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I. INTRODUCTION

It is well known that the perturbations calculations[1] of quantum field theory give to the proton and the neutron anomalous magnetic moments (due to pions) which differ from those given by experimental measurements. The perturbation calculation in higher order cannot be relied upon because of the large value of the coupling constant and in fact, for the neutron the discrepancy increases if the fourth order term is included[2].

In this paper we use an alternative method, making use of the Mass Operator formalism of Schwinger[3] to evaluate the nucleon magnetic moments. Such a calculation for electrons has recently been done[4] and served to show some numerical discrepancies in earlier calculations[5].

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The Mass Operator describes the mass or self-energy of the particle in interaction with a external electromagnetic field. Taking a constant (in space and time) external magnetic field acting on the particle, the expectation value of this operator, computed to the desired order of approximation, contains a term linear in the magnetic field, and the magnetic moment is the coefficient of this term.

The calculations thus carried out in second order in the coupling constant agrees with the perturbations calculations[6].

The anomalous magnetic moment of nucleons due to pions can arise by the two following possibilities:

$$N \longrightarrow N_c + \pi \longrightarrow N \qquad \text{(a)}$$

$$N \longrightarrow N + \pi_c \longrightarrow N \qquad \text{(b)}$$

where the index c denotes the particle which interacts with the constant external electromagnetic field, and N stands for nucleon.

In the next section we calculate the contributions of these two cases to the magnetic moments of nucleons.

2. CONTRIBUTION OF (a) (IN SECOND ORDER APPROXIMATION)

Using the property that pions are pseudoscalars in strong interaction, we have for the expression of mass operator:

$$\mathcal{M}^{(2)} = \frac{i\, G_o\, e}{4\pi^3} \int d^4k \;\; d^4k' \, \gamma_5 \, e^{ikx} \, [\gamma P + m]^{-1} \gamma_5 \, e^{-ik'x} [k^2 + M^2]^{-1} \, \delta(k-k') \qquad (1)$$

where

$G_o = \frac{G^2}{4\pi \hbar c}$, is the strong pion-nucleon coupling

e = nucleon charge

m = nucleon mass

M = pion mass

k = pion four-momentum

$P_\mu = -i\hbar \frac{\partial}{\partial x_\mu} - e A_\mu (x)$

It is necessary to observe that $\mathcal{M}$ (mass operator) is a relativistic scalar matrix in the same algebra as the γ-matrices and is an operator function of operator P . This last satisfies

$$\left[P_\mu , P_\nu \right]_- = i F_{\mu\nu} \tag{2}$$

Taking a constant(in space and time)external electromagnetic field, we get

$$\left[P_\mu , F_{\nu\lambda}\right]_- = 0 \tag{3}$$

Introducing,

$$\mathcal{F} = \frac{1}{2} \sigma_{\mu\nu} F_{\mu\nu} \tag{4}$$

we have, in matrix notation:

$$\left[\gamma P, \mathcal{F}\right]_- = 2i \gamma F P \tag{5}$$

Using the fact that e^{ikx} behaves like a translation operator,

$$e^{ikx} \left[\gamma P + m \right]^{-1} e^{-ikx} = \left[\gamma (P - k) + m\right]^{-1}$$

using (2), (4) and the anticommutativity of γ-matrices[7], we obtain

$$\left[\gamma (P - k) + m\right]^{-1} = \frac{m - \gamma(P - k)}{m^2 + (P - k)^2 - \mathcal{F}} \tag{6}$$

The mass operator then has the form:

$$\mathcal{M}^{(2)} = \frac{i\, G_0\, e}{4\pi^3} \int \frac{d^4 k}{k^2 + M^2} \gamma_5 \frac{m - \gamma (P - k)}{m^2 + (P - k)^2 - \mathcal{F}} \gamma_5 \tag{7}$$

This integral is divergent for great values of k. Introducing the ultra-violet cut-off

$$C\ (k^2) = \frac{\lambda m^2}{k^2 + \lambda m^2}$$

(where λ is the dimensionless cut-off parameter).

