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The Riemann Conjecture and the Advanced Calculus Methods for Physics

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2/Abstract: We present a set of lectures on topics of advanced calculus in one real and complex variable with several new results and proofs on the subject, specially with a detailed proof - always missing in the literature - of the Cissoti explicitly integral formula conformally representing a polygon onto a disc. Besides we present - in the paper appendix - a new study embodied with a mathematical physicist perspective, on the famous Riemann conjecture on the zeros of the Zeta function, reducing its proof to a conjecture on the positivity of a numerical series.

Introduction

In this initial, we introduce with detailed proofs all mathematical objects and tools of the basic advanced calculus (Real Numbers, Uniform Convergence, Power Series, Fourier Series, Cauchy theorem in Functions of one Complex Variable, etc.), all of them needed to formulate the results exposed in next chapters of this set of complementary advanced calculus lectures notes.

1.2. On the real numbers: an abstract overview ([1] and [3]).

We start this section by trying to solve the simple algebraic equation of second order through rational numbers $x \in \mathbb{Q}$ (the attempt for understanding of Pytaghoras theorem for a rectangle triangle with equal lenght catets was the ancestor of this inquirie)

$$x^2 = 2 \tag{1.1}$$

Let us suppose that there is a rational number in the irreducible prime form $x = m/n$ satisfying eq.(1.1). Obviously m^2 is an even number, so four is a divisor of m [note that if m were odd, m^2 would be odd!]. As a consequence n is even. A contradiction with the prime number irreducibility of m and n. So $x \notin Q$.

We must thus introduce a new notion of mathematical objects called the real numbers, mathematical concept making the basis of all western mathematical continuum thinking in terms of logical objects: the so called real numbers. The most suitable mathematical procedure is the constructive method due to the mathematician G. Dedekind ([1]). Let us briefly sketch the basics of his idea.

Firstly, we define mathematical objects called Dedekind cuts which are sets of rational numbers.

Definition 1. A set $[\alpha]$ of rational numbers is a Dedekind cut if it satisfies the following conditions:

a) $[\alpha]$ is a non empty set of rational numbers

b) if $p \in [\alpha]$ and $q < p$, then $q \in [\alpha]$

c) there is no maximal element $\bar{x} \in [\alpha]$. Namely if for all $p \in [\alpha]$ and $p < \bar{x}$ with $\bar{x} \in [\alpha]$, then $[\alpha] = \varnothing$.

It is easy to accept that the above set of assertives defines non trivial mathematical objects. For instance the set $[1/3] = \{x \in \mathbb{Q} \mid x < 1/3\}$ is a Dedekind cut in the context of Definition 1.

As an extended exercise (see for instance ref.[1]), one can show that there is a somewhat intuitive (althought highly non trivial to prove!) order relation among the set of all Dedekind cuts. Namely: $[\alpha] \prec [\beta]$ if there is an element $y \in \beta$ which is maximal for $[\alpha]$, for instance, one has the following result on basis of the above mentioned order relation among the Dedekind cuts.

Theorem 1. Let $[\alpha]$ and $[\beta]$ be Dedekind cuts. We have thus the following set of order relations (which turns the set of Dedekind cuts an Ordered Complete Field [3]).

a) $[\alpha] = [\beta]$, $[\alpha] \prec [\beta]$ or $[\beta] \prec [\alpha]$.

b) If $[\alpha] \prec [\beta]$ and $[\beta] \prec [\gamma]$, then $[\alpha] \prec [\gamma]$.

c) $[\alpha] \oplus [\beta] \overset{\text{def}}{\equiv} \{x \in \mathbb{Q} \mid x = u + v,$ with $u \in [\alpha]$ e $v \in [\beta]\}$ is a Dedekind cut.

d) $[\alpha] \cdot [\beta] \overset{\text{def}}{\equiv} \{x \in \mathbb{C} \mid x = u \cdot v,$ with $u \in [\alpha], v \in [\beta]\}$ is a Dedekind cut.

e) If $[\alpha] \neq [0]$, there is a unique $[\beta]$ such that $[\alpha] \cdot [\beta] = [1]$. We call the Dedekind cut $[\beta]$ by the suggestive notation $[\beta] = [\alpha^{-1}]$.

f) $([\alpha] \cdot [\beta]) \cdot [\gamma] = [\alpha] \cdot ([\beta] \cdot [\gamma])$.

g) $[\alpha]([\beta] \oplus [\gamma]) = ([\alpha] \cdot [\beta] + [\alpha] \cdot [\gamma])$.

Now we can see that the set of real numbers as rigorous mathematical object defined as the set of all Dedekind cuts makes real numbers such as abstract objects in Modern Mathematics as ever one could hardly imagine. In Modern Calculus it is common to consider a further abstraction by taking into account the extended real numbers with a point called infinite point, denoted by ∞, and satisfing the operational rules

a) $[\alpha] + [\infty] = +\infty$

b) $[\alpha] - [\infty] = -\infty$

c) if $[\alpha] > [0]$, then $[\alpha][+\infty] = +\infty$, $[\alpha][-\infty] = -\infty$

d) if $[\alpha] < [0]$, then $[\alpha][+\infty] = -\infty$, $[\alpha][-\infty] = +\infty$

e) $[\alpha] \cdot [\alpha^{-1}] = [\alpha][(-\infty)^{-1}] = 0$

Let us now announcing the basic result of the rigorous theory of real numbers in this framework of Dedekind cuts:

Theorem 2 (Dedekind). Let be given two sets of real numbers A and B such that

a) $(B)^C = A$

b) $A \cap B = \{\varnothing\}$

c) $A \neq \varnothing$ and $B \neq \{\varnothing\}$

d) If $[\alpha] \in A$ and $[\beta] \in B$, then $[\alpha] \prec [\beta]$. Then there is only one Dedekind cut $[\gamma]$ such that $[\alpha] \prec [\gamma]$ and $[\gamma] \prec [\beta]$ ($\exists\, \gamma \in R \mid x \leq \gamma \leq y$, for any $x \in [\alpha]$ and $y \in [\beta]$).

As an important result of the above theorem one can show that for any given non-empty set of real numbers E with a upper bound M (which is a real number $M \in R$, such that all elements x of $E\, [x] \prec [M]$, $\forall\, [x] \in E$), there is a real number $[y]$ such that for any $[x] \in E$ with $[x] \prec [y]$ and given another (y') satisfying the above property, necessarily $[y] \prec [y']$. We call this real number $[y]$ as the supremum of E, namely: $\sup E = [y]$.

We have the following algebric theorem producing a plenitude of real numbers.

Theorem 3. Let $P_n(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ be a polinomial of degree n with all the coeficients belonging to the set of integers. If the algebric equation $p_n(x) = 0$ has roots in $\mathbb{Q}$, then theses roots are necessarily integers numbers.

Proof. Let $x = p/q$, with p and q primes irreducibles and being a root of our given polinomial equation. By elementary arithmetic manipulations we have

$$0 = q^n\, P_n(p/q) = p^n + a_1 p^{n-1} q + a_2 p^{n-2} q^2 + \cdots + a_n q^n. \tag{1.2}$$

As a consequence

$$p^n = -a_1 p^{n-1} q - \cdots - a_n q^n. \tag{1.3}$$

So q is a divisor of p^n, and thus q is a prime factor of p. A contradiction. ■

As an application of this beautiful elementary theorem, one can show that the real number $\sqrt[3]{7}$ (the cubic root of seven) is not a rational number by just considering the equation $x^3 = 7$ (the roots'candidates satisfying $x < 2$ are $x = 0$ or 1 and obviously are not its roots!).

