NOTAS DE FISICA

VOLUME II

Nº 5

NOTE ON THE NON RELATIVISTIC EQUATION FOR SPIN $\frac{1}{2}$ AND 1 PARTICLES WITH ANOMALOUS MAGNETIC MOMENT.

by

J. J. GIAMBIAGI

CENTRO BRASILEIRO DE PESQUISAS FISICAS

Av. Wenceslau Braz, 71

RIO DE JANEIRO

1955

Notas de Física - Volume II - Nº 5

NOTE ON THE NON RELATIVISTIC EQUATION FOR SPIN $\frac{1}{2}$ AND 1 PARTICLES WITH ANOMALOUS MAGNETIC MOMENT* **

J. J. Giambiagi

Centro Brasileiro de Pesquisas Físicas
Rio de Janeiro, D.F.

June 15, 1955

INTRODUCTION

Foldy and Wouthuysen obtained the non relativistic equation for the electron using a unitary transformation which eliminates the odd terms of Dirac equation to any desired order of approximation. Then, they could impose the condition $\beta\psi = \psi$ and in this way they could obtain the non relativistic equation to any desired order, which was not possible with the old method of elimination of the small components. This method was extended independently by Case and by Tiomno and Giambiagi to spin 1 particles.

In both types of particles, the spin-orbit interaction term is given by

$$\frac{e}{2\, m^2} (g - 1) \vec{\mathcal{E}} \wedge (\vec{P} + e \vec{\mathfrak{A}}) . \vec{S} \tag{1}$$

where g is the gyromagnetic factor and s the spin of the parti-

* Publication sponsored by the Conselho Nacional de Pesquisas.

** To be published in An. Acad. Bras. Ci. (1955).

cle, $\vec{\mathcal{E}}$ being the electric field, and $\vec{\mathcal{A}}$ the vector potential.

This formula was first obtained by Thomas and later discussed by Frenkel. Thomas showed that expression (1) results from a relativistic effect, and should be a general effect in the classical theory.

It seemed of interest to verify that formula (1) is also valid in quantum theory when one introduces in the relativistic wave equations, an extra term representing an anomalous magnetic* moment. For spin $\frac{1}{2}$ we add in the present work, the usual Pauli term of anomalous magnetic moment and obtain the non relativistic approximation with F-W method. In spin 1 case, we start with Kemmer equation in ten components and introduce the extra term in a way exactly analogous to that used in Dirac case and then we write the result in equation in six components, as in previous works. Following the method developed in references (2) and (3) we obtain the non relativistic equation.

It is found that Thomas formula is also valid in quantum theory for both types of particles with an anomalous magnetic moment.

Some considerations are added about the divergence term which appears in both type of particles.

SPIN $\frac{1}{2}$.

We add to the Dirac equation

$$\gamma^{\mu} \partial_{\mu} \psi + m \psi = 0$$

$$x_0 = i\,t \;;\; \gamma^0 = \beta \;;\; \gamma^i \beta \vec{d} \;; \qquad \hbar = c = 1 \tag{2}$$

the anomalous magnetic moment term

$$\frac{\lambda}{2}\,F^{\mu\nu} S_{\mu\nu} \qquad S^{\mu\nu} = \frac{1}{2}\,\frac{\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu}}{2i}$$

where λ is a parameter whose value will be fixed later, but whose order of magnitude we assume to be 1/m.

Proceeding in the usual way we get the following hamiltonian

$$H = \beta m - e\varphi + \vec{\alpha}\,(\vec{P} - e\vec{A}) + \lambda\beta\,\vec{\mathcal{H}}\,\vec{S} - i\,\frac{\lambda\beta\,\vec{\mathcal{E}}\cdot\vec{\alpha}}{2}$$

In order to obtain the non-relativistic aproximation, we make a Foldy-Wouthuysen transformation to eliminate the odd terms from the hamiltonian.

The terms corresponding to magnetic moment, spin-orbit interaction and the quadrupole term are generated with the first transformation.