We have:

$$\mathcal{M}^{(2)} = \frac{i\ G_0\ e\ \lambda m^2}{4\pi^3} \int \frac{d^4k}{k^2+M^2} \frac{1}{k^2+\lambda m^2} \gamma_5 \frac{m - \gamma(P-k)}{m^2 + (P-k)^2 - \mathcal{F}} \gamma_5 \tag{8}$$

Making use of the identity:

$$\frac{1}{a} = i \int_0^\infty e^{-ixa}\ dx \tag{9}$$

Then

$$\mathcal{M}^{(2)} = \frac{G_0\ e\ \lambda m^2}{4\pi^3} \int_0^\infty ds\ dt\ dq \int d^4k\ \exp\left\{-is(k^2+M^2) - it(k^2+\lambda m^2)\right\} X$$

Where,

$$X = \gamma_5 \left[m - \gamma(P-k) \right] \exp\left\{-iq(m^2 + (P-k)^2 - \mathcal{F})\right\} \gamma_5$$

Using (3) we bring $\mathcal{M}$ to the expression:

$$\mathcal{M}^{(2)} = \frac{G_0\ e\ \lambda m^2}{4\pi^3} \int_0^\infty ds\ dt\ dq \int d^4k\ \exp\left\{-is\ (k^2+M^2) - it(k^2 + \lambda m^2)\right\} \eta$$

Where

$$\eta = \left[m + \gamma(P-k)\right](1 + iq\ \mathcal{F}) \exp\left\{-iq\left[m^2 + (P-k)^2\right]\right\}$$

In the problem of magnetic moments we are interested only in terms of $\mathcal{M}$ which are of the first degree in external field. This has been used to obtain the last relation.

We can carry now the integration over k. Firstly, we use in the k-exponentials the identity.

$$(s+t)k^2 + q(P-k)^2 = (s+t+q)\left[k - \frac{qP}{s+t+q}\right]^2 + \frac{P^2 q\,(s+t)}{s+t+q} \qquad (10)$$

Care has to be taken because we use exponentials of operators and the usual algebraic relations are not valid in general. The k-exponential, by using (10), is put like:

$$e^{A+B}$$

Where

$$\begin{cases} A = -i(s+t+q)\left[k - \dfrac{q\,P}{s+t+q}\right]^2 \\ B = -i\,\dfrac{P^2 q\,(s+t)}{s+t+q} \end{cases} \qquad (11)$$

To first order in external field:

$$e^{A+B} = e^{A}\; e^{B}\; e^{-\frac{1}{2}[A,B]} \qquad (12)$$

Expanding the exponential with the commutator and retaining again only the first power in F we obtain after some calculations:

$$2e^{A+B} = \left\langle \exp\left\{-i\,\frac{P^2 q(s+t)}{s+t+q}\right\} \prod_{j=1}^{\circ} \exp\left\{-i(s+t+q)\left[k_j - \frac{q\,P_j}{s+t+q}\right]^2\right\}\right\rangle_{+P} \qquad (13)$$

The notation $\prod_{j=1}^{\circ}$ indicates product of four factors affected by the j-index, in the last factor we must change the sign(time factor)[7], $\langle\,\rangle_{+P}$ indicates that an average must be taken of the quantity in braces and the same quantity with the order of P factors reversed.

We make now the translation:

$$\bar{k}_\lambda = k_\lambda - \frac{qP_\lambda}{s+t+q} \qquad (14)$$

This brings $\mathcal{M}$ to the expression:

$$\mathcal{M}^{(2)} = \frac{G_0\, e\, \lambda\, m^2}{8\pi^3} \int_0^\infty ds\, dt\, dq\, \exp\left\{-isM^2 - im^2(q+\lambda t)\right\} \int d^4\bar{k}\;\; \bar{\eta}\;\; \exp\,(\overline{A+B})$$

where

$$\bar{\eta} = \left[m + (1 - \frac{q}{s+t+q})\gamma P - \gamma \bar{k} \right] (1 + iq\mathfrak{Z})$$

and $\exp\,(\overline{A+B})$ is given by (13), (14). The k-integrals are:

$$K = \int_{-\infty}^{\infty} d\bar{k}_1\, d\bar{k}_2\, d\bar{k}_3\, d\bar{k}_0\;\; \bar{\eta} \exp\left\{-i\bar{k}_1^2\,(s+t+q)\right\} \exp\left\{-i\bar{k}_2^2\,(s+t+q)\right\} \exp\left\{-i\bar{k}_3^2(s+t+q)\right\} \exp\left\{i\bar{k}_0^2\,(s+t+q)\right\}$$

