V^i_\mu . \qquad (13)$$

Using (12) and (13) in eq. (11) we get

$$\int dk_o \left[\alpha_i + \sum_\ell k_o V^\ell_o \frac{\partial a_\ell}{\partial u^i}\right] = 0 . \qquad (14)$$

By writing these equations in the system $p \to \infty$ (defined in Ref. 5) we obtain [11]

$$\left.\begin{aligned} &\int d\nu \left(a + \nu \frac{\partial a}{\partial \nu}\right) = \nu a \Big|_{-\infty}^{\infty} = 0 \\ &\int d\nu \left(b + \nu \frac{\partial a}{\partial u^2}\right) = 0 \\ &\int d\nu \left(c + \nu \frac{\partial a}{\partial u^3}\right) = 0 \end{aligned}\right\} \qquad (15)$$

It is also possible to deduce other relations for the case in which the kinematical configuration makes Δ_μ time-like (see Ref. 5). In this case we have

$$\left.\begin{aligned} &\int dv \left(c + v\frac{\partial c}{\partial v} \right) = vc \Big|_{-\infty}^{\infty} = 0 \\ &\int dv \left(b + v\frac{\partial c}{\partial u^2} \right) = 0 \\ &\int dv \left(a + v\frac{\partial c}{\partial u^1} \right) = 0 \end{aligned}\right\} \qquad (16)$$

Eqs. (15) and (16) provide us with a set of relations of the superconvergence type, which must be valid for covariance reasons. We should note that the first of (15) and (16) are trivial [11].

Eqs. (10) for p higher than 1 give relations involving higher derivatives of the invariant functions.

4. FURTHER COVARIANCE CONSIDERATIONS

We shall now put the problem of covariance of sum rules on a more general basis.

One always starts from a commutator of two currents,

$$f_{\alpha\ldots\mu\ldots}(x) = \left[j_{\alpha\ldots}(x),\ j_{\mu\ldots}(0) \right] . \qquad (17)$$

and, of course, one does not know in general how actually to compute it. What one does is to take a specific model for the calculation, obtaining then

$$m_{\alpha\ldots\mu\ldots}(x) = \left[j_{\alpha\ldots}(x),\ j_{\mu\ldots}(0) \right]_{\text{model}} \qquad (18)$$

In general, f(x), does not coincide with m(x), i.e.,

$$g(x) = f(x) - m(x) \neq 0 \ .$$

Nevertheless, the assumption which leads to the sum rule is that g(x) is actually zero for equal times ($x_o = 0$). Specifically,

$$\delta(x_o) f_{o\beta \ldots \mu \ldots} = \delta(x_o) m_{o\beta \ldots \mu \ldots} \qquad (19)$$

or

$$\frac{\partial \theta}{\partial x^\alpha} \cdot g_{\alpha \ldots \mu \ldots} = 0 \, . \qquad (20)$$

In other words, the hypothesis is equivalent to the statement that the scalar product in eq. (20) gives in fact the zero tensor. The tensor g must then satisfy covariance conditions (7):

$$f_{\mu \ldots} \partial^\mu \left[\delta^{(p-1)}(x_o) x_{i_1} \ldots x_{i_p} \right] = m_{\mu \ldots} \partial^\mu \left[\delta^{(p-1)}(x_o) x_{i_1} \ldots x_{i_p} \right] . \qquad (21)$$

Then, with eq. (19) which contains the sum rule, there are always associated other relations given by eqs. (21). When the equal time commutator of the model is a tensor, the right-hand side of eq. (21) is zero, and the latter reduces to eq. (7). On the other hand, when the equal time commutator of the model is not a tensor, as in the last two examples of Sec. 2, the complete eq. (21) must be satisfeid. Any covariant part which is extracted from the model automatically drops out from eq. (21).

5. DISCUSSION

The sum rule $\int t_o dk_o$ is, in fact, a set of infinite sum rules, one for each reference system. This set can be expressed in a compact way by

$$\int d\tau\, n^{\mu}\, t_{\mu}(k^{\nu} + \tau n^{\nu}) = \text{inv. (independent of n)} . \qquad (22)$$

Each election of the time-like vector n is associated with the integral $\int t_o dk_o$ in a particular frame.

The n-independence of the evaluated integral in (22) is equivalent to its covariance.

If we pick a particular reference frame characterized by the time-like vector n_o, with $n_o = (1,000)$ in that system, the sum rule relative to that system in covariant notation is

$$\int d\tau\, n_o^{\mu}\, t_{\mu}(k^{\nu} + \tau n_o^{\nu}) = \text{invariant form factor.} \qquad (23)$$

The covariance conditions for this sum rule reads:

$$\int d\tau \left[n^{\mu} t_{\mu}(k^{\nu} + \tau n^{\nu}) - n_o^{\mu}\, t_{\mu}(k^{\nu} + \tau n_o^{\nu}) \right] = 0 \qquad (24)$$

for arbitrary time-like n .