This one is e^{iS} where

$$S = \frac{i}{2m}\,\beta\,\vec{\alpha}\,(\vec{P} + e\,\vec{a})$$

We shall consider only the extra terms of the hamiltonian, as the others give the normal terms(see reference 1)

$$\lambda\; e^{iS}\beta\,(\vec{\mathcal{H}}\,\vec{S} - i\,\frac{\vec{\mathcal{E}}\cdot\vec{\alpha}}{2})\;e^{-iS} = \lambda\beta(\vec{\mathcal{H}}\cdot\vec{S} - i\frac{\vec{\mathcal{E}}\cdot\vec{\alpha}}{2}) + i\left[S, \lambda\beta(\vec{\mathcal{H}}\,\vec{S} - i\frac{\vec{\mathcal{E}}\cdot\vec{\alpha}}{2})\right] + O(\frac{1}{m^2})$$

The first commutator is odd and will be eliminated in the next transformation, so we keep only the second.

$$\lambda\left[S, \frac{\beta\,\vec{\mathcal{E}}\,\vec{\alpha}}{2}\right] = \frac{\lambda}{m}\vec{S}\cdot\frac{\vec{\mathcal{E}}\wedge(\vec{P} + e\,\vec{a}) - (\vec{P} + e\,\vec{a})\wedge\vec{\mathcal{E}}}{2} + \frac{\lambda}{4m}\,\mathrm{div}\,\vec{\mathcal{E}}.$$

In a central field

$$\text{rot}\,\vec{\mathcal{E}} = 0 \quad \text{so} \quad \vec{\mathcal{E}} \wedge \vec{P} + \vec{P} \wedge \vec{\mathcal{E}} = \frac{1}{i}\, \text{rot}\, \vec{\mathcal{E}} = 0$$

and we get

$$\lambda \left[\vec{S},\ \frac{\beta\, \vec{\mathcal{E}}.\vec{\alpha}}{2} \right] = \frac{\lambda}{m}\, \vec{S}.\, \vec{\mathcal{E}} \wedge (\vec{P} + e\, \vec{A}) + \frac{\lambda}{4m}\, \text{div}\, \vec{\mathcal{E}}$$

By choosing

$$\lambda = \frac{e}{2m}\, \xi$$

we get, considering only the even part of (3)

$$\frac{e}{2m}\ \xi\, (\vec{\mathcal{H}} . \vec{S}) + \frac{e}{2m^2}\, \xi\ \ \vec{S}.\, \vec{\mathcal{E}} \wedge (\vec{P} + \ p\, \vec{A}) + \frac{e\, \xi}{8\, m^2}\, \text{div}\, \vec{\mathcal{E}}$$

The total hamiltonian will be, adding the normal terms

$$H = \beta\, m + \frac{\beta\, (\vec{P} + e\, \vec{\mathcal{A}})^2}{2\, m} + \frac{e}{2m}(2 + \xi)\, \beta\, \vec{\mathcal{H}}.\, S +$$

$$+ \frac{e}{2m^2}\, (2 + \xi\, -1)\ \vec{S}.\, \vec{\mathcal{E}} \wedge (\vec{P} + e\, \vec{\mathcal{A}}) + \frac{e}{8m^2}(2 + \xi - 1)\, \text{div}\, \vec{\mathcal{E}}$$

We obtain the non-relativistic equation by imposing

$$\beta\psi = \psi$$

We get then,

$$\left.\begin{array}{lll}
\text{Magnetic moment} & = & \frac{e}{2\, m}(2 + \xi)\, (\vec{\mathcal{H}}\ \vec{S}) \\[2ex]
\text{Spin-orbit} & = & \frac{e}{2m^2}(2 + \xi - 1)\ \vec{S}\ \vec{\mathcal{E}} \wedge (\vec{P} + e\, \vec{A}) \\[2ex]
\text{divergence term} & & \frac{e}{8m^2}\, (2 + \xi\, -1)\, \delta^{ij}\, \frac{\partial\, \mathcal{E}_i}{\partial\, x_i}
\end{array}\right\} \qquad (4)$$

<u>SPIN 1</u>

In order to introduce an anomalous magnetic moment, we wri te Proca equations

$$\partial_t \vec{E} = m\,\vec{A} + \vec{\partial} \wedge \vec{H}$$

$$-\,\partial_t \vec{A} = m\,\vec{E} + \vec{\partial}, \qquad\qquad \partial_t = \frac{\partial}{\partial t} - i\,e\,\varphi$$

$$o = m\,\vec{H} - \vec{\partial} \wedge \vec{A} \qquad\qquad \partial_i = \frac{\partial}{\partial x_i} + i\,e\,a_i$$

$$o = m\,V + \vec{\partial}\,.\,\vec{E}$$

(In Kemmer's form (See ref. 6) these equations are explicitly written for a negative charge).

