Notas de Física - Volume XII - Nº 5

SPINOR CURVATURES IN THE THEORY OF THE GRAVITATIONAL FIELD

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(Received March 10, 1966)

1. INTRODUCTION

Two spinor approaches have been proposed so far to describe the gravitational field as a generally covariant field theory. The first proposal is that of Weyl [1] and independently of Infeld and van der Waerden [2], recently reviewed by Bade and Jehle [3]. This approach has been subsequently exploited by Bergmann [4]. The second point of view is represented by the work of Witten [5] which uses an analogy with the spinor representation of the electromagnetic field; this proposal has been studied in great details by Penrose [6], with subsequent applications in practical problems.

In the first of these two approaches the basic variable describing the field is a set of four linearly independent Hermitian matrices, from which the metric field can be uniquely constructed. From the transformation law of the derivatives of any given spinor with respect to the unimodular spin transformation group it is possible to construct the spin affine connection, and from the expression of second order covariant derivative (properly antisymmetrized) of any spinor it is possible to form the components of the spin curvature. This last quantity turns out to be a mixed second rank skew symmetric tensor and second rank spin tensor with specific symmetry properties in the spin space.

The second approach uses a coordinate free point of view in the sense that the basic variables characterizing the system are defined by their transformation law respect to the unimodular spin transformation group, that is, they do not exhibit any tensor index. In this formalism the above defined Hermitian spin matrices are just intermediate variables which allow us to relate the theory with the conventional tensor calculus. The point of departure of this method is obtained from the analogy of the relationships between a second rank skew symmetric tensor (such as $F_{\mu\nu}$) and its spinor equivalent. For general relativity we have in place of $F_{\mu\nu}$ the Riemannian curvature $R_{\mu\nu\rho\sigma}$, but the same kind of calculations can be done in this case as was done for the electromagnetic field. It is found that the spinor equivalent of the Riemann tensor split out into two independent spinor curvatures.

It is the aim of this paper to derive the relationships between these two approaches. It has been shown presently that both methods can be related in an unambiguous way, the basic variables of each one of them, taken here as the correspondent spin curvatures, can be translated in terms of the variables of the other method by means of simple formulas.

At the present time the use of each one of these formalisms appears to be a question of particular advantage, for instance in the problem of quantization via the classical Hamiltonian formulation of the gravitational field the method of Bergmann is necessary [7]. However, if we use the Lagrangian formulation as the first step towards the quantization we can use both methods as well. The formalism of Witten and Penrose has found its best applications for the specification of the initial value problem when we use as the initial hypersurface a null surface instead of a space-like surface. [8]

In regard to the notation we shall denote spinor indices by capital latin letters, a notation which presently is more frequently used than any other else. The spinors used in this paper (and in all the references) are two-component spinors. The tensor indices are denoted by greek letters.

2. SPINOR APPROACHES IN GENERAL RELATIVITY

Witten [5] and Penrose [6] have shown that the spinor components of the Riemann tensor

$$R_{A\dot{E}\,B\dot{F}\,C\dot{G}\,D\dot{H}} = \sigma^{\mu}_{A\dot{E}}\,\sigma^{\nu}_{B\dot{F}}\,\sigma^{\lambda}_{C\dot{G}}\,\sigma^{\tau}_{D\dot{H}}\,R_{\mu\nu\lambda\tau},$$

can be splitted out into two independent spinor curvatures χ_{ABCD}

and $\Phi_{AB\dot{C}\dot{D}}$ according to the relation

$$R_{A\dot{E}\,B\dot{F}\,C\dot{G}\,D\dot{H}} = \frac{1}{2}\Big\{\chi_{ABCD}\,\epsilon_{\dot{E}\dot{F}}\,\epsilon_{\dot{G}\dot{H}} + \phi_{AB\dot{G}\dot{H}}\,\epsilon_{CD}\,\epsilon_{\dot{E}\dot{F}} + \phi_{\dot{E}\dot{F}CD}\,\epsilon_{AB}\,\epsilon_{\dot{G}\dot{H}} + \chi_{\dot{E}\dot{F}\dot{G}\dot{H}}\,\epsilon_{AB}\,\epsilon_{CD}\Big\}\,. \tag{1}$$

