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We present a current algebra for the two-site Bose-Hubbard model and use it to get the quantum dynamics of the currents. For different choices of the Hamiltonian parameters we get different currents dynamics. We generalize the Heisenberg equation of motion to write the second derivative of an operator.

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Introduction - Since the first experimental verification of the Bose-Einstein condensation (BEC) [1–3], occurred more than seven decades after its theoretical prediction [4, 5], a great effort in the theoretical and experimental viewpoint has been made in the study of this quantum many body physical phenomenon [6–14]. The early experimental realization of a two-well BEC condensate was made only two years after the experimental verification of the BEC to study the interference between two freely expanding condensates [15, 16], and their results had direct implications in the study of the atom laser and the Josephson effect [17, 18] for BEC. Some models were used to study some behaviors of these systems as for example the quantum phase transitions, the classical analysis and the quantum dynamics [19–22]. We are considering here the two-site Bose-Hubbard model, also known as the *canonical Josephson Hamiltonian* [8], that has been an useful model in understanding tunneling phenomena using two BEC [23–29]. This model is integrable in the sense that it can be solved by the quantum inverse scattering method (QISM) [30–40] and it has been discussed in different ways using this method [32–39]. In this context this model is a particular case of the bosonic multi-state model studied in [41]. The experimental quantum dynamics and the classical analysis of this model was performed by [42–44]. In this letter we will discuss the current algebra for the two-site Bose-Hubbard model and use it to study the quantum dynamics of the currents. This method can be applied to many systems that present microscopic tunneling phenomenon. The model is described by the Hamiltonian

$$\hat{H} = \frac{K}{8}(\hat{N}_1 - \hat{N}_2)^2 - \frac{\Delta\mu}{2}(\hat{N}_1 - \hat{N}_2) - \frac{\mathcal{E}_J}{2}(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1), \quad (1)$$

where \hat{a}_1^\dagger , \hat{a}_2^\dagger , denote the single-particle creation boson operators in the two wells and $\hat{N}_1 = \hat{a}_1^\dagger \hat{a}_1$, $\hat{N}_2 = \hat{a}_2^\dagger \hat{a}_2$, are the corresponding number of particles boson operators. These bosons operators satisfies the canonical commutation relations

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} \hat{I}, \quad [\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0, \quad (2)$$

and

$$[\hat{N}_i, \hat{a}_j] = -\delta_{ij} \hat{a}_j, \quad [\hat{N}_i, \hat{a}_j^\dagger] = +\delta_{ij} \hat{a}_j^\dagger, \quad (3)$$

where \hat{I} is the identity operator.

The coupling K provides the strength of the s -wave scattering interaction between the bosons, $\Delta\mu$ is the external potential and \mathcal{E}_J is the amplitude of tunneling.

Symmetries - The Hamiltonian (1) is invariant under the \mathbb{Z}_2 mirror transformation $\hat{a}_j \rightarrow -\hat{a}_j$, $\hat{a}_j^\dagger \rightarrow -\hat{a}_j^\dagger$, and under the global $U(1)$ gauge transformation $\hat{a}_j \rightarrow e^{i\alpha} \hat{a}_j$, where α is an arbitrary c -number and $\hat{a}_j^\dagger \rightarrow e^{-i\alpha} \hat{a}_j^\dagger$, $j = 1, 2$. For $\alpha = \pi$ we get again the \mathbb{Z}_2 symmetry. The global $U(1)$ gauge invariance is associated with the conservation of the total number of atoms $\hat{N} = \hat{N}_1 + \hat{N}_2$ and the \mathbb{Z}_2 symmetry is associated with the parity of the wave function by the relation

$$\hat{P} |\Psi\rangle = (-1)^N |\Psi\rangle, \quad (4)$$

$$|\Psi\rangle = \sum_{n=0}^N C_{n,N-n} \frac{(a_1^\dagger)^n}{\sqrt{n!}} \frac{(a_2^\dagger)^{N-n}}{\sqrt{(N-n)!}} |0,0\rangle, \quad (5)$$

where \hat{P} is the parity operator and $[\hat{H}, \hat{P}] = 0$.