$$(\partial^{\mu}\Gamma_{\mu} + m)\psi = o \quad \text{with} \quad \psi = \begin{pmatrix} \vec{A} \\ \vec{E} \\ \vec{H} \\ V \end{pmatrix}$$

are ten by ten matrices

$$\Gamma_o = \left|\begin{array}{ccc|ccc|cccc} 0 & 0 & 0 & \multicolumn{3}{c|}{-i\,\mathbb{1}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & & & & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & & & & 0 & 0 & 0 & 0 \\ \hline \multicolumn{3}{c|}{i\,\mathbb{1}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right|$$

$$\vec{\Gamma} = \left|\begin{array}{ccc|ccc|ccc|c} 0 & 0 & 0 & 0 & 0 & 0 & \multicolumn{3}{c|}{-i\,\vec{m}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & & & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & & & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \vec{\lambda} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \hline \multicolumn{3}{c|}{i\,\vec{m}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \multicolumn{3}{c|}{\vec{\lambda}} & 0 & 0 & 0 & 0 \end{array}\right|$$

where $\mathbb{1}$ is the unitary 3 x 3 matrix

$$m_k = \frac{tij - tji}{i} \qquad i,\ j,\ k \qquad \vec{\lambda} = (1,\ 1,\ 1)\ (tij)_{rs} = \delta_{ir} \quad \delta_{js}$$

We introduce an anomalous magnetic moment

$$\lambda \frac{\mathfrak{F}^{\mu\nu}}{2} S_{\mu\nu} \qquad \text{with} \qquad S^{\mu\nu} = \frac{\Gamma^{\mu}\Gamma^{\nu} - \Gamma^{\nu}\Gamma^{\mu}}{i}$$

Proca equations for a particle with anomalous magnetic moment will be: (λ is a constant whose value will be fixed later, but we shall assume it to be of order 1/m.)

$$\begin{aligned}
\partial_t \vec{E} &= m\vec{A} + \vec{\partial} \wedge H + i\lambda(\vec{\mathcal{H}} \wedge \vec{A} - \vec{\mathcal{E}}.\ ) \\
\partial_t \vec{A} &= m\vec{E} + \vec{\partial} V + i\ \lambda(\vec{\mathcal{V}} \wedge \vec{E} + \vec{\mathcal{E}} \wedge H\ ) \\
o &= m\vec{H} - \vec{\partial} \wedge \vec{A} + i\lambda\ (\vec{\mathcal{H}} \wedge \vec{H} - \vec{\mathcal{E}} \wedge E) \qquad (5) \\
o &= m - \vec{\partial}.\ \vec{E} - i\ \lambda\ \ \vec{\mathcal{E}}\ .\ \vec{A}
\end{aligned}$$

This equations can be written also:

$$m\ F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu}\ A_{\mu} + i\ \lambda\ \ (\mathfrak{F}_{\mu\rho}\ F^{\rho}_{\nu} - \mathfrak{F}_{\nu\rho}\ F^{\rho}_{\mu}\ )$$

$$\partial^{\nu} F_{\nu\mu} = m\ A_{\mu} + i\ \lambda\ \mathfrak{F}_{\nu\mu}\ \ A^{\nu}$$

In order to obtain the non relativistic limit, is betther to write eqs (5) with six components instead of ten (see ref. 2 and 3).

We must replace in the two first equations, the expressions of H and V found in the last two. V is obtained immediately, while for H it is necessary to solve the linear equation

$$m\ \vec{H} + i\ \lambda\ \vec{\mathcal{H}} \wedge\ \ \vec{H} = \vec{\partial} \wedge \vec{A} + i\ \lambda \vec{\mathcal{E}}\ \wedge\ \vec{E} = \vec{Q}$$

The solution is

$$\vec{H} = \frac{\vec{Q}\ m^2 + i\,m\,\lambda\ (\vec{Q} \wedge \vec{\mathcal{H}}) - \lambda^2 \vec{\mathcal{H}}(\vec{Q}.\ \vec{\mathcal{H}}\ )}{m^3 - \lambda^2\ \ \mathcal{H}^2\ \ m}$$

We have also

$$V = -\frac{1}{m}\vec{\partial}_{\circ}\vec{E} + \frac{i\lambda}{m}\vec{\xi}.\vec{A}$$