From the symmetries of $R_{\mu\nu\lambda\tau}$ it follows that these spinor curvatures satisfy the conditions

$$\chi_{ABCD} = \chi_{BACD} = \chi_{ABDC} = \chi_{CDAB}\,, \tag{2-1}$$

$$\phi_{AB\dot{C}\dot{D}} = \phi_{BA\dot{C}\dot{D}} = \phi_{AB\dot{D}\dot{C}} = \phi_{\dot{C}\dot{D}AB}\,. \tag{2-2}$$

The spinor components of the Ricci tensor $R_{\mu\nu}$ are given by

$$R_{B\dot{F}\,D\dot{H}} = \sigma^{\mu}_{B\dot{F}}\,\sigma^{\nu}_{D\dot{H}}\,R_{\mu\nu}\,, \tag{3}$$

or equivalently by

$$R_{B\dot{F}\,D\dot{H}} = \frac{1}{2}\,\epsilon^{AC}\,\epsilon^{\dot{E}\dot{G}}\,R_{A\dot{E}\,B\dot{F}\,C\dot{G}\,D\dot{H}}\,. \tag{4}$$

In this paper we shall use the Hermitian spin matrices $\sigma^{\mu}_{A\dot{B}}$ satisfying the "anticommutation" relations with a factor 2 in the right hand side, this notation is the same as those used in the reference [4].

$$\sigma^{\mu M\dot{N}}\,\sigma^{\nu}_{\dot{N}V} + \sigma^{\nu M\dot{N}}\,\sigma^{\mu}_{\dot{N}V} = 2\,g^{\mu\nu}\,\delta^{M}_{V}\,. \tag{5}$$

so that

$$\sigma^{\mu M\dot{N}}\,\sigma_{M\dot{N}} = 2\,g^{\mu\nu}. \tag{6}$$

Using Eq. (6) it is simple to prove that the relations (3) and (4) are actually equivalent.

The scalar curvature is represented in terms of spinors by means of the formula,

$$R = \frac{1}{2}\,\epsilon^{BD}\,\epsilon^{\dot{F}\dot{H}}\,R_{B\dot{F}\,D\dot{H}}\,. \tag{7}$$

As result of the algebraic condition on the components of the

Riemann tensor

$$R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0 ,$$

it is possible to show that the contracted χ spinor is a real number [6],

$$\chi_{AB}{}^{AB} = \chi_{\dot{A}\dot{B}}{}^{\dot{A}\dot{B}} = 2\lambda , \tag{8}$$

$$\chi_{ABC}{}^{B} = \lambda\ \epsilon_{AC} . \tag{9}$$

Multiplying Eq. (1) by $\epsilon^{AC}\ \epsilon^{\dot{E}\dot{G}}$ we can show that

$$R_{B\dot{F}D\dot{H}} = \frac{1}{2}\left[\lambda\ \epsilon_{BD}\epsilon_{\dot{F}\dot{H}} - \phi_{BD\dot{F}\dot{H}}\right] . \tag{10}$$

Contracting both members with the ϵ spinors, and using (7) we obtain

$$R = \lambda . \tag{11}$$

This result will be used in what will follow. The spinor components of the Einstein tensor $G_{\mu\nu}$ are given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R , \tag{12-1}$$

$$G_{A\dot{C}B\dot{D}} = -\frac{1}{2}\left[\lambda\ \epsilon_{AB}\ \epsilon_{\dot{C}\dot{D}} + \phi_{AB\dot{C}\dot{D}}\right] . \tag{12-2}$$

For empty spaces $G_{\mu\nu} = 0$, which gives the two conditions (since ϕ and χ are independent of each other),

$$\phi_{AB\dot{C}\dot{D}} = 0 , \tag{13}$$

$$\lambda = 0 . \tag{14}$$

Therefore, for empty spaces it follows from (9) and (14) that χ_{ABCD} is symmetric in BD, this along with the symmetry properties given by Eqs. (2-1) imply that χ_{ABCD} is totally symmetric. We shall call this totally symmetric spinor curvature by the letter ψ in order to distinguish from the general situation of spaces filled with matter,

$$\chi_{ABCD} = \psi_{ABCD} .$$

Then, in the empty space the gravitational field is described entirely in terms of a totally symmetric fourth rank spinor, which turns clear that we treat with a theory of a field with spin 2.