There is also the permutation symmetry of the atoms of the two wells if we have $\Delta\mu = 0$ and when we turn on $\Delta\mu$ we break the symmetry. The wave function (5) is symmetric under this permutation

$$\hat{\mathcal{P}} |\Psi\rangle = \sum_{n=0}^N C_{N-n,n} \frac{(a_1^\dagger)^{N-n}}{\sqrt{(N-n)!}} \frac{(a_2^\dagger)^n}{\sqrt{n!}} |0,0\rangle = |\Psi\rangle, \quad (6)$$

where $\hat{\mathcal{P}}$ is the permutation operator and $[\hat{H}, \hat{\mathcal{P}}] = 0$ if $\Delta\mu = 0$ [32].

In the antisymmetric case $\Delta\mu \neq 0$ we can change the bias of one well. In this case it is called a tilted two-well potential [27, 45]. In the Fig. 1 we represent the two BEC by a two-well potential for the case $\Delta\mu \neq 0$. We get the two-site Bose-Hubbard model when we consider each BEC as a site.

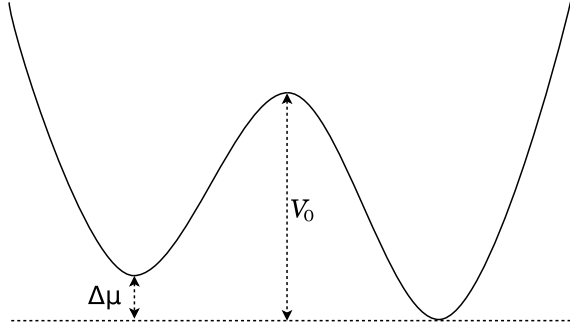


FIG. 1. A two-well potential representation of the two-site Bose-Hubbard model for the case $\Delta\mu \neq 0$ and barrier height V_0 . We are considering one condensate for each well.

Current Algebra - The total particles number boson operator, $\hat{N} = \hat{N}_1 + \hat{N}_2$, is a conserved quantity and it is commutable compatible operator (CCO) with the particles number boson operators in each well, $[\hat{N}, \hat{N}_1] = [\hat{N}, \hat{N}_2] = [\hat{N}_1, \hat{N}_2] = 0$. The number of particles boson operators in each well don't commute with the Hamiltonian and their time evolution is dictated by the Josephson tunneling current operator,

$$\hat{\mathcal{J}} = \frac{1}{2i}(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1) \quad (7)$$

in coherent opposite phases because of the conservancy of \hat{N} , with

$$[\hat{H}, \hat{N}_1] = +i\varepsilon_J \hat{\mathcal{J}}, \quad [\hat{H}, \hat{N}_2] = -i\varepsilon_J \hat{\mathcal{J}}, \quad (8)$$

and

$$\frac{d\hat{N}_1}{dt} = -\frac{\varepsilon_J}{\hbar} \hat{\mathcal{J}}, \quad (9)$$

$$\frac{d\hat{N}_2}{dt} = +\frac{\varepsilon_J}{\hbar} \hat{\mathcal{J}}. \quad (10)$$

Here it is worth to note that the two BEC are entangled by the tunneling of the particles and we can study the quantum phase transition of the system using tools of the quantum information [20, 21].

The tunneling current $\hat{\mathcal{J}}$ together with the imbalance current $\hat{\mathcal{I}}$,

$$\hat{\mathcal{I}} = \frac{1}{2}(\hat{N}_1 - \hat{N}_2), \quad (11)$$

and the coherent correlation tunneling current operator $\hat{\mathcal{T}}$,

$$\hat{\mathcal{T}} = \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1), \quad (12)$$

span the current algebra

$$[\hat{\mathcal{T}}, \hat{\mathcal{J}}] = +i\hat{\mathcal{I}}, \quad [\hat{\mathcal{T}}, \hat{\mathcal{I}}] = -i\hat{\mathcal{J}}, \quad [\hat{\mathcal{J}}, \hat{\mathcal{I}}] = +i\hat{\mathcal{T}}. \quad (13)$$