These expressions are to be replaced in the two first of (5) and keeping terms only up to $1/m^2$, the result is

$$\partial_t \vec{E} = m\vec{A} + \frac{1}{m}\vec{\partial}\wedge\vec{\partial}\wedge\vec{A} + \frac{i\lambda}{m}\vec{\xi}(\vec{\partial}.\vec{E}) + i\lambda\vec{\mathcal{H}}\wedge\vec{A} + \frac{i\lambda\,\vec{\partial}\wedge(\vec{\xi}\wedge\vec{E})}{m}$$

$$-\partial_t \vec{A} = m\vec{E} - \frac{1}{m}\vec{\partial}(\vec{\partial}.\vec{E}) + \frac{i\lambda}{m}\vec{\partial}(\vec{\xi}.\vec{A}) + i\lambda\vec{\mathcal{H}}\wedge\vec{E} + \frac{i\lambda}{m}\vec{\xi}\wedge\vec{\partial}\wedge\vec{A}$$

These equations can be written in a matricial form

$$i\beta\frac{\partial\psi}{\partial t} = \left[m - e\beta\varphi - \frac{\partial^2}{2m} + \frac{e}{2m}\vec{\mathcal{H}}.\vec{m} + \frac{e}{2m}Q^{ij}\partial_i\partial_j\right]$$

$$+\lambda\left[\vec{\mathcal{H}}.\vec{m} - \frac{i}{m}(\vec{\xi}.\vec{m}\,\vec{\partial}.\vec{m} - \partial_i\,\xi_j\,t^{ij})\left(\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \end{array}\right) - \frac{i}{m}(\vec{\partial}.\vec{m}\,\vec{\xi}.\vec{m} - \right.$$

$$\left. -\xi_i\,\partial_j\,t^{ij})\left(\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right)\right]$$

where β is the proyection of Γ° on the subspace $\vec{A}$ $\vec{E}$ and

$\theta = \left(\begin{array}{c|c} 1 & \\ \hline & -1 \end{array}\right)$ $\quad Q_{ij} = t_{ij} + t_{ji} - \delta_{ij}$ $\quad \psi = \begin{pmatrix}\vec{A}\\ \vec{E}\end{pmatrix}$

we have also used

$$i\,\vec{a}\wedge\vec{b} = \vec{a}.\vec{m}.\vec{b} \qquad \vec{a},\vec{b} \text{ being any vector.}$$

which can be easily verified.

The first bracket leads to the normal non-relativistic equation and will not be considered. (See references 2 and 3). We write the final formula for it.

$$i\frac{\partial\psi}{\partial t} = m - e\varphi + \frac{(\vec{P} + e\vec{\mathcal{A}})^2}{2m} + \frac{e}{2m}\vec{\mathcal{H}}.\vec{m} \qquad (6)$$

In order to bring B to diagonal form, the following transformation has to be made.

$$U = \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} 1 & -i\,1 \\ \hline -i\,1 & 1 \end{array}\right) \tag{7}$$

$$\left(\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right) \longrightarrow \frac{1}{2} \left(\begin{array}{c|c} i\,1 & 1 \\ \hline 1 & -i\,1 \end{array}\right) \left(\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \end{array}\right) \longrightarrow \frac{1}{2} \left(\begin{array}{c|c} -i\,1 & 1 \\ \hline 1 & i\,1 \end{array}\right) \beta \longrightarrow \left(\begin{array}{c|c} 1 & \\ \hline & -1 \end{array}\right)$$

We will keep only the even part, as the odd is eliminated in the non relativistic aproximation.

The secon bracket will be equal to

$$\lambda \vec{\mathcal{H}} . \vec{m} + \frac{\lambda}{2m} \left(\begin{array}{c|c} 1 & \\ \hline & -1 \end{array}\right) \left\{ (\vec{\partial} . \vec{m}\, \vec{\mathcal{E}} . \vec{m} - \vec{\mathcal{E}} . \vec{m}\, \vec{\partial} . \vec{m}) - t^{ij} (\mathcal{E}^i{}_j - \partial_i \mathcal{E}_j) \right.$$