At this point appears curious to note the fact that the famous Fermat theorem about the impossibility of satisfying the polinomial relationship $a^n + b^n = c^n$, for $n \geq 3$ and a, b, c belonging to positive integers is equivalent to the assertive that for $n \geq 3$ the equation $\left(1-\alpha^n\right)^{-\frac{1}{n}} = x$ for $x > 1$ and $x \in \mathbb{Q}$ does not have solution in $\mathbb{Q}$. Since if this would be the case for some $\alpha = \frac{p}{q}$ and $x = \frac{r}{s}$ (with $r > s$), one would trivially obtain the relationship $(qr)^n = (pr)^n + (qs)^n$. (Note that $(qr)^n - (qs)^n = q^n(r^n - s^n) > 1$). It becomes worth to have a concrete representation of the real numbers as sequences of rational numbers – an usual axiomatic concept.

We left as an exercise of our reader to show that every real number in $[0, 1]$ (and consequently all R!) can be expressed by a sequence of rational numbers called the binary expansion $x = \sum_{n=1}^{\infty} b_k/2^k$ where b_k are either 0 or 1. It is clear that the set of these sequences has the same cardinality (can be put in a bijective correspondence) with the set of all functions of I in the set $\{0, 1\}$ which has cardinality $2^{\#(Q)}$, this prove that $\#(R) = 2^{\#(Q)}$, where $\#(A)$ means the cardinal number of the set A (see references). It is interesting to call the reader attention that there is a hypothesis in abstract set theory which still remains an open problem: the called continuous hypothesis which conjectures that there is no cardinal number between the cardinality of the integers (or the rationals) $\#(Q) = \aleph_0$ (aleph zero) and the cardinality of the real numbers $\aleph_1 = 2^{\aleph_0}$ (the aleph one).

After having defined precisely in an axiomatic context the set of real numbers, denoted from here on by R, let us introduce some objects in this abstract set.

Definition 3. A number $\bar{a}$ is called the limit supremum (lim sup for short) of a given enumerable set of real numbers $\{x_n\}$ (which can be tought as the range of an application of the Positive Integers I^+ on $R \Leftrightarrow x_n = f(n)$, $f \in \mathcal{F}(I^+, R)$ = set of real valued functions defined on the integers) if for any *real* number $\varepsilon > 0$, we have always that

a) *There is an infinite number* of terms of the given sequence which are greater (or equal) than $\bar{a} - \varepsilon$.

b) Solely a *finite number* of term (elements) of the sequence $\{x_n\}$ are greater (or equal) than $\bar{a} + \varepsilon$.

We introduce the notation $\limsup(x_n) = \bar{a}$.

The limit infimum of a given sequence is defined by the relationship

$$\liminf(x_n) = -\limsup(-x_n). \tag{1.4}$$

One thus says that a given sequence $\{x_n\}$ converges for a given real number $\bar{x}$ if $\limsup(x_n) = \liminf(x_n) = \bar{x}$.

From these definitions, one can turn the abstract definition of real numbers in terms of the more concrete sequence of rational number (the decimal/binary expansion, etc.) so familiar to the elementary Calculus students.

Let us show that the sequence below defined

$$x_n = \sum_{k=0}^{n} \left(\frac{1}{k!}\right) \tag{1.5}$$

defines an irrational number (it converges - there is $\lim\{x_n\}$ denoted here on by e).

Firstly we left as an exercise to our reader to prove that

$$0 < e - x_n < \frac{1}{n \cdot n!}\cdot \tag{1.6}$$

Let us suppose now that $e \in \mathbb{Q}$. So there is p and q primes irreducible with $q > 1$. As a consequence

$$0 < q!(e - x_q) < \frac{1}{q} \tag{1.7}$$

or

$$0 < \left[q! \cdot e - q!\left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{q!}\right)\right] < \frac{1}{q}\cdot \tag{1.8}$$

From the above inequality [note that $q! \cdot e$ is an integer and $q!\left[1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{q!}\right]$ is an integer too!] one obtains that there is an integer between 0 and 1 if e is a rational number. A contradiction. ■

At this point we invite our reader to remember with paper and pencil all those sections of one-variable calculus books related to convergence of series and sequences (Cauchy criterium, the $\varepsilon - \delta$ definitions, etc.).

As a final point of this introductory review on axiomatic concepts, about real numbers, let us introduce the definition of double sequences and appropriate convergence criterions for them.

One call a double sequence $a_{n,m}$ as the range of a real-valued application of the set $Z^+ \times Z^+$ (the Cartesian Product of the positive integers). Namely

$$\begin{aligned} f\colon Z^+ \times Z^+ &\longrightarrow R \\ (n,m) &\longrightarrow f(n,m) = a_{n,m} \end{aligned} \tag{1.9}$$

For instance

$$a_{n,m} = (-1)^m \, \frac{n}{(n+1)!} \tag{1.10}$$

$$a_{n,m} = m \, \cos\big[(2n+1)\pi\big] \tag{1.11}$$

$$a_{n,m} = \frac{1}{\big(Am^2 + 2Bmn + Cn^2\big)^p} \text{ (with real numbers } A > 0,\ B^2 < AC,\ p > 1) \tag{1.12}$$

In our more oriented advanced calculus context we stand for the usual Cauchy definition for the convergence as a practical criterium to test convergence of sequences.

Definition 3. We say that the double sequence $\{a_{nm}\}$ converges to a limit $\bar{a}$ if for any $\varepsilon > 0$ ($\forall\, \varepsilon > 0$), there is ($\exists$) an integer-function of ε ($N(\varepsilon)$) such that ($|$) for any n and m greater than $N(\varepsilon)$ ($n, m > N(\varepsilon)$) it is true the relation (inequality) below

$$\Big\{\forall\, \varepsilon > 0, \exists\, N(\varepsilon) \mid |a_{n,m} - \bar{a}| < \varepsilon, \text{ for } n > N(\varepsilon), m > N(\varepsilon) \text{ means that } \lim_{\substack{m\to\infty \\ n\to\infty}} a_{n,m} = \bar{a}\Big\} \tag{1.13}$$

Since it is harder to apply the usual elementary calculus criteriums for test the convergence of double sequences, we present the integral criterium which appears the most suitable one for practical applications within advanced calculus context.

Lemma 1. Let $a_{n,m} = f(m,n) \in Z^+$ be such that defines a function of two-variables with domain in $R^2_+ = \{(x,y) \mid x \geq 0; y \geq 0\}$, namely $f(x,y)$ and such that

a) $f(x,y)$ is continuous and Improper Riemann integrable in R^2_+

b) $f(x,y)$ is monotone non increasing ($f(x+h,y+r) \leq f(x,y)$ for any $h,r \geq 0$).

We have thus:

c) the double series $\sum_{n,m} a_{n,m}$ and the improper (Riemann) integral $\iint_{R^2_+} f(x,y)dxdy$, both converge and diverge at the same time.

Proof. Let us observe firstly that by a simple application of the mean value theorem under the hypothesis a) and b) we have the estimate

$$a_{n+1,m+1} \leq \int_n^{n+1} dx \int_m^{m+1} dy\, f(x,y) \leq a_{n,m}\,. \tag{1.14}$$

The result below stated as an solved exercise complete the proof of Lemma 1.