Terms of odd degree in $\bar{k}_\lambda$ will be integrate to zero, since they transforms to their negatives under reflections $\bar{k}_\lambda \to -\bar{k}_\lambda$; dropping such terms we get:

$$K = \left[m + (1 - \frac{q}{s+t+q})\gamma P\right] (1 + iq\mathfrak{Z}) \prod_{j=1}^{3} \mathcal{G}_j \;\; \mathcal{G}_0$$

where:

$$\mathcal{G}_j = \int_{-\infty}^{\infty} d\bar{k}_j\;\; \exp\left\{-i(s + t + q)\bar{k}_j^2\right\} = \sqrt{\frac{\pi}{i(s+t+q)}}$$

$$\mathcal{G}_0 = \mathcal{G}_j^*$$

Hence

$$\mathcal{M}^{(2)} = \frac{-i\, G_0\; e\; \lambda\, m^2}{4\pi} \int_0^\infty ds\; dt\; dq\; \exp\left\{-isM^2 - im^2(q+\lambda t)\right\} \frac{1}{2} \langle\beta\rangle_{+P}\;\; \frac{1}{(s+t+q)^2}$$

$$\langle\beta\rangle_{+P} = \left\langle \left[m + (1 - \frac{q}{s+t+q})\gamma P\right](1 + iq\mathfrak{Z}) \exp\left\{-\frac{i\, P^2 q\, (s+t)}{s + t + q}\right\}\right\rangle_{+P} \qquad (15)$$

Using the new variables s', u, w given by:

$$\begin{cases} s' = s + t + q \\ u = \dfrac{q}{s + t + q} \\ w = \dfrac{s}{s + t} \end{cases}$$

We have in (15),

$$\mathcal{M}^{(2)} = \frac{-i\, G_{p}\, e \lambda m^2}{8\pi} \int_0^{\infty} ds' \int_0^1 (1-u)\, du \int_0^1 dw \exp\left\{-is'\left[m^2 u + m^2 \lambda (1-u)(1-w) + M^2 w (1-u)\right]\right\} \langle \beta' \rangle_{+P}$$

$$\beta' = \left[m + (1-u)\gamma P\right] (1 + is' u \mathfrak{F}) \exp\left\{-i P^2 u s' (1-u)\right\}$$

Carrying out the symmetrization:

$$\frac{1}{2} \langle \beta' \rangle_{+P} = (m + i m s' u \mathfrak{F} + i s' u (1-u) \gamma P \mathfrak{F}) \exp\left\{-i P^2 u s' (1-u)\right\} +$$
$$+ \frac{1}{2} (1-u) \left\{\gamma P, \exp\left\{-i P^2 u s' (1-u)\right\}\right\}$$

Terms of first degree in F need not be symmetrized since, this would generate higher powers of F; the last symbol stands for anticommutator.

We must take the expectation value of mass-operator in a state solution of Dirac equation,

$$(\gamma P + m)\psi = 0$$

Because of this, we must use the relation.

$$(\gamma P)^2 = \mathfrak{F} - P^2$$

In order to replace P^2 by $(\gamma P)^2$ in the exponentials of the last equation. Retaining only the first power of F, and calculating the anticommutator on account of (5) and:

$$\left[\mathfrak{F}, (\gamma P)^2\right]_- = 0$$

We are left to,

$$\frac{1}{2}\langle\beta\rangle_{+P} = \left\{ m(1+isu^2 \mathfrak{Z}) + (1-u)\gamma P + \frac{isu}{2}(1-u)\left[(1+u)\gamma P \mathfrak{Z} - \right.\right.$$

$$\left.\left. - (1-u)\mathfrak{Z}\gamma P\right]\right\} \exp\left[isu\,(1-u)(\gamma P)^2\right]$$

we have dropped the prime from s. Integrating over s. (we use (9) and the integral obtained from her by differentiation with respect to the parameter a)

$$\mathcal{M}^{(2)} = \frac{-i\,G_0\,e\,\lambda m^2}{4\pi} \int_0^1 (1-u)du \int_0^1 dw.\ S \qquad (18)$$

$$S = \frac{-i(m+(1-u)\gamma P)}{m^2u + m^2\lambda(1-u)(1-w) + M^2w\,(1-u) - u(1-u)\,(\gamma P)^2} -$$

$$- \frac{imu^2\mathfrak{Z} + \frac{iu}{2}(1-u)\left[(1+u)\gamma P\mathfrak{Z} - (1-u)\mathfrak{Z}\gamma P\right]}{\left[m^2u + m^2\lambda(1-u)(1-w) + M^2w\,(1-u) - u(1-u)(\gamma P)^2\right]^2}$$