From the Bianchi identities it follows that this spinor satis fies the equation

$$\partial^{A\dot{E}} \psi_{ABCD} = 0 . \qquad (15)$$

This equation is formally similar to that obtained for the Maxwell field, where in place of a fourth rank symmetric spinor we have a second order symmetric spinor ϕ_{AB}, the free field equations being

$$\partial^{A\dot{E}} \phi_{AB} = 0 .$$

These equations are particular cases of the basic zero rest mass equation for spin $\frac{1}{2}n$,

$$\partial^{A\dot{E}} \theta_{AB...N} = 0 ,$$

where $\theta_{AB...N}$ is completely symmetric. The unique difference between the previous equations is due to the fact that the gravita tional theory is non linear, in other words the derivative appearing in the left hand side of Eq. (15) is a covariant derivative, so that the Klein Gordon equation obtained by iteration of this equation will contain self source terms.

Now, we turn over the spinor representation outlined in the Introduction. It is known that the derivatives of any given spinor do not transform as a spinor respect to the unimodular spin transformation group; however, it is possible to construct a covariant derivative

$$u^A_{;\rho} = \frac{\partial u^A}{\partial x^\rho} + \Gamma_\rho{}^A{}_B\, u^B,$$

by introducing the spin affine connection $\Gamma_\rho{}^A{}_B$, such that this new derivative $u^A{}_{;\rho}$ possess the correct transformation law,

$$u'^A_{;\rho} = M^A{}_K\, u^K{}_{;\rho}\ , \quad \det M = 1\ .$$

Which implies that the Γ_ρ transform as

$$\Gamma_\rho'{}^A{}_B = M^A{}_K\, \Gamma_\rho{}^K{}_L\, M^{-1\,L}{}_B - M_{,\rho}{}^A{}_K\, M^{-1\,K}{}_B\ ,$$

under the unimodular spin transformation group.

We require that the covariant derivative of the ϵ spinor vanishes

$$\epsilon^{KM}_{;\rho} = \Gamma_\rho{}^K{}_R\, \epsilon^{RM} + \epsilon^{KR}\, \Gamma_{\rho R}{}^M = 0\ .$$

This implies that the spin matrix $\Gamma_\rho{}^K{}_R$ has null trace,

$$\Gamma_\rho{}^K{}_K = \Gamma_{\rho K}{}^K = 0\ .$$

We also require that the covariant derivative of $\sigma_\mu{}^{K\dot{M}}$ vanish, which allow us to obtain an explicit form for the quantities Γ_ρ in terms of the usual Christoffel symbols.[4]

The components of the spin curvature are defined as usually by means of the mixed covariant derivatives,

$$u^A_{;\alpha\beta} - u^A_{;\beta\alpha} = P^A_{\alpha\beta}{}_B\, u^B\ ,$$

$$P^A_{\alpha\beta}{}_B = \Gamma_{\alpha}{}^A{}_{B,\beta} - \Gamma_{\beta}{}^A{}_{B,\alpha} - \Gamma_\alpha{}^A{}_K\, \Gamma_\beta{}^K{}_B + \Gamma_\beta{}^A{}_K\, \Gamma_\alpha{}^K{}_B. \quad (16)$$

From the relation (16) it follows that the $P^A_{\alpha\beta}{}_B$ have a null trace with respect to the spinor indices,

$$P_{\alpha\beta}{}^{A}{}_{A} = P_{\alpha\beta\ A}{}^{A} = 0 \ ; \qquad (17)$$

besides this the $P^{A}_{\alpha\beta\ B}$ is skew symmetric in the tensor indices.

We have,

$$P_{\alpha\beta}{}^{K}{}_{R} = \epsilon^{KM} P_{\alpha\beta MR} \ ,$$

as consequence of the condition (17) it follows from this equation that the covariant spin components of $P_{\alpha\beta}$ are symmetric,

$$P_{\alpha\beta MK} = P_{\alpha\beta KM} \ . \qquad (18)$$

We will have the opportunity of using this condition later on.