With the identification $\hat{L}_x \equiv \hbar\hat{\mathcal{T}}$, $\hat{L}_y \equiv \hbar\hat{\mathcal{J}}$, and $\hat{L}_z \equiv \hbar\hat{\mathcal{I}}$ we can write (13) in the standard compact way of the momentum angular algebra

$$[\hat{L}_k, \hat{L}_l] = i\hbar\varepsilon_{klm}\hat{L}_m. \quad (14)$$

We have two Casimir operators for that current algebra. One of them is the total number of particles \hat{N} , related to the $U(1)$ symmetry

$$\hat{C}_1 = \hat{N}, \quad (15)$$

and the another one is related to the momentum angular algebra and the $O(3)$ symmetry

$$\hat{C}_2 = \hat{\mathcal{T}}^2 + \hat{\mathcal{I}}^2 + \hat{\mathcal{J}}^2. \quad (16)$$

We can show that \hat{C}_2 is just a function of \hat{C}_1

$$\hat{C}_2 = \frac{\hat{C}_1}{2} \left(\frac{\hat{C}_1}{2} + 1 \right). \quad (17)$$

The Casimir operators (15) and (16), the boson number of particles in each well \hat{N}_1 , \hat{N}_2 , and the imbalance current operator, $\hat{\mathcal{I}}$, are CCO and so they have the same set of eigenfunctions and can simultaneous have well-defined values

$$\hat{C}_2 |n_1, n_2\rangle = \frac{N}{2} \left(\frac{N}{2} + 1 \right) |n_1, n_2\rangle, \quad (18)$$

$$\hat{\mathcal{I}} |n_1, n_2\rangle = \frac{1}{2} (n_1 - n_2) |n_1, n_2\rangle. \quad (19)$$

Using the commutation relations of the currents (13) it is easy to calculate the anticommutators

$$[\hat{\mathcal{T}}, \hat{\mathcal{I}}]_+ = 2\hat{\mathcal{I}}\hat{\mathcal{T}} - i\hat{\mathcal{J}}, \quad (20)$$

$$[\hat{\mathcal{T}}, \hat{\mathcal{J}}]_+ = 2\hat{\mathcal{J}}\hat{\mathcal{T}} + i\hat{\mathcal{I}}, \quad (21)$$

$$[\hat{\mathcal{J}}, \hat{\mathcal{I}}]_+ = 2\hat{\mathcal{I}}\hat{\mathcal{J}} + i\hat{\mathcal{T}}. \quad (22)$$

We will use these anticommutators together with the commutators (13) in the calculus of the currents quantum dynamics.

Current Quantum Dynamics - We can rewrite the Hamiltonian (1) using the currents operators

$$\hat{H} = \frac{K}{2}\hat{\mathcal{I}}^2 - \Delta\mu\hat{\mathcal{I}} - \varepsilon_J\hat{\mathcal{T}}. \quad (23)$$

The quantum dynamic of the currents are determined by the current algebra (13), their commutation relations with the Hamiltonian and the parameters. If the Hamiltonian is not explicitly time-dependent it is not time-dependent, $\frac{d\hat{H}}{dt} = \frac{\partial\hat{H}}{\partial t} = 0$, and the system is closed (conservative). It is also important to note that the Hamiltonian is the same in the Heisenberg and Schrödinger

pictures, $\hat{H}_H = \hat{H}_S$. Using this fact we can write the second derivative of the average value of an operator \hat{O} in the Heisenberg picture as

$$\frac{d^2\hat{O}}{dt^2} = \left(\frac{i}{\hbar}\right)^2 [\hat{H}, [\hat{H}, \hat{O}]], \quad (24)$$

or as

$$\frac{d^2\hat{O}}{dt^2} = \frac{i}{\hbar} [\hat{H}, \frac{d\hat{O}}{dt}]. \quad (25)$$