It can easily be shown that

$$\vec{\partial} . \vec{m}\, \vec{\mathcal{E}} . \vec{m} - \vec{\mathcal{E}} . \vec{m}\, \vec{\partial} . \vec{m} = \left(\delta^{ij} - \frac{t^{ij} + t^{ji}}{2} \right) \frac{\partial \mathcal{E}_i}{\partial x_i} + \vec{m} . \frac{\vec{\mathcal{E}} \wedge \vec{\partial} - \vec{\partial} \wedge \vec{\mathcal{E}}}{2i}$$

and

$$\left[\mathcal{E}^i \partial_j - \partial_i \mathcal{E}_j \right] t^{ij} = - \left[\vec{m} \frac{\vec{\mathcal{E}} \wedge \vec{\partial} - \vec{\partial} \wedge \vec{\mathcal{E}}}{2i} - \frac{t^{ij} + t^{ji}}{2} \frac{\partial \mathcal{E}_i}{\partial x_j} \right]$$

Replacing both expressions in the preceding one, and choosing

$$\lambda = \frac{e}{2m} \mathcal{E}$$

we get, assuming again rot $\vec{\mathcal{E}} = 0$ and $\beta \psi = \psi$

$$\frac{e}{2m} \mathcal{E}\, \vec{\mathcal{H}} . \vec{m} + \frac{e\mathcal{E}}{2m^2} \vec{\mathcal{E}} \wedge (\vec{P} + e\, \vec{a}) + \frac{e\mathcal{E}}{4m^2} \delta^{ij} \frac{\partial \mathcal{E}_i}{\partial x_i} \tag{8}$$

from (6) and (8) it follows

$$i \frac{\partial \psi}{\partial t} = m - e\, \psi + \frac{(\vec{P} + e\, \vec{a})^2}{2m} + \frac{e}{2m} (1 + \mathcal{E})\, \vec{\mathcal{H}} . \vec{m}$$

$$+ \frac{e\, \mathcal{E}}{2m^2} \vec{m} . \vec{\mathcal{E}} \wedge (\vec{P} + e\, \vec{a}) + \frac{e\, \mathcal{E}}{4m^2} \delta^{ij} \frac{\partial \mathcal{E}_i}{\partial x_i}$$

We are explicitly interested in

$$\left.\begin{array}{ll} \text{magnetic moment} & \frac{e}{2m}(1+\varepsilon)\,\vec{\mathcal{K}}\cdot m \\ \text{spin orbit term} & \frac{e}{2m^2}\,(1+\varepsilon-1)\vec{\mathcal{E}}\wedge(\vec{P}+e\,\vec{\alpha}) \\ \text{divergence term} & \frac{e}{4m^2}\,(1+\varepsilon-1)\delta^{ij}\;\frac{\partial\mathcal{E}_i}{\partial x_i} \end{array}\right\} \qquad (9)$$

Conclusion

Expressions (4) and (9) for the spin orbit interaction are identical with formula (1)if we put $g = 2+\varepsilon$ and $g = 1+\varepsilon$ for spin $\frac{1}{2}$ and 1 as the case is. So we have proved the validity of Thomas formula in the considered cases.

It is worthwhile to add some considerations about the divergence term. In the Dirac case it has the following form

$$\frac{e}{8m^2}\,\mathrm{div}\,\vec{\mathcal{E}} = \frac{e}{2m^2}\;\frac{S^{oi}\,S^{oj}+S^{oj}\,S^{oi}}{2}\;\frac{\partial\mathcal{E}_i}{\partial x_i} \quad \text{as } S^{oi}\,S^{oj}+S^{oj}\,S^{oi}=\frac{\delta^{ij}}{2}$$

One could expect that this will be the case with spin 1. It is in fact so, with

$$S^{oi} = \frac{\Gamma^o\Gamma^i - \Gamma^i\Gamma^o}{i}$$

In this case, as the original representation was with ten components, one has to consider the proyection of this even operator on the subspace $\vec{A}$, $\vec{E}$ and here make the transformation defined in formula (7).

It is the verified that

$$S^{oi}\,S^{oj} + S^{oj}\,S^{oi} = \delta^{ij}$$

and formula 10 is valid also in this case.

Acknowledgment.

The author would like to express his appreciation to Professor J. Tiomno for many useful discussions on the subject.

1. Foldy L. - Wouldhuysen S.A., P. R. 78 29 1950.
2. Case K.L., P. R. 96 1323 1954.
3. Giambiagi J. - Tiomno J., Anais da Ac.Bras. de Ciências 26, 327 1954.
4. Thomas L., Phil. Mag. 3 11 1927.
5. Frenkel J., Zeit. F. Phys. 37 2430 1927.
6. Kemmer N., Proc. Roy. Soc. 173 99 1939.

* (pag. 2, 8th line) Following a suggestion made by Prof. L. Marquez.