Using the property that the covariant derivatives of the σ_μ vanish, we can derive directly the following relations among the components of the spinor curvature $P_{\alpha\beta}$ and the Riemann curvature:

$$R_{\alpha\beta\lambda\mu}\, \sigma^{\lambda K\dot{M}} - P_{\alpha\beta}{}^{\dot{M}}{}_{\dot{Q}}\, \sigma_{\mu}^{\dot{Q}K} - \sigma_{\mu}{}^{\dot{M}Q}\, P_{\alpha\beta Q}{}^{K} = 0 \ ,$$

$$R_{\alpha\beta\lambda\mu}\, \sigma^{\lambda}_{K\dot{M}} + P_{\alpha\beta K}{}^{Q}\, \sigma_{\mu Q\dot{M}} + \sigma_{\mu K\dot{Q}}\, P_{\alpha\beta}{}^{\dot{Q}}{}_{\dot{M}} = 0 \ .$$

It is possible to solve the above relations for both curvatures [4]

$$R_{\alpha\beta\lambda\mu} = \frac{1}{4}\Big[P_{\alpha\beta}{}^{\dot{M}}{}_{\dot{Q}}\left(\sigma_{\lambda}{}^{\dot{Q}R}\, \sigma_{\mu R\dot{M}} - \sigma_{\mu}{}^{\dot{Q}R}\, \sigma_{\lambda R\dot{M}}\right)$$

$$+\left(\sigma_{\mu Q\dot{R}}\, \sigma_{\lambda}{}^{\dot{R}M} - \sigma_{\lambda Q\dot{R}}\, \sigma_{\mu}{}^{\dot{R}M}\right) P_{\alpha\beta M}{}^{Q}\Big], \qquad (19)$$

$$P_{\alpha\beta Q}{}^{K} = \frac{1}{4}\, \sigma^{\lambda}_{Q\dot{R}}\, \sigma^{\mu\dot{R}K}\, R_{\alpha\beta\lambda\mu} \ . \qquad (20)$$

The Ricci tensor takes the form

$$R_{\beta\mu} = -\frac{1}{4}\Big[\sigma_{\mu}^{\dot{K}M}\, \sigma^{\rho}_{M\dot{R}}\, P_{\rho\beta}{}^{\dot{R}}{}_{\dot{K}} - P_{\rho\beta}{}^{\dot{R}}{}_{\dot{K}}\, \sigma^{\rho\dot{K}M}\, \sigma_{\mu M\dot{R}} +$$

$$+\, \sigma_{\mu}^{\dot{K}M}\, P_{\rho\beta M}{}^{R}\, \sigma^{\rho}_{R\dot{K}} - \sigma^{\rho\dot{K}M}\, P_{\rho\beta M}{}^{R}\, \sigma_{\mu R\dot{K}}\Big] \ , \qquad (21)$$ [9]

and the scalar curvature the form

$$R = -\frac{1}{2}\left[\sigma^{\lambda}_{K\dot{M}}\, P^{\dot{M}}_{\lambda\rho}{}_{\dot{R}}\, \sigma^{\rho\dot{R}K} + \sigma^{\rho\dot{K}M}\, P_{\lambda\rho M}{}^{R}\, \sigma^{\lambda}_{R\dot{K}}\right]. \tag{22}$$

Using the relations [10]

$$\sigma^{\rho\dot{K}M}\, P_{\rho\alpha M}{}^{R}\, \sigma_{\lambda R\dot{K}} = -\,\sigma_{\lambda}^{\dot{K}M}\, P_{\rho\alpha M}{}^{R}\, \sigma^{\rho}_{R\dot{K}} \;,$$

together with the Hermitian conjugate, we can simplify the Eq. (21),

$$R_{\beta\mu} = -\frac{1}{2}\left[\sigma_{\mu}^{\dot{K}M}\, \sigma^{\rho}_{M\dot{R}}\, P_{\rho\beta}{}^{\dot{R}}{}_{\dot{K}} + \sigma_{\mu}^{\dot{K}M}\, P_{\rho\beta M}{}^{R}\, \sigma^{\rho}_{R\dot{K}}\right]. \tag{23}$$

In the next section we are going to derive the relationships between this approach and Penrose's formalism.

3. CORRESPONDENCE BETWEEN THE TWO SPINOR FORMALISMS

Multiplication of the Eq. (1) by adequate combinations of the ϵ spinors give the following result

$$\chi_{ABCD} = \frac{1}{2}\, \epsilon^{\dot{H}\dot{G}}\, \epsilon^{\dot{F}\dot{E}}\, R_{A\dot{E}\;B\dot{F}\;C\dot{G}\;D\dot{H}} \,, \tag{24}$$

$$\phi_{AB\dot{G}\dot{H}} = \frac{1}{2}\, \epsilon^{CD}\, \epsilon^{\dot{E}\dot{F}}\, R_{A\dot{E}\;B\dot{F}\;C\dot{G}\;D\dot{H}} \,. \tag{25}$$

These relations represent the inverses of the fundamental relation (1). From the symmetries of the Riemann tensor it follows all the symmetries of the spinors χ_{ABCD} and $\phi_{AB\dot{C}\dot{D}}$ written in the Eqs. (2). These symmetries can also be easily seen from the relations (24) and (25).