The integration over w gives

$$\mathcal{M}^{(2)} = -\frac{G_0\,e\,\lambda m^2}{4\pi}\int_0^1 (1-u)du.\ W \qquad (19)$$

where:

$$W = \frac{m+(1-u)\gamma P}{(1-u)M^2 - (1-u)\lambda\,m^2} \log\frac{m^2u + (1-u)M^2 - u(1-u)(\gamma P)^2}{m^2u + m^2\lambda(1-u) - u\,(1-u)(\gamma P)^2} +$$

$$+ \frac{mu^2\mathfrak{Z} + \frac{u}{2}(1-u)\left[(1+u)\gamma P\mathfrak{Z} - (1-u)\mathfrak{Z}\gamma P\right]}{\left(m^2u + m^2\lambda(1-u) - u(1-u)(\gamma P)^2\right)\left(m^2u + (1-u)M^2 - u\,(1-u)(\gamma P)^2\right)}$$

Introducing the approximation of ultraviolet cut-off:

$$(1-u)\,M^2-(1-u)\lambda\, m^2 \simeq (u-1)\lambda\, m^2$$

$$m^2 u + m^2\lambda(1-u) - u\,(1-u)(\gamma P)^2 \simeq m^2\lambda(1-u)$$

We are interested only in the field dependent terms of mass-operator and may forget the term in logarithm:

$$W_F = \frac{m u^2 \mathfrak{F} + \frac{u}{2}\left[(1+u)\gamma P \mathfrak{F} - (1-u)\mathfrak{F}\gamma P\right](1-u)}{m^2\lambda(1-u)\left[m^2 u + (1-u)M^2 - u\,(1-u)(\gamma P)^2\right]} + O\left(\frac{1}{\lambda}\right)$$

Then,

$$\mathcal{M}_F^{(2)} = -\frac{G_0 e}{4\pi}\int_0^1 u\,du\,\frac{m u\,\mathfrak{F} + \frac{1-u}{2}\left[(1+u)\gamma P\mathfrak{F} - (1-u)\mathfrak{F}\gamma P\right]}{m^2 u + (1-u)M^2 - u(1-u)(\gamma P)^2} \tag{20}$$

The variation of nucleon mass due to their interaction with the extern al field is:

$$\left(\Delta_F m\right)^{(2)} = \langle \Psi \mid \mathcal{M}_F^{(2)} \mid \Psi \rangle \tag{21}$$

Ψ being a solution of (16). After simple transformations we get,

$$\left(\Delta_F m\right)^{(2)} = -\frac{G_0\, e\, m}{4\pi}\langle \mathfrak{F}\rangle \int_0^1 \frac{u^3\,du}{Bu^2 + 2Qu + R} \tag{22}$$

where

$$\langle \mathfrak{F}\rangle = \langle \Psi \mid \mathfrak{F} \mid \Psi \rangle$$

$$B = m^2$$

$$2Q = -M^2$$

$$R = M^2$$

The discriminant of the quadratic denominator in the integrand of (22) is negative on account of,

$$m \sim 7M$$

Taking the correct solution for the u-integral in this case,we obtain:

$$\left(\Delta_F m\right)^{(2)} = - \frac{e\hbar}{2mc} \frac{G_0}{2\pi} f\left(\frac{M^2}{m^2}\right) \langle \mathcal{J} \rangle \qquad (23)$$

f(x) being the following function:

$$f(x) = \frac{x(x-1)}{2} \log \frac{1}{x} + \frac{1+2x}{2} - \frac{x^2(3-x)}{\sqrt{4x-x^2}} \cos^{-1} \frac{\sqrt{x}}{2} \qquad (24)$$

In (23) we include also c, $\hbar$ which were assumed unity in earlier expression of $\mathcal{M}^{(2)}$ (equation (1)).

The contribution for the second-order anomalous nucleon moment is then:

$$\mu_N^{(2)} = - \frac{G_0}{2\pi} f\left(\frac{M^2}{m^2}\right) \qquad (25)$$

in nuclear magnetons.

3. CONTRIBUTION OF (b) (in second order)

We have now explicitely a charged pion, and so we must take for the square of coupling constant the value 2 G_0 instead of G_0. The Mass-Operator has the form,

$$\mathcal{M}^{(2)} = \frac{2i\, G_0\, e}{4\pi^3} \int d^4p\, e^{ipx}\, \gamma_5 \frac{1}{\gamma p + m} \gamma_5\, e^{-ipx} \frac{1}{p^2 + M^2} \qquad (26)$$

we integrate now, over the nucleon four-momentum p, the symbols have the same meaning as in (1), and e is the pion charge. We take this charge like positive, for obtaining the result for negative charge we must change the sign.