It is direct to generalize the Eqs. (24) and (25) for higher derivatives. We can get similar equation in the

another pictures. The pictures preserve the commutation relations between the operators in the sense that if we have $[\hat{A}_S, \hat{B}_S] = \hat{C}_S$ in the Schrödinger picture we get the same relation in the Heisenberg picture, $[\hat{A}_H, \hat{B}_H] = \hat{C}_H$, and in the interaction picture, $[\hat{A}_I, \hat{B}_I] = \hat{C}_I$. The same is true for the anticommutators, and so the pictures preserve the algebra. We can see from Eqs. (23) and (24) that the Casimir operators (15) and (16) are also conserved quantities, $[\hat{H}, \hat{C}_1] = [\hat{H}, \hat{C}_2] = 0$. Using the Eq. (24) or (25) we found the following equations for the quantum dynamics of the three currents

$$\frac{d^2\hat{\mathcal{I}}}{dt^2} + \frac{(\varepsilon_J)^2}{\hbar^2}\hat{\mathcal{I}} = -\frac{\varepsilon_J K}{\hbar^2}\hat{\mathcal{I}}\hat{\mathcal{T}} + \frac{\varepsilon_J \Delta\mu}{\hbar^2}\hat{\mathcal{T}} + i\frac{\varepsilon_J K}{2\hbar^2}\hat{\mathcal{J}}, \quad (26)$$

$$\begin{aligned} \frac{d^2\hat{\mathcal{J}}}{dt^2} + \frac{1}{\hbar^2} \left[(\Delta\mu)^2 + (\varepsilon_J)^2 + \frac{K^2}{4} \right] \hat{\mathcal{J}} = & -\frac{K^2}{\hbar^2}\hat{\mathcal{I}}^2\hat{\mathcal{J}} - i\frac{K^2}{\hbar^2}\hat{\mathcal{I}}\hat{\mathcal{T}} + 2\frac{K\Delta\mu}{\hbar^2}\hat{\mathcal{I}}\hat{\mathcal{J}} - \frac{K\varepsilon_J}{\hbar^2}\hat{\mathcal{J}}\hat{\mathcal{T}} \\ & + i\frac{K\Delta\mu}{\hbar^2}\hat{\mathcal{T}} - \frac{i}{2}\frac{K\varepsilon_J}{\hbar^2}\hat{\mathcal{I}}, \end{aligned} \quad (27)$$

$$\begin{aligned} \frac{d^2\hat{\mathcal{T}}}{dt^2} + \frac{1}{\hbar^2} \left[(\Delta\mu)^2 + \frac{K^2}{4} \right] \hat{\mathcal{T}} = & -\frac{K^2}{\hbar^2}\hat{\mathcal{I}}^2\hat{\mathcal{T}} + i\frac{K^2}{\hbar^2}\hat{\mathcal{I}}\hat{\mathcal{J}} + 2\frac{K\Delta\mu}{\hbar^2}\hat{\mathcal{I}}\hat{\mathcal{T}} \\ & - \frac{K\varepsilon_J}{\hbar^2}\hat{\mathcal{I}}^2 + \frac{K\varepsilon_J}{\hbar^2}\hat{\mathcal{J}}^2 + \frac{\Delta\mu\varepsilon_J}{\hbar^2}\hat{\mathcal{I}} - i\frac{K\Delta\mu}{\hbar^2}\hat{\mathcal{J}}. \end{aligned} \quad (28)$$

We can see from the Eqs. (26), (27) and (28) that a complete analysis with all Hamiltonian's parameters is very complicated because the currents are coupled on the right hand side of these equations. To simplify our analysis we will make some choices of the parameters. Different choices of the ratio between the Hamiltonian's parameters gives us different dynamics for the currents. We will consider two simple cases in the Rabi regime, $K/\varepsilon_J \ll N^{-2}$ [8, 25, 37]. Consequently, in the extreme Rabi regime we can neglect K and consider the no interaction limit $K \rightarrow 0$.