By the Eq. (10) along with Eqs. (7) and (11) we obtain

$$\phi_{BD\dot{F}\dot{H}} = -\,2R_{B\dot{F}D\dot{H}} + \frac{1}{2}\, \epsilon^{KL}\, \epsilon^{\dot{N}\dot{M}}\, \epsilon_{\dot{F}\dot{H}}\, \epsilon_{BD}\, R_{K\dot{N}L\dot{M}} \,. \tag{26}$$

Thus, the components of the mixed curvature spinor $\phi_{BD\dot{F}\dot{H}}$ are entirely given as function of the spin components of the Ricci tensor, a fact to be expected since $\phi_{BD\dot{F}\dot{H}}$ vanishes in the empty space.

Using Eq. (24) and the definition of the spin components of the Riemann tensor as well as the relation (20), we can write

$$\chi_{ABCD} = -2\, \sigma^{\alpha\dot{F}}{}_{A}\, \sigma^{\beta}{}_{B\dot{F}}\, P_{\alpha\beta CD} \,. \tag{27}$$

In this relation we recall the symmetry of $P_{\alpha\beta CD}$ with respect to the spinor indices, which makes clear the same symmetry for χ_{ABCD}.

Next we derive the expression of $\phi_{AB\dot{C}\dot{D}}$ in function of the spinor curvature $P_{\alpha\beta A}{}^{B}$. A straightforward calculation using the Eq. (26) gives the following result

$$\phi_{BD\dot{F}\dot{H}} = \left(-2\, \sigma^{\alpha}_{B\dot{F}}\, \sigma^{\beta}_{D\dot{H}} + g^{\alpha\beta} \epsilon_{\dot{F}\dot{H}}\, \epsilon_{BD}\right) R_{\alpha\beta} \,. \tag{28}$$

If in this formula [11] we substitute $R_{\alpha\beta}$ by its value in function of $P_{\alpha\beta}$ and its Hermitian conjugate as given by Eq. (23) we have reached our intention, that is, to express $\phi_{BD\dot{F}\dot{H}}$ in terms of $P_{\alpha\beta}$ and $P^{\dagger}_{\alpha\beta}$.

The next step is related to the obtention of the inverse relations, that is, to the obtention of $P_{\alpha\beta A}{}^{B}$ as function of the spinor curvatures $\phi_{BD\dot{F}\dot{H}}$ and χ_{BDFH}.

We use the relation,

$$R_{\alpha\beta\lambda\mu} = \frac{1}{16}\, \sigma_{\alpha}{}^{A\dot{E}}\, \sigma_{\beta}{}^{B\dot{F}}\, \sigma_{\lambda}{}^{C\dot{G}}\, \sigma_{\mu}{}^{D\dot{H}}\, R_{A\dot{E}\, B\dot{F}\, C\dot{G}\, D\dot{H}} \,.$$

From this equation and from (20) we can show that

$$P_{\alpha\beta C}{}^{V} = \frac{1}{16}\, \epsilon^{\dot G\dot H}\, \epsilon^{VD}\, \sigma_\alpha{}^{A\dot E}\, \sigma_\beta{}^{B\dot F}\, R_{A\dot E\, B\dot F\, C\dot G\, D\dot H}\ .$$

This together with the Eq. (1) gives

$$P_{\alpha\beta C}{}^{V} = \frac{1}{16}\, \epsilon^{VD} \Big\{ \sigma_\alpha{}^{A}{}_{\dot F}\, \sigma_\beta{}^{B\dot F}\, \chi_{ABCD} + \sigma_\alpha{}^{\dot E}{}_{B}\, \sigma_\beta{}^{B\dot F}\, \phi_{\dot E\dot F CD} \Big\}\ , \tag{29}$$

It can be easily seen from this last relation that the trace of $P_{\alpha\beta\ C}{}^{V}$ vanishes as consequence of the symmetries of $\phi_{AB\dot C\dot D}$ and χ_{ABCD}.