The calculations involved in the derivation of magnetic moment using (26) are essencialy the same as in section -2. The magnetic moment is given by.

$$\mu_N^{(2)} = \frac{G_o}{\pi} \varphi\left(\frac{M^2}{m^2}\right) \tag{27}$$

in nuclear magnetons. Here φ is the function,

$$\varphi(x) = \frac{1}{2} - x + \frac{x(2-x)}{2} \log \frac{1}{x} - \frac{x(2-4x+x^2)}{\sqrt{4x - x^2}} \cos^{-1} \frac{\sqrt{x}}{2} \tag{28}$$

4. TOTAL CONTRIBUTION FOR PROTON AND NEUTRON

Introducing the notation,

$$\begin{aligned} \pi I_1(x) &= \varphi(x) \\ \pi I_2(x) &= \frac{1}{2} f(x) \end{aligned} \tag{29}$$

We have for I_1, I_2 the same results as found by perturbations methods[6]. For an earlier paper about this see for instance K.M. Case[1] his results agree also with the ours.

In the new notation;

Contribution of (a): $\mu^{(2)} = - G_o I_2 \left(\frac{M^2}{m^2}\right)$ (30)

Contribution of (b): $\mu^{(2)} = G_o I_1 \left(\frac{M^2}{m^2}\right)$ (31)

For obtaining the total contribution for proton and neutron we proceed like in perturbation theory, adding the contributions of each graph be longing to self-energy of the proton and the neutron in external field, this procedure is well known, we write the results:

$$\mu_p^{(2)}(\pi) = G_o (I_1 - I_2) \tag{32}$$

$$\mu_n^{(2)}(\pi) = - G_o (I_1 + 2I_2) \tag{33}$$

In (32), (33) we disregard the differences in mass between the π-mesons and between proton and neutron;
The I's in (32), (33) has the common argument $\left(\frac{m_\pi^2}{m^2}\right)$. We have put π like argument in this equations because this is the pion contribution to the moments of p and n.
Replacing in (32), (33) the values of I_1, I_2 we have,

$$\mu_p^{(2)}(\pi) = \frac{G_o}{\pi}\left[\frac{1}{4} - \frac{3x}{2} + \frac{x}{4}(5-3x)\log\frac{1}{x} - \frac{4x-11x^2+2x^3}{2\sqrt{4x-x^2}}\cos^{-1}\frac{\sqrt{x}}{2}\right] \quad (34)$$

$$\mu_n^{(2)}(\pi) = -\frac{G_o}{\pi}\left[1 + \frac{x}{2}\log\frac{1}{x} - x.\frac{(2-x)}{\sqrt{4x-x^2}}\cos^{-1}\frac{\sqrt{x}}{2}\right] \quad (35)$$

where,

$$x = \left(\frac{m_\pi}{m}\right)^2$$

Taking the ratio of m_π to m like 0,15 we have,

$$x \sim 0{,}023$$

Since x is very small we may retain only their first power in (34),(35); this gives:

$$\mu_p^{(2)}(\pi) \simeq \frac{G_o}{\pi}\left[\frac{1}{4} - \frac{3x}{2} + \frac{5x}{4}\log\frac{1}{x} - \sqrt{x}\cos^{-1}\frac{\sqrt{x}}{2}\right] \quad (36)$$

$$\mu_n^{(2)}(\pi) \simeq -\frac{G_o}{\pi}\left[1 + \frac{x}{2}\log\frac{1}{x} - \sqrt{x}\cos^{-1}\frac{\sqrt{x}}{2}\right] \quad (37)$$

The numerical computation leads to,

$$\mu_p^{(2)}(\pi) = 0{,}031 \frac{G^2}{4\pi\hbar c} \quad (38)$$

$$\mu_n^{(2)}(\pi) = -0{,}260 \frac{G^2}{4\pi\hbar c} \quad (39)$$

These values are not the same as the experimental ones, the signs are correct but the magnitude of μ_p is too small while the magnitude of μ_n is too large in comparison with the experimental values.

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1. K.M. Case, Phys. Rev. 76, 1 (1949).
2. Nakabayasi, Sato, Akiba, Prog. Th. Phys. 12, 250 (1954).
3. J. Schwinger, Proc. Nat. Ac. Sci. 37, 452, 455 (1951).
4. C. H. Sommerfield, Ann. Phys. 5, 26 (1958).
5. R. Karplus, N. Kroll, Phys. Rev. 77, 536 (1950).
6. S. N. Gupta, Phys. Rev. 111, 1436 (1958).
7. We use the same notation as Sommerfield[4]

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = -2\delta_{\mu\nu}$$

$$a_\mu b_\mu = \vec{a}\cdot\vec{b} - a_0 b_0$$