Exercise – Prove that the Improper Riemann Integral below is finite (for $\delta_1, \delta_2 \in R^+$ fixed)

$$\lim_{\substack{A_1\to\infty\\A_2\to\infty}} \left\{ \int_{\delta_1}^{A_1} dx \int_{\delta_2}^{A_2} dy\ g(x)h(y)\, f(x,y) \right\} < +\infty \tag{1.15}$$

if

a) $g(x)$ and $h(y)$ are monotone non increasing continuously differentiable functions vanishing at the infinite

b) the indefinite integrals

$$\int^{\xi} f(\xi',\eta)d\xi' = \Omega_1(\xi,\eta); \qquad \int^{\eta} f(\xi,\eta')d\eta' = \Omega_2(\xi,\eta)$$

are bounded functions in $[\delta_1,\infty) \times [\delta_2,\infty)$

Answer to exercise:

By an application of the elementary double integration theorems on the exchange order of integration, we have that

$$\int_{\delta_1}^{A_1} d\xi \int_{\delta_2}^{A_2} d\eta\ g(\xi)h(\eta)\, f(\xi,\eta)$$

Fubini-Toneli theorem

$$\begin{aligned}
&= \int_{\delta_2}^{A_2} d\eta\, h(\eta) \left[g(A_1)f(A_1,\eta) - g(\delta_1)f(\delta_1,\eta) - \int_{\delta_1}^{A_1} d\xi\ g'(\xi)\Omega_1(\xi,\eta) \right] \\
&= g(A_1) \left(\int_{\delta_2}^{A_2} d\eta\, h(\eta)\, f(A_1,\eta) \right) - g(\delta_1) \left(\int_{\delta_2}^{A_2} d\eta\, h(\eta)\, f(\delta_1,\eta) \right) \\
&\quad - \left(\int_{\delta_2}^{A_2} d\eta \int_{\delta_1}^{A_1} d\xi\, h(\eta) g'(\xi)\, \Omega_1(\xi,\eta) \right) \qquad (1.16) \\
&= g(A_1) \left(h(A_2)\Omega_2(A_1,A_2) + \int_{\delta_2}^{A_2} d\eta(-h'(\eta))\Omega_2(A_2,\eta) \right) \\
&- g(\delta_1) \left(h(A_2)\Omega_2(A_1,A_2) + \int_{\delta_2}^{A_2} d\eta(-h'(\eta))\Omega_2(A_2,\eta) \right) \\
&\quad - \left(\int_{\delta_2}^{A_2} d\eta \int_{\delta_1}^{A_1} d\xi\ h(\eta) g'(\xi)\, \Omega_1(\xi,\eta) \right) \qquad (1.17)
\end{aligned}$$

Since the functions $g(x)$, $h(y)$ vanish at $x, y \to +\infty$ and the estimates below hold true:

$$\begin{aligned}
&\left| \int_{\delta_2}^{A_2} d\eta \int_{\delta_1}^{A_1} d\xi\ h(\eta)(-g'(\xi))\Omega_1(\xi,\eta) \right| \\
&\quad \le \sup_{(x,y)\in R_+^2} |\Omega_1(x,y)h(y)| \left(\int_{+\delta_1}^{A_1} d\xi(-g'(\xi)) \right) \\
&= \left(\sup_{(x,y)\in R_+^2} |\Omega_1(x,y)h(y)| \right) \big(g(\delta_1) - g(A_1) \big) < +\infty \qquad (1.18)
\end{aligned}$$

and,

$$\begin{aligned}
&\left| \int_{\delta_2}^{A_2} d\eta(-h'(\eta)\Omega_2(A_2,\eta) \right| \\
&\quad \le \left(\sup_{(x,y)\in R_+^2} |\Omega_2(x,y)| \right) \big(h(\delta_2) - h(A_2) \big) \le +\infty, \qquad (1.19)
\end{aligned}$$

we have that the improper Riemann integral as defined by the limit of A_1 and $A_2 \to +\infty$ in the left-hand side of eq.(1.15) exists.

As another exercise apply these results for the double infinite series eq.(1.12) and generalize them for the case of n-uple infinite series.

Exercise – Show that the limit below exists

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e,\ \text{with } 2 < e < 3.$$

Answer: Let us consider the sequence of rational numbers

$$x_n = \left(1+\frac{1}{n}\right)^n,\ \text{with } n = 1, 2, \ldots\ .$$

We note the obvious identity for a given $k \leq n$

$$\frac{1}{n^k}\left(\frac{n(n-1)\ldots(n-k+1)}{k!}\right) = \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{(k-1)}{n}\right). \tag{1.20}$$

On the basis of eq.(1.20), one can see that (the Newton Binomial Theorem!)

$$\begin{aligned} x_n = \left(1+\frac{1}{n}\right)^n &= 1 + \frac{n}{1!}\left(\frac{1}{n}\right) + \cdots + \frac{n(n-1)\ldots(n-k+1)}{k!}\left(\frac{1}{n}\right)^k + \cdots + \\ &\quad \frac{n(n-1)\ldots(n-(n-1))}{n!}\left(\frac{1}{n}\right)^n \\ &= 1 + \frac{1}{1!} + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \ldots \\ &\quad \frac{1}{k!}\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{(k-1)}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\ldots\left(1-\frac{(k-1)}{n}\right) \end{aligned} \tag{1.21}$$

We remark now that if we consider the case of $n \to n+1$ in eq.(1.21), the new k-th term takes the form

$$\frac{1}{k!}\left(1-\frac{1}{(n+1)}\right)\ldots\left(1-\frac{(k-1)}{n+1}\right) \tag{1.22}$$

which is greater then the k-term corresponding to n-th term in eq.(1.21).

As consequence, we have showed that $\{x_n\}$ is a monotone non-decreasing sequence ($x_n \leq x_{n+1}\,,\ \forall\, n \in Z^+$) and consequently has a limit (see any text-book on Elementary Calculus). Let us show that the bound below is correct

$$x_n \leq b_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \tag{1.23}$$

This result is straightforward since

$$\frac{1}{k!}\overbrace{\left(1-\frac{1}{n}\right)}^{<1}\dots\overbrace{\left(1-\frac{(n-1)}{n}\right)}^{<1}\leq\frac{1}{n!}\,(1)\dots(1)=\frac{1}{k!} \tag{1.24}$$

Additionally, one has the immediate estimate

$$\frac{1}{k!}=\frac{1}{1.2\dots k}\leq\frac{1}{1.2.2\dots 2}=\frac{1}{2^{k+1}} \tag{1.25}$$

which by its turn leads to the bound below and thus showing our claim:

$$b_n<1+\left(1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{n-1}}\right)=3-\frac{1}{2^{n-1}}\Rightarrow e<3 \tag{1.26}$$

Exercise – For any positive integer n, let R^n be set of all n-ordered uples ($R^n=R\times\cdots\times R$, the Cartesian Product Set) $x=(x_1,\dots,x_n)$, with $x_n\in R$, for $1\leq k\leq n$. Show that if one defines the operations below

$$\text{a)}\quad x+y=(x_1+y_1,\dots,x_n+y_n) \tag{1.27}$$

$$\text{b)}\quad \alpha\, x=(\alpha\, x_1,\dots,\alpha\, x_n),\quad x\in R \tag{1.28}$$

$$\text{c)}\quad x\cdot y=\sum_{i=1}^{n}x_iy_i \tag{1.29}$$

one has that R^n is a vectorial space and the lenght function

$$\text{d)}\quad |x|=(x\cdot x)^{1/2}=\left(\sum_{n=1}^{n}x_k^2\right)^{1/2} \tag{1.30}$$

satisfies the relations

$$\begin{aligned}\text{d-1)}\quad &|x|=0\Leftrightarrow x=0.\quad |x\cdot y|\leq|x|\,|y|;\quad |x+y|\leq|x|+|y|;\\ &|x-z|\leq|x-y|+|y-z|;\quad \big||x|-|y|\big|\leq|x-y|;\\ &|x+y|^2+|x-y|^2=2\big(|x|^2+|y|^2\big)\end{aligned} \tag{1.31}$$

Exercise – Let $x>1$ and $y>0$. Show that there is a unique real number z such that $x^z=y$. This number is called the logarithm of the number y in the “log basis” x.

Answer: Show that the set $E = \{t \in R \mid x^t < y\}$ is non empty and is upper bounded.