As the last subject of this section we shall discuss the case of empty spaces. In this situation we have seen that the spinor curvature $\phi_{AB\dot C\dot D}$ vanish, and the remaining curvature χ_{ABCD} is completely symmetric. This spinor as it is obvious is then directly related to the components of the Weyl tensor.

The Eq. (27) takes the form,

$$\psi_{ABCD} = -2\, \sigma^{\alpha\dot F}{}_{A}\, \sigma^{\beta}{}_{B\dot F}\, P_{\alpha\beta CD}\ , \tag{30}$$

and the inverse given by (29) the form,

$$P_{\alpha\beta C}{}^{V} = \frac{1}{16}\, \epsilon^{VD}\, \sigma_\alpha{}^{A}{}_{\dot F}\, \sigma_\beta{}^{B\dot F}\, \psi_{ABCD}\ . \tag{31}$$

The equations (9), (11) and (14) which imply in the symmetry of ψ_{ABCD} in BD can be used in order to obtain the following conditions on the components of $P_{\alpha\beta CD}$.

$$\sigma^{\alpha\dot F}{}_{A}\, \sigma^{\beta}{}_{\dot F}{}^{D}\, P_{\alpha\beta CD} = 0\ . \tag{32}$$

This relation together with its Hermitian conjugate are equivalent to the property that the scalar curvature vanishes, which is a result similar to the Eqs. (9) and (14).

The Bianchi identities written in terms of the $P_{\alpha\beta V}{}^{C}$ take the form (see Eqs. (15) and (30))

$$\sigma^{\alpha\dot{F}}{}_{A}\,\sigma^{\beta}_{B\dot{F}}\;\partial^{A\dot{E}}\,P_{\alpha\beta CD} = 0 \qquad (33)$$

where [12]

$$\partial^{A\dot{E}} = \sigma_{\mu}{}^{A\dot{E}}\,\partial^{\mu} \quad .$$

The relations (32) and (33) are the conditions on the components of the spinor-tensor $P_{\alpha\beta AB}$ for the case of empty spaces.

It is well known that there are four invariants of the Weyl tensor in empty spaces. Presently we can represent these invariants as the real and imaginary parts of the two complex invariants of Witten's formalism,

$$I = \psi_{ABCD}\,\psi^{ABCD} \;, \qquad (34)$$

$$J = \psi_{AB}{}^{CD}\,\psi_{CD}{}^{EF}\,\psi_{EF}{}^{AB} \;. \qquad (35)$$

Following our line in this paper we write these invariants in terms of the components of $P_{\alpha\beta AB}$

$$I = 32\;P_{\alpha\beta CD}\,P^{\alpha\beta CD} \;, \qquad (36)$$

the second set of invariants is represented by the more complicated relations,

$$J = -\,8\;\sigma^{\alpha\dot{F}}{}_{A}\,\sigma^{\beta}{}_{B\dot{F}}\,\sigma^{\gamma C\dot{T}}\,\sigma^{\delta}{}_{\dot{T}}{}^{D}\,\sigma^{\lambda E\dot{R}}\,\cdot$$

$$\cdot\,\sigma^{\tau}{}_{\dot{R}}{}^{F}\,P_{\alpha\beta CD}\,P_{\gamma\delta EF}\,P_{\lambda\tau}{}^{AB} \;. \qquad (37)$$

These invariants can also be written in function of the spin components of the Weyl tensor, which in this circunstance are identical to the spin components of the Riemann tensor. The real and imaginary parts of such expression can in turn be written as function of the canonical variables g_{mn} and p^{mn} of Dirac's theory[13]. Therefore, it is possible to write the invariants in complex form, called before by I and J as function of the canonical degrees of freedom for the field. Since the details of this computation are straightforward we will not reproduce them here.

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9. We are defining the Ricci tensor by $R_{\beta\mu} = g^{\alpha\lambda} R_{\alpha\beta\lambda\mu}$, which differs of the definition of the reference (4) by a sign.
10. In the notation of reference (4) these equations read as $tr(\sigma^{\rho} P_{\rho\alpha} \gamma_{\lambda}) = -tr(\sigma_{\lambda} P_{\rho\alpha} \gamma^{\rho})$.
11. We recall that $g^{\alpha\beta} = 1/2\, \sigma^{\alpha}_{A\dot{B}}\, \sigma^{\beta\, A\dot{B}}$.
12. The symbol ∂^{μ} here as in Eq. (15) stands for covariant differentiation.
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