In the first analysis we will consider the symmetric case, $\Delta\mu = 0$. For this case the current $\hat{\mathcal{T}}$ is a conserved quantity, $[\hat{\mathcal{H}}, \hat{\mathcal{T}}] = 0$, but this don't means that we don't have tunneling. We can see from Eqs. (9) and (10) that the quantum dynamic of \hat{N}_1 , \hat{N}_2 , and $\hat{\mathcal{I}}$ only depend of the current $\hat{\mathcal{J}}$ and the amplitude of tunneling ε_J . The current dynamics for these currents are the dynamic of the simple harmonic oscillator (SHO)

$$\frac{d^2\hat{\mathcal{I}}}{dt^2} + \omega_{\mathcal{I}}^2\hat{\mathcal{I}} = 0, \quad (29)$$

$$\frac{d^2\hat{\mathcal{J}}}{dt^2} + \omega_{\mathcal{J}}^2\hat{\mathcal{J}} = 0, \quad (30)$$

where $\omega_{\mathcal{I}} = \omega_{\mathcal{J}} = \frac{\varepsilon_J}{\hbar}$ is the natural frequency of the

SHO. The period of the oscillations is $T = \frac{2\pi\hbar}{\varepsilon_J}$. In analogy with the classical SHO, the ratio between the elastic constant \mathcal{K} and the mass m is $\frac{\mathcal{K}}{m} = \frac{\varepsilon_J^2}{\hbar^2}$.

In the second analysis we will break the symmetry, $\Delta\mu \neq 0$, to consider the antisymmetric case. For this case the currents dynamics are

$$\frac{d^2\hat{\mathcal{I}}}{dt^2} + \left(\frac{\varepsilon_J}{\hbar}\right)^2 \hat{\mathcal{I}} = \frac{\varepsilon_J \Delta\mu}{\hbar^2} \hat{\mathcal{T}}, \quad (31)$$

$$\frac{d^2\hat{\mathcal{T}}}{dt^2} + \left(\frac{\Delta\mu}{\hbar}\right)^2 \hat{\mathcal{T}} = \frac{\varepsilon_J \Delta\mu}{\hbar^2} \hat{\mathcal{I}}, \quad (32)$$

$$\frac{d^2\hat{\mathcal{J}}}{dt^2} + \frac{(\Delta\mu)^2 + \varepsilon_J^2}{\hbar^2} \hat{\mathcal{J}} = 0. \quad (33)$$

The Eq. (31) is a linear inhomogeneous equation similar to a classical undamped forced (driven) SHO with natural frequency of the SHO $\omega_{\mathcal{I}} = \varepsilon_J/\hbar$, external force $\hat{\mathcal{T}}$, mass $m = \hbar^2/\varepsilon_J\Delta\mu$, elastic constant $\mathcal{K} = \varepsilon_J/\Delta\mu$ and period of the free oscillations $T = \frac{2\pi\hbar}{\varepsilon_J}$. The Eq. (32) is similar to the Eq. (31) when we exchange the currents and the parameter $\Delta\mu$ by ε_J . The Eq. (33) describes a SHO with natural frequency $\omega_{\mathcal{J}} = \sqrt{(\Delta\mu)^2 + \varepsilon_J^2}/\hbar$ and

period of the oscillations $T = \frac{2\pi\hbar}{\sqrt{(\Delta\mu)^2 + \varepsilon_J^2}}$. We have two possible dynamics for this case.

Summary - We have showed that a current algebra appears when we calculate the quantum dynamics of the tunneling of the atoms. We generalize the Heisenberg equation to write the second derivative of an operator. Then we calculated the quantum dynamics of these currents and show that different dynamics appear when we consider different choices of the parameters of the Hamiltonian. The strength of the parameter K determines the non linearity of the currents dynamics. For specific choice of the parameter we get analogue equations to the classical simple harmonic oscillator and the undamped forced (driven) simple harmonic oscillator with the natural frequencies dependent of the parameters of the Hamiltonian.

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