Exercise – Let $\{x_n\}$ be a sequence of positive real numbers. We have that $\sum\limits_{n=1}^{\infty} x_n$ converges if and only if $\sum\limits_{n=1}^{\infty} 2^n x_{2^n} = x_1 + 2x_2 + 4x_n + \dots$ converges (prove it!). Apply this result to analyze the convergences of the following series

a) $\sum\limits_{n=2}^{\infty} \frac{1}{n\,\ell g\, n\, \ell g(\ell g n)}$ (diverges)

b) $\sum\limits_{n=3}^{\infty} \frac{1}{n\,\ell g\, n\, (\ell g(\ell g n))^2}$ (converges)

1.3 On sequence of functions of one variable – an overview ([1], [2], [3]).

Let us be given a sequence of real valued functions $\{f_n(x)\}$ with a common domain $E \subset R$ of the extended real line and such that for any given $\bar{x} \in E$, the numerical sequence $\{f_n(x)\}$ converges. One can show (exercise) that $\bigcup\limits_{x\in[a,b]} \{\lim\limits_{n\to\infty} f_n(x)\}$ defines a function $f(x)$, denoted by $f(x) = \lim\limits_{n\to\infty} f_n(x)$ and called the pointwise limit of the functional sequence $\{f_n(x)\}$. Note that for a given $\bar{x} \in E$, and any $\varepsilon > 0$, $\exists\, N_0(\varepsilon, x)$ such that $|f_n(x) - f(x)| < \varepsilon$ for $n \geq N_0(\varepsilon, x)$. In the all important cases one can show that it is possible to consider N_0 as an unique function of ε and uniformly for $x \in E$. One thus says that the convergence of the functional sequence $\{f_n(x)\}$ is uniform on E and it is represented by the notation $f(x) \overset{\text{unif.conv.}}{=} \lim\limits_{n\to\infty} f_n(x)$.

As an example of the above cited uniform convergence let us consider the sequence

$$f_n(x) = 1 + x + \dots + x^n \tag{1.32}$$

$$E = \big[-1/2, 1/2\big] \tag{1.33}$$

Since (the usual Geometrical series!)

$$\begin{aligned}
&\max_{x\in\left[-\frac{1}{2},\frac{1}{2}\right]} \left| \frac{1}{1-x} - (1 + x + \dots + x^n) \right| \\
&= \max_{x\in\left[-\frac{1}{2},\frac{1}{2}\right]} \left| x^n \left(\frac{1}{1-x} \right) \right| \\
&= 2^{-n} \cdot 2 = 2^{n-1},
\end{aligned} \tag{1.34}$$

we can see that eq.(1.32) converges uniformly for $f(x) = 1/(1-x)$ on E (eq.(1.33)).

We have the obvious quite important K. Weierstrass criterium for uniform convergence: If $\sup\limits_{x\in E} |f_n(x) - f(x)| = M_n$ is a sequence of real numbers converging to zero, then $f_n(x)$ converges uniformly for $f(x)$.

The basic operational result in such setting of uniform convergence is the following:

a) If $f_n(x) \in C(E)$ then $f(x) \in C(E)$

b) $\displaystyle\int_a^x f_n(\xi)\, d\xi \to \int_a^x f(\xi)\, d\xi$ for any $(a, x) \in E \times E$.

c) If $f_n \in C^1(E)$ and $f_n'(x) \overset{\text{conv. unif.}}{\longrightarrow} G(x)$ for $x \in E$ and there is $x_0 \in E$ such $f_n(x_0)$ converges. Then

$$f_n(x) \overset{\text{conv. unif.}}{\longrightarrow} \int_{x_0}^x d\xi\; G(\xi) + \big(\lim_{n\to\infty} f_n(x_0)\big) \tag{1.35}$$

An important sequence of uniformly convergent real functions are these associated to the Taylor theorem with remainders

Theorem 4 (Taylor). Let $f(x)$ be a real function on $[0, b]$ (just for simplicity) and possessing $(n+1)$-derivatives there. Let $x \in [0, b]$ be an arbitrary point of $[0, b]$. Then we have the equality

$$f(x) = f(0) + f'(0)x + \cdots + \frac{x^n}{n!}\, f(0) + \frac{x^{n+1}}{(n+1)!}\, f^{(n+1)}\,(\theta(x)\cdot x) \tag{1.36}$$

with the function $\theta(x)$, such that $0 < \theta(x) < 1$.

On the basis of the above theorem one is lead to the following easy result

Theorem 5 (Power Series). Let $f(x) \in C^\infty([0, b])$ and

a) $\{a_n\} = \left\{\dfrac{f^{(n)}(0)}{n!}\right\}$ be a sequence of real numbers with

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{R} < +\infty \tag{1.37}$$

then the sequence of polinomial functions

$$f_n(x) = \sum_{n=1}^{n} a_n\, x^k \tag{1.38}$$

converges uniformly for $f(x)$ in $-R < x < R$.

Proof. Let us estimate the following difference for $\varepsilon > 0$

$$\max_{[-R+\varepsilon, R-\varepsilon]} |f(x) - f_n(x)| = \max_{[-R+\varepsilon, R-\varepsilon]} \left| \sum_{m=n+1}^{\infty} a_m \, x^m \right|$$
$$\leq \sum_{m=n+1}^{\infty} |a_n|(R-\varepsilon)^m = M_n \to 0 \tag{1.39}$$

since

$$\sum_{m=0}^{\infty} |a_n|(R-\varepsilon)^m < +\infty.$$

By the Weierstrass criterium we have our Theorem 4 proved. ■

We call a given function in $C^{\infty}(E)$ with $E \subset R$, an analytical function at a point $x_0 \in E$ if there is an open interval $I_{x_0} = (x_0 - \varepsilon, \, x_0 + \varepsilon) \subset E$ such that $f(x)$ can be expanded as a power series centered at x_0 for any $x \in I_{x_0}$: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$.

An useful criterium for analyticity besides eq.(1.32) is the theorem of Pringshein that says that if $\lim_{n\to\infty} \left(\big[\sup_{a \leq x \leq b} |f^{(n)}(x)| \big]^{1/n} / n \right) = 0$, then $f(x)$ is an analytic function on $[a, b]$.

Exercise – On the uniform convergence of sum of functions in R^N. Let now be given a sequence of real valued functions $u_n(x)$ in a given common domain $\Omega \subset R^N$. Let us consider the sum $\big(u_n(x) \equiv u_n(x_1, \ldots, x_N)\big)$.

$$S_N(x, [u_n]) = \sum_{n=0}^{N} a_n \, u_n(x). \tag{1.40}$$

If we suppose that $u_{n+1}(x)/u_n(x)$ never vanishes in Ω and

$$\left(\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \right) \underbrace{\lim_{n\to\infty} \left(\sup \left| \frac{u_{n+1}(x)}{u_n(x)} \right| \right)}_{g(x)} < L < 1 \tag{1.41}$$

then $\left\{ \sum_{n=0}^{N} a_n \, u_n(x) \right\}$ converges uniformly for a function $\overline{u}(x)$ in any domain $W \subset g^{-1}([0, R])$, with $\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ (this is a real-version of ours of the famous theorem of Burmann-

Lagrange in Complex Calculus to be seen in the end of this article). Of couse, the determination of the coefficients a_n makes recourse to integrals representation in the complex plane as we are going to show.

Exercise – Extend all the above presented results for the R^N case.

We now apply the above results to solve second order linear differential equations with analytical variable coefficients.

Our aim now is to determine two linearly independent solutions of the following variable-coefficients homogeneous second order ordinary differential equation

$$y'' + a(x)y' + b(x)y = 0 \tag{1.42}$$

in a form of power series if the function $a(x)$ and $b(x)$ have convergent power series around a given point x_0 which will allways be considered to be the origin $x_0 = 0$ without lost of generality.