For example, Fubini's sum rule is given by eq. (23) corresponding to a system such that $P.n_o \longrightarrow \infty$, $\frac{k.n_o}{P.n_o} \rightarrow 0$ and $\frac{\Delta n_o}{P.n_o} \longrightarrow 0$. Actually, the sum rule relative to the most general $P.n \longrightarrow \infty$ system is

$$\int d\nu\, (a + \alpha b + \beta c) = \text{invariant} \qquad (25)$$

where the amplitudes are functions of the variable (ν; $\frac{1}{2} k^2 + \alpha\nu$; $\Delta k + \beta\nu$), the parameters α, β being arbitrary. It should also be noted that in eq. (25) the integration variable appears in all three scalars μ^i. It is possible to do a change of variable so as to obtain again a one-scalar integration. In fact, if we introduce $\omega^1 = \mu^1$, $\omega^2 = \mu^2 - \alpha\mu^1$, $\omega^3 = \mu^3 - \beta\mu^1$, then the integration is carried out on ω^1 with fixed values of

ω^2 and ω^3. Further, if we modify the invariant decomposition of t_μ as follows:

$$t_\mu = a' P_\mu + b(k_\mu - \alpha P_\mu) + c(\Delta_\mu - \beta P_\mu)$$

then eq. (25) is simply

$$\int d\omega^1 \, a' = \text{inv.} \qquad (25')$$

which is again of Fubini's type. Eq. (25') corresponds, in Fubini's method, to the assumption of unsubtracted dispersion relations in ν, for fixed values of ω^2 and ω^3.

Another frame which is usually adopted to work in is the Breit system. In this reference frame, $n^\mu = P^\mu$ and the sum rule reads:

$$\int d\tau \left(a + \frac{\tau}{P^2} b \right) = \text{form factor} \qquad (26)$$

where

$$a = a\left[\tau; \frac{1}{2}\left(k^2 - \frac{P.k^2}{P^2} + \frac{\tau^2}{P^2}\right); \Delta.k\right]$$

and similarly for b. As before, a change of variable can be used to eliminate the double τ dependence of the amplitudes. Using the new variables $\rho^1 = \mu^1$; $\rho^2 = \mu^2 - \frac{(\mu^1)^2}{\rho^2}$; $\rho^3 = \mu^3$, and the invariant decomposition

$$t_\mu = a' P_\mu + b\left(k_\mu - \frac{k.P}{P^2} P_\mu\right) + c\Delta_\mu.$$

Eq. (26) takes the form

$$\int d\rho^1 \, a'(\rho^1; \rho^2; \rho^3) = \text{form factor} \qquad (26')$$

Eqs. (25, (25'), (26), (26'), as any other particular sum rule, have to be supplemented by the covariance conditions.

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APPENDIX

SOME MORE GENERAL COVARIANCE CONSIDERATIONS

We have discussed the covariance of products of the type $f_\mu \partial^\mu \theta$. This is a particular case of products of the type

$$f_{\mu_1\cdots\mu_m} \partial^{\mu_1}\cdots\partial^{\mu_s} \theta = \delta^{(s-1)}(x_o)\, f_{oo\ldots o\mu_{s+1}\cdots\mu_m}. \qquad \text{(A-1)}$$

As in Sec. 2, we easily find the general condition for covariance:

$$f_{\mu_1\cdots\mu_m} \partial^{\mu_1}\cdots\partial^{\mu_s} \left[\delta^{(p-1)}(x_o)x_{i_1}\cdots x_{i_p}\right] = 0 \qquad \text{(A-2)}$$

In particular, for s = 0 and p = 1,

$$f_{\mu_1\ldots}\, \delta(x_o)x_i = 0 \qquad \text{(A-3)}$$

is the condition for covariance of the product $f_{\mu_1\ldots}\theta(x_o)$. (See Ref. 1) formula (2.26), which is the Fourier transform of (A.3)). Eq. (A-3) means that for $x_o = 0$, $f_{\mu_1\ldots}$ can have, at most, a singularity of $\delta^3(x)$ - type (see also Ref. 3)). For s = 1 we get back our formula (7).

We will now use conditions (A.2) to examine the covariance of Fubini's procedure [4]. Here, instead of reducing the sum rule directly from $f_\mu \partial^\mu \theta$, use is made of the identity

$$\partial^\mu(\theta f_\mu) = \partial^\mu \theta . f_\mu + \theta \partial^\mu f_\mu . \qquad \text{(A-4)}$$

The Fourier transform of this expression in the $p \to \infty$ system leads to Fubini's result. For the covariance of all the steps in Fubini's method, it is necessary that each term of (A-4) be covariant.

The covariance condition for the first term of the right-hand side is formula (7). That of the second term is given by (A-3). So, for the covariance of the left-hand side we must have simultaneously

$$f_\mu \partial^\mu \left[\delta^{(p-1)} x_{i_1} \ldots x_{i_p}\right] = 0 \qquad \text{(A-5)}$$

and

$$\partial^\mu f_\mu \left[\delta^{(p-1)} x_{i_1} \ldots x_{i_p}\right] = 0 \; . \qquad \text{(A-6)}$$

Thus, conditions (A-5) and (A-6), for the covariance of Fubini's procedure, are more restrictive than ours in view of the additional eqs. (A-6). We could also impose in (A-4) that $\theta . f_\mu$ be a vector quantity and $\theta . \partial^\mu f_\mu$ a scalar as in Ref. 1, thus obtaining

$$\partial^\mu . f_\mu \left[\delta^{(p-1)} x_{i_1} \ldots x_{i_p}\right] = 0 \qquad \text{(A-6)}$$

$$f_\mu \left[\delta^{(p-1)} x_{i_p} \ldots x_{i_p}\right] = 0 \quad . \qquad \text{(A-7)}$$

Eqs. (A-6) and (A-7) are, for $p = 1$, the Fourier transforms of conditions found in Ref. 1. These conditions areagain more restrictive (for the covariance of the sum rule) than our condi tions (7) as the latter can be deduced from (A-6) and (A-7).

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9. It is possible to show that, as a consequence of eqs. (7), $f_o(x)$ has to be zero outside the light cone.

10. The covariance of this example has also been discussed in Ref. 2.

11. We have taken the limits in a straightforward way, i.e., in the same manner in which Fubini's sum rule $\int a(\nu)d\nu$ = form factor is obtained from eq. (3). For more careful consideration see Ref. 5).

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