Let us thus consider the power series expansions of the variable coefficients on the common interval $-R < x < R$

$$a(x) = \sum_{n=0}^{\infty} a_n\, x^n \tag{1.43-a}$$

$$b(x) = \sum_{n=0}^{\infty} b_n\, x^n \tag{1.43-b}$$

and the searched solution written as a power series

$$y(x) = \sum_{n=0}^{\infty} y_n\, x^n. \tag{1.43-c}$$

Here the coefficients y_n are defined from the recurrence relationship

$$(n+1)(n+2)\, y_{n+2} = -\left\{ \sum_{k=0}^{n} \left[(k+1)a_{n-k}\, y_{k+1} + b_{n-k}\, y_k \right] \right\}. \tag{1.43-d}$$

Let us show that eq.(1.43-c) has the same radius of convergence R.

Let $\varepsilon > 0$ be given such that for some $M > 0$, we have the bounds

$$|a_n|(R-\varepsilon)^n \leq M \tag{1.44-a}$$

$$|b_n|(R-\varepsilon)^n \leq M \tag{1.44-b}$$

We have the estimate (with $r = R - \varepsilon$)

$$(n+1)(n+2)|y_{n+1}| \leq \frac{M}{r^n}\left\{\sum_{k=0}^{n}\big[(k+1)|y_{k+1}| + |y_k|\big]r^k\right\} + \big(M|y_{n+1}|r\big) \tag{1.45}$$

We now define the majorant series z_n by the equation

$$(n+1)(n+2)z_{n+2} = \left\{\frac{M}{r^n}\left[\sum_{k=0}^{n}\big((k+1)z_{k+1} + z_k\big)r^k\right]\right\} + Mz_{n+1}\,r \tag{1.46}$$

Obviously

$$|y_n| \leq z_n\,. \tag{1.47}$$

Let us consider eq.(1.46) for $n = n-1$ and $n = n-2$ respectively

$$n(n+1)z_{n+1} = \left\{\frac{M}{r^{n-1}}\left(\sum_{k=0}^{n-1}\big[(n+1)z_{k+1} + z_k\big]r^k\right]\right\} + Mz_n\,r \tag{1.48-a}$$

$$(n-1)nz_n = \left\{\frac{M}{r^{n-2}}\left(\sum_{k=0}^{n-2}\big[(n+1)z_{k+1} + z_k\big]r^k\right]\right\} + Mz_{n-1}\,r \tag{1.48-b}$$

By multiplying eq.(1.48-a) by r, we have that

$$r\,n(n+1)z_{n+1} = \left\{\frac{M}{r^{n-2}}\left[\sum_{k=0}^{n-2}\big[(k+1)z_{k+1} + z_k\big]r^k\right]\right\} + Mz_n\,r^2$$
$$+\left(\left[\overbrace{(n-1+1)z_{n-1+1} + z_{n-1}}^{(nz_n+z_{n-1})rM}\right]\frac{M}{r^{n-2}}\right)r^{n-1} \tag{1.49-a}$$

which leads to

$$\begin{aligned} rn(n+1)z_{n+1} &= (n-1)nz_n - Mz_{n-1}r + rM(nz_n + z_{n-1}) + Mz_nr^2 \\ &= \big[(n-1)n + rMn + Mr^2\big]z_n \end{aligned} \tag{1.49-b}$$

As a consequence one has immediately that

$$\lim_{n\to\infty} \frac{z_{n+1}}{z_n} = \frac{(n-1)n + rMn + Mr^2}{r(n)(n+1)} = \frac{1}{r} \tag{1.50}$$

Showing thus our result eq.(1.43-c)/eq.(1.43-d). ■

After having determined two linearly independent solutions with integration constants y_1 and y_2 left undetermined one can use the Lagrange result to write a particular solution of the non homogeneous equation (1.42) as a functional of these two linearment independent homogeneous solutions $y_1(x)$ and $y_2(x)$

$$y_p(x) = C_1(x)y_1(x) + C_2(x)y_2(x)$$

where the Lagrange multipliers are given by

$$\frac{dC_1(x)}{dx} = -f(x)y_2(x)/(y_1y_2' - y_1'y_2)(x) \tag{1.51-a}$$

$$\frac{dC_2(x)}{dx} = +f(x)y_1(x)/(y_1y_2' - y_1'y_2)(x) \tag{1.51-b}$$

where $f(x)$ denotes the non-homogeneous term.

It is useful to know that if it is possible to determine one homogeneous solution of eq.(1.42), the other solution will be always given by Lagrange integral representation formula

$$y_2(x) = y_1(x)\left\{\int \exp\left(-\int^x a(\xi)\,d\xi\right)\left(y_1(x)\right)^{-1} dx\right\}. \tag{1.52}$$

At this point it is worth to write explicitly formulae for the coefficients of an inverse power series (Exercise) (see eq.(1.16a)

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)^{-1} = \sum_{n=0}^{\infty} b_n x^n \tag{1.53}$$

where

$$b_0 = \frac{1}{y_0} \tag{1.54-a}$$

$$b_n = \frac{(-1)^n}{(y_0)^{n+1}} \begin{vmatrix} y_1 & y_2 & \dots & y_n & \\ y_0 & y_1 & y_2 & \dots & y_{n-1} \\ 0 & y_0 & y_1 & \dots & y_{n-1} \\ 0 & \dots & & y_1 & y_2 \end{vmatrix} \tag{1.54-b}$$

Formal solutions of non-linear second order ordinary differential equations can in principle be easily written. Let us scketch such procedure

$$\frac{d^2y}{d^2x} = F\left(x, y(x), \frac{dy}{dx}(x)\right). \tag{1.55}$$

Here $F(x, u, v)$ is a $C^\infty(\Omega)$ function of 3-variables in a given domain $\Omega \subset R^3$.

Let us call $y(x_0) = y_0$ and $y'(x_0) = v_0$. We have that $y''(x_0) = F(x_0, y_0, v_0)$. The next coefficients are easily determined from eq.(1.55) through derivatives. Namely

$$\frac{d^3y}{dx^3} = \frac{\partial F}{\partial x}(x_0, y_0, v_0) + \frac{\partial F}{\partial u}(x_0, y_0, v_0)v_0 + \frac{\partial F}{\partial v}(x_0, y_0, v_0)y''(x_0) \tag{1.56}$$

One must now analyze the convergence of the resulting power series case by case.

A further application of power series is the determination of linearment independent solutions of a second order ordinary differential equation of the form (the so called regular singular point)

$$x^2y''(x) + x\,a(x)y'(x) + b(x)y(x) = 0. \tag{1.57}$$

Here

$$a(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{for} \quad |x| < R \tag{1.58-a}$$

$$b(x) = \sum_{n=0}^{\infty} b_n x^n \quad \text{for} \quad |x| < R \tag{1.58-b}$$

Let us try a solution of eq.(1.57) in the (Laurent-Frobenius) power series generalized form

$$x^{-r}\,y(x) = \sum_{n=0}^{\infty} y_n(r)x^n \quad \text{for} \quad 0 < |x| < R. \tag{1.58-c}$$

The coefficients $y_n(r)$ are functions of the parameter r and satisfy the recurrence relationship below

$$y_n(r)\big[(r+n)(r+n-1)+(r+n)a_0+b_0\big]$$
$$= -\left\{\sum_{k=0}^{n-1} y_k\big[(r+k)a_{n-k}+b_{n-k}\big]\right\}. \tag{1.58-d}$$

Now one can show (exercise!) that the function $y(x,r) = x^r\left[\sum\limits_{n=0}^{\infty} y_n(r)x^n\right]$ satisfy the non-homogeneous equation

$$x^2\, y''(x,r) + xa(x)y'(x,r) + b(x)y(x,r) = r(r-1) + ra_0 + b_0\,. \tag{1.59}$$

Let us obviously choose the free parameter r as solution of the algebric (indicial) equation

$$r(r-1) + r\,a_0 + b_0 = 0. \tag{1.60}$$

We have now the two cases

a) If the two roots r_1 and r_2 of eq.(1.60) have their difference not being an integer number, then $y(x,r_1)$ and $y(x,r_2)$ are two linearment ($\ell.i$) solutions of eq.(1.57).

b) If $r_1 = r_2$ (a double root), then $y(r,r_1)$ and $\lim\limits_{r\to r_1}\left(\dfrac{d}{dr}\,y(r,r)\right)$ is another solution (just derive in relation to the r-formula eq.(1.59) and take the limit of r approaching r_1 to obtain another (formal) solution).

c) Now if $r_1 - r_2 \in Z$, and the recurrence relationship eq.(1.58-d) is ill-defined for the root r_2. The two linearment independent solutions are thus given by $\lim\limits_{r\to r_2}\big[(r-r_2)y(r,r)\big]$ or $\lim\limits_{r\to r_2}\dfrac{d}{dr}\big((r-r_1)y(,r)\big)$. (Note that $y(x,r_1)$ is a linear combination of the two previous considered solutions).

Exercise – The reader should reads the paper about elementary special functions (Bessel, Legendre, etc...) on any elementary book of calculus ([2]).

Exercise – Show that the recurrence relation for the equation

$$xy'' - 3y' + xy = 0 \tag{1.61}$$

is given by

$$y_n(r) = -\left(\frac{1}{(r+4-n)(r+n)}\right) y_{n-2}(r), \quad n \geq 1. \tag{1.62}$$

As a consequence $r_2 = 0$ is a problematic indicial root. So the solutions are given explictly by

$$\begin{aligned} y(x,r) = y_0\, x^r \Big\{ r - \left(\frac{r}{(r-2)(r+2)}\right) x^2 + \frac{1}{(r-2)(r+2)(r+4)}\, x^4 \\ - \frac{1}{(r-2)(r+2)^2(r+4)(r+6)}\, x^6 + \ldots \Big\} \end{aligned} \tag{1.63}$$

$$y_1(x) = y(x,0) = -\frac{1}{2^4}\, x^4 + \frac{1}{2^6\, 3}\, x^6 - \ldots \tag{1.64}$$

$$\begin{aligned} y_2(x) &= \left(\frac{d}{dr}\, y(r,r)\right)\Big|_{r=0} = y_1(x)\, \ell\, nx \\ &+ 1 + \frac{1}{2^2}\, x^2 + \frac{1}{2^6}\, x^4 \\ &- \frac{1}{1!\, 3!\, 2^6} \left(1 + \frac{1}{2} + \frac{1}{3}\right) x^6 + \ldots \end{aligned} \tag{1.65}$$

1.4 – Basics of Fourier Series in one-variable.

In this section we wish to highlight the main basic points of the important theory of Fourier Series for piecewise continuous functions in the interval $[-\pi, \pi]$ for definiteness. Let us start our study by introducing the following definition.

Definition 4. A function $f(x)$ defined on the closed interval $[-\pi, \pi]$ is called piecewise continuous if the interval $[-\pi, \pi]$ can be splitted into a finite number of disjoint subintervals $\left([-\pi, \pi] = \bigcup_{i=1}^{N} [x_i, x_{i+1})\right)$ such that on each subinterval (x_i, x_{i+1}), the function $f(x)$ is a continuous function on the open interval (x_i, x_{i+1}) and the function $f(x)$ possesses (finite) limits at the left end $\left(\lim\limits_{\substack{h\to 0\\ h>0}} f(x_i + h) = f^+(x_i) < +\infty\right)$ and at the right end $\left(\lim\limits_{\substack{h\to 0\\ h>0}} f(x_{i+1} - h) = f^-(x_{i+1}) < +\infty\right)$.

The Fourier Expansion of a piecewise continuous function $f(x)$ on the interval $[-\pi,\pi]$ is defined by the following sequence of functions in $C^{\infty}([-\pi,\pi])$

$$S_N(x,[f]) = \frac{a_0[f]}{2} + \sum_{n=1}^{N} a_n[f]\cos(nx) + b_n[f]\sin(nx) \tag{1.66-a}$$

where the Fourier coefficients are defined by the integrations below in a suggestive notation by keeping the functional dependence of the given function $f(x)$ explicitly on these coefficients

$$a_0[f] = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \tag{1.66-b}$$

$$a_n[f] = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\cos(nx)\,dx \tag{1.66-c}$$

$$b_n[f] = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin(nx)\,dx \tag{1.66-d}$$

It is useful to re-write the above expansion into the so called complex notation

$$\overline{S}_N(x,[f]) = \sum_{n=-N}^{N} c_n[f]e^{inx} \tag{1.67-a}$$

with the Complex Fourier Coefficients

$$C_n[f] = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}\,dx. \tag{1.68}$$

In order to study the uniform convergence of the above written series of functions let us state the following lemma called the Bessel inequality

$$\frac{(a_0[f])^2}{4} + \sum_{n=1}^{N}\left((a_n[f])^2 + (b_n[f])^2\right) \le \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx \tag{1.69-a}$$

or

$$\sum_{n=-N}^{N} |C_n[f]|^2 \le \frac{1}{2\pi}\left(\int_{-\pi}^{\pi}|f(x)|^2\,dx\right) \tag{1.69-b}$$

Its proof is get easily from the obvious inequality

$$\left(\int_{-\pi}^{\pi}|f(x)|^2\,dx\right) - 2\pi\left(\sum_{n=-N}^{N}|C_n[f]|^2\right)$$

$$= \int_{-\pi}^{\pi}\left|f(x) - \sum_{n=-N}^{N} C_n[f]e^{inx}\right|^2 dx \ge 0 \tag{1.70}$$

With this result in hands let us show the uniform convergence of the "Fourier Partial sums" eq.(1.66-a) under the hypothesis that $f(x) \in C^1([-\pi,\pi))$ (a continuously differentiable function on $[-\pi,\pi]$: $f'(x) \in C([-\pi,\pi])$ and such tht $f(\pi) = f(-\pi)$; $f'(+\pi) = f'(-\pi)$).

This result is a simple consequence of the Weierstrass criterium applied to sequence of functions eq.(1.66-a) through the following chains of estimates

$$\begin{aligned}
\sup_{x\in[-\pi,\pi]} |S_N(x,[f])| &\leq \sum_{n=0}^{N} \sup_{x\in[-\pi,\pi]} |a_n[f]\cos(nx) + b_n[f]\sin(nx)| \\
&\leq \sqrt{2}\left\{\sum_{n=0}^{N} \left(a_n[f])^2 + (b_n[f])^2\right)^{1/2}\right\} \\
&\leq \sqrt{2}\left(\sum_{n=0}^{N} \frac{1}{n}\cdot n\left((a_n[f])^2 + (b_n[f])^2\right)^{1/2}\right\} \\
&\leq \sqrt{2}\left(\sum_{n=1}^{N}\left(\frac{1}{n^2}\right)\right)^{1/2}\left(\sum_{n=1}^{N}(na_n[f])^2 + (nb_n[f])^2\right)^{1/2} \\
&\leq \sqrt{2}\left(\frac{\pi^2}{6}\right)^{1/2}\left(\int_{-\pi}^{\pi} dx(f'(x))^2\right) < +\infty. \qquad (1.71)
\end{aligned}$$

As a consequence $M_n = \sup_{x\in(-\pi,\pi)} |S_N(x,[f])|$ is a non-decreasing bounded monotone sequence of real numbers, which concludes our proof. ■

As an exercise the reader should extend theses results to a general interval: $C^1([a,b])_{per}$, i.e: $f\big(x + \overbrace{(b-a)}^{T}\big) = f(x)$.

As a simple consequence of the above exposed result on the uniform convergence of the Fourier Series. It is possible to give an advanced calculus proof of the famous Weierstrass theorem as a simple "one line remark", however under the restrictive condition that the function $f(x)$ (to be approximated uniformly by polinomials) should be in $C^1([-1,1])$.

We take a given function $f(x) \in C^1([-1,1])$ and consider the following (even extension) function on S^1: $g(\theta) = f(\cos\theta)$. We easily verify that $g'(\theta) = \big(-\sqrt{1-x^2}\big)f(x)$. As a consequence

$$
\begin{aligned}
f(x) = g(\theta) &\overset{\text{unif. conv.}}{=} \lim_{N\to\infty} \left\{ \frac{a_0[g]}{2} + \sum_{n=1}^{N} a_n[g] \cos(n\theta) \right\} \\
&\overset{\text{unif. conv.}}{=} \lim_{N\to\infty} \left\{ \frac{a_0[g]}{2} + \sum_{n=1}^{N} a_n[g] \times \overbrace{\cos(n \arccos x)}^{T_n(x) \equiv \text{ Tchebichef Polinomials}} \right\} \\
&= \lim_{N\to\infty} (B_0 + B_1 x + B_2 x^2 + \cdots + B_n x^n)
\end{aligned} \tag{1.72}
$$

Here the coefficients $a_n[g]$ are easily written as integrals with integrands depending explicitly of the given function $f(x)$

$$
a_n[g] = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)\ \cos(n \arccos x) dx}{\sqrt{1-x^2}} \, . \tag{1.73}
$$

Note that the Tchebichef Polinomials are given explicitly by the Newton Binomial formula

$$
T_n(x) = \sum_{\substack{\ell=0 \\ (\ell = \text{ even})}}^{n} \left((-1) \binom{n}{\ell} (x)^{n-\ell} \, (1-x^2)^{\ell/2} \right) . \tag{1.74}
$$

The general case of the Weierstrass theorem is easily obtained from the following result due to Féjer ([3]).

Féjer Theorem - Theorem 7 – The "Césaro weighted sum" of the Fourier Partial sum always converges uniformly for $f(x)$, if $f(x) \in C([\pi, \pi])$. Namely

$$
\sigma^N(x, [f]) = \frac{a_0[f]}{2} + \sum_{n=0}^{N-1} \left(\frac{N-n}{N} \right) (a_n[f] \cos(nx) + b_n[f] \sin(nx)) \tag{1.75}
$$

Finally we left to our diligent readers to prove that the Bessel inequality is an equality for piecewise functions in $[-\pi, \pi]$. In other words:

$$
\lim_{N\to\infty} \left\{ \int_{-\pi}^{\pi} \big(f(x) - S_N(x, [f])\big)^2 \, dx \right\} = 0. \tag{1.76}
$$

As a last result, let us give a detailed proof of the pointwise convergence of the Fourier Series for piecewise continous by differentiable functions as an extended solved exercise.

Firstly, we write (exercise) the following (Dirichlet) integral formula for the N-the Fourier sum

$$S_N(x,[f]) = \frac{1}{2\pi}\left\{\int_0^{\pi} dt \left(\frac{\sin[(N+\frac{1}{2})t]}{\sin(t/2)}\right)[f(t+x)+f(x-t)]\right\}. \tag{1.77}$$

After this step one has the following useful lemma.

The Riemann-Lebesgue Lemma – Let $f(x)$ be a continuously differentiable function on $[-\pi,\pi]$. Then we have the validity of the limits below

$$\lim_{N\to\infty}\left\{\int_{-\pi}^{\pi} f(x)\,\mathrm{sen}(Nx)dx\right\} = \lim_{N\to\infty}\left\{\int_{-\pi}^{\pi} f(x)\cos(Nx)dx\right\} = 0. \tag{1.78}$$

Its proof comes from the following set of estimates for instance

a) $$\int_{-\pi}^{\pi} f(x)\,\mathrm{sen}(Nx) = \frac{1}{2}\left\{\int_{-\pi}^{\pi}\Big[\overbrace{f(x)-f(x+\pi/N)}^{g_N(x)}\Big]\sin(Nx)dx\right\} \tag{1.79}$$

b) $$\lim_{N\to\infty} g_N(x) \overset{\text{unif. conv.}}{=} 0, \text{ since: } |g_N(x)| \le 2\sup_{x\in[-\pi,\pi]}|f(x)| \text{ and}$$

$$\sup_{x\in[-\pi,\pi]}|g_N(x)-g_M(x)| \le \sup_{x\in[-\pi,\pi]}\left|\overbrace{f\left(x+\frac{\pi}{N}\right)}^{(|a-b|-|a-c|)\le|b-c|} - f\left(x+\frac{\pi}{M}\right)\right|$$
$$\le \pi\left(\frac{1}{N}-\frac{1}{M}\right)\sup_{x\in[-\pi,\pi]}|f'(x)| \to 0 \tag{1.80}$$

Now it is really easy to see that

$$\left[S_n(x,[f]) - \left(\frac{f(x^+)+f(x^{-1})}{2})\right)\right] =$$
$$= \frac{1}{\pi}\int_{0^+}^{\pi} dt\,\frac{\sin\left(N+\frac{1}{2})t\right)}{\sin\left(\frac{1}{2}\,t\right)}\left(\frac{t}{2}\right)\left\{\frac{[f(x+t)-f(x^+)]+[f(x-t)-f(x)]}{t}\right\} \tag{1.81}$$

On basis of eq.(1.81) and the Riemann-Lebesgue lemma we have that eq.(1.81) goes to zero if the upper and lower derivatives at a given point $x\in[-\pi,\pi]$ as defined below

[note that one can easily remove the differentiability condition on the Riemann-Lebesgue Lemma - prove it!]

$$\lim_{\substack{t\to 0\\ t>0}} \left(\frac{f(x+t)-f(x^+)}{t} \right) \stackrel{\text{def}}{\equiv} (D\, f(x))^+ \tag{1.82}$$

$$\lim_{\substack{t\to 0\\ t>0}} \left(\frac{f(x-t)-f(x^-)}{t} \right) \stackrel{\text{def}}{\equiv} (D\, f(x))^- \tag{1.83}$$

are such that the sum is finite (note that the individual terms, in principle, do not need to be finite)

$$\big(D\, f(x)\big)^+ + \big(D\, f(x)\big)^- < +\infty.$$

Exercise – Analyze the existence and convergence of the Fourier Series of the function

$$f(x) = \begin{cases} ax\ \text{sen}^2\left(\frac{1}{x}\right) + bx\cos^2\left(\frac{1}{x}\right) - \pi < x < 0 \\ 0, \quad 0 < x < \pi \end{cases}$$

We suppose now that there is a non-zero $f(x) \in C([-\pi,\pi])$ such its Fourier coefficients are all vanishing, i.e.

$$\int_{-\pi}^{\pi} f(x) \begin{Bmatrix} \sin(Nx) \\ \cos(Nx) \end{Bmatrix} dx = 0 \tag{1.83}$$

We are going to show that $f(x) \equiv 0$ or $[-\pi,\pi]$, and thus producing a proof the uniqueness of the Fourier Coefficients of a given continuous function.

In order to show this result let us suppose that there is $x_0 \in [-\pi,\pi]$ with $f(x_0) > 0$ and $\delta > 0$ such that $f(x) > \bar{\varepsilon} > 0$. Here $\bar{\varepsilon} = \min\limits_{x_0-\delta<x<x_0+\delta} f(x)$ and $f(x) > 0$ in $I(x_0) = [x_0-\delta,\, x_0+\delta]$. All these assertions are true since $f(x) \in C([-\pi,\pi])$. Let us consider the Trigonometric Polinomial $P_n(x) = (y(x))^n$, where $y(x) = (1+\cos(x-x_0)-\cos\delta)$. On the interval $I(x_0)([x-x_0| \leq \delta)$ obviously $P_n(x) \geq 1$. For any closed interval $J \subset I(x_0)$ we have that $y(x) > 1$. Note either that $|y(x)| < 1$ for $x \in [-\pi,\pi]\backslash I(x_0)$. Now we can see that

$$0 \stackrel{\text{(hypothesis)}}{=} \int_{-\pi}^{\pi} f(x)\, P_n(x)\, dx = \left(\int_{I(x_0)} f(x)\, P_n(x)\, dx \right) + \left(\int_{[-\pi,\pi]\backslash I(x_0)} f(x)\, P_n(x)\, dx \right). \tag{1.84}$$

By the other side we have the estimates written below, which are in contradiction with the above written equation (1.84)

$$\left|\int_{[-\pi,\pi]\backslash I(x_0)} f(x)\, P_n(x)\, dx\right| \leq 2\pi \left(\sup_{x\in[-\pi,\pi]} |f(x)|\right)\left(\sup_{[-\pi,\pi]\backslash I(x_0)} |y(x)|^n\right) \stackrel{n\to\infty}{\longrightarrow} 0 \quad (1.85\text{-a})$$

and

$$\left|\int_{I(x_0)} f(x)\, P_n(x)\, dx\right| > \bar{\varepsilon}\int_J (y(x))^n\, dx \geq \bar{\varepsilon}(\text{lengh}(J)) \times (\min_{x\in J} y(x))^n \stackrel{n\to\infty}{\longrightarrow} +\infty \quad (1.85\text{-b})$$

Solved Exercises – Section 1.2

Exercise – The use of Logical Quantifiers in Mathematical Proofs.

It is very important to understand the basics of the logic quantifiers. Let us introduce some elementary concepts. A statement frame $P(x)$ is a mathematical sentence which is true or false as it stands and it is written in the form of logical quantifiers. For instance the phrases "for each x" (or "for all x") is customarily represented as $\forall\, x$. The existencial quantification is represented by the symbol $\exists\, x$ and mean "there exists an x". The negative logical quantifiers (not) is usually represent by the symbol $\sim \exists\, x$, $\sim \forall\, x$, etc and is fully applied to a setence and shouldn't be considered a connective at all. For instance the statement frame that a real function $f(x)$ is continuous at the point x_0 is written as $(\forall\, \varepsilon)\ (\exists\, \delta)\ (\forall\, x)\ \overbrace{(\text{if } |x - x_0| < \delta \text{ then } (\Rightarrow)\, |f(x) - f(x)| < \varepsilon)}^{P(x,f)}\cdot$

Exercise – Show in details that the negation of a statement begining with a string of logical quantifiers, one simply change each quantifier to the opposite kind and move the negation sign to the end of the string thus

$$\sim \left[\forall(x)\exists(y)(\forall\, \delta)(P(x,y,z))\right] \Leftrightarrow (\exists(x))(\forall\, y)(\exists\, z)(\sim P(x,y,z)).$$

The reader should analyze carefully the proof of the following fundamental theorem on convergence uniform and translating the result as frame statements with the use of logical quantifiers.

Theorem 8. Let $f_n(x) \in C([a,b],R)$, $n = 1, 2, \ldots$ and assume that the sequence $\{f_n(x)\}$ converges uniformly to a given function $f(x)$. Then $f(x)$ is a continuous function on (a,b).

Proof. Let $x_0 \in [a,b]$. We must show that $\forall\, \varepsilon > 0$, $\exists\, \delta = \delta(\varepsilon, x_0)$ such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$. This result is a simple consequence of the following estimate

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned} \tag{1.85-c}$$

Since for a given $\varepsilon > 0$, one chooses $N_0(\varepsilon)$ such $|f_n(x) - f(x)| < \varepsilon/3$ for $n \geq N_0(\varepsilon)$ due to the hypothesis of uniform convergence of the sequence $\{f_n(x)\}$ and $\delta(x_0, \varepsilon)$ is such that $|f_{N_0}(x) - f_{N_0}(x_0)| < \varepsilon/3$ for $|x - x_0| < \delta$. ■

Another useful result is the following Theorem:

Theorem 9. If $f_n(x) \in C([a,b],R)$ converges uniformly for $f(x)$, then for $a \leq \alpha \leq \beta \leq b$, we have the limit exchange

$$\lim_{n\to\infty} \int_\alpha^\beta f_n(x)\, dx = \int_\alpha^\beta \big(\lim_{n\to\infty} f_n(\xi)\big)\, d\xi = \int_\alpha^\beta f(\xi)\, d\xi. \tag{1.85-d}$$

Proof. Let $\varepsilon > 0$ be given such $N \geq N_0(\varepsilon)$ we have that $|f_n(x) - f(x)| < \varepsilon\big/(b-a)$ for $x \in [a,b]$ uniformly. We see now the validity of the estimate for $\forall\, n \geq N_0(\varepsilon)$

$$\begin{aligned} \left| \int_\alpha^\beta f_n(\xi)\, d\xi - \int_\alpha^\beta f(\xi)\, d\xi \right| &\leq \int_\alpha^\beta d\xi\, |f_n(\xi) - f(\xi)| \\ &\leq \frac{\varepsilon}{(b-a)}\, |\beta - \alpha| \leq \varepsilon. \ ■ \end{aligned} \tag{1.85-e}$$

As a exercise the reader should show the Cauchy criterion for the uniform convergence of functions. A sequence of real functions (not necessarily continuous!) converges uniformly on a interval $[a,b]$ if and only if for every $\varepsilon > 0$, there is an integer $N_0(\varepsilon)$ such that $|f_n(x) - f_m(x)| < \varepsilon$ for $\forall\, x \in [a,b]$ whenever $n \geq N_0(\varepsilon)$ and $m \geq N_0(\varepsilon)$.

Exercise – A choice function for a set A is a function f which associates to each nonempty subset E of A an element e of $E: f(E) \in E$. In informal words, f chooses an element out of each nonempty subset of X. Think on the assertive (the Axiom of Choice): "For every set there is a choice function".

Solved Exercises in Elementary Calculus Manipulations – Section 1.3 – A formal solution of Laplace Equation.

Exercise – The Laplace Equation is Spherical Coordinates ([3]).

In Application to Eletrostatic Problem in R^3 and Quantum Mechanics, it is useful to find the structure of solutions of the Laplace Equation in a spherical coordinates (r, θ, φ) in a separated form. Power series are instrumental tools in such applied studies. Let us thus search a solution for the Laplace Equation in spherical coordinates

$$\frac{1}{r}\frac{\partial^2}{\partial^2 r}\left(r\,U(r,\theta,\varphi)\right) + \frac{1}{r^2 \operatorname{sen}\theta}\frac{\partial}{\partial\theta}\left(\operatorname{sen}\theta\,\frac{\partial U(r,\theta,\varphi)}{\partial\theta}\right) + \frac{1}{r^2\operatorname{sen}^2\theta}\frac{\partial^2 U(r,\theta,\varphi)}{\partial^2\phi} = 0 \tag{1.86}$$

in the form

$$U(r,\theta,\varphi) = \frac{w(r)}{r}\,P(\theta)Q(\phi) \tag{1.87}$$

After substituting the "ansatz" eq.(1.87) into the Laplace Equation eq.(1.86) and multiplying the obtained relation by the over all factor $r^2\,\sin^2\,\theta/w(r)P(\theta)Q(\phi)$, one obtains the set of decoupled ordinary second order differential equations

$$\begin{cases} \dfrac{1}{Q(\phi)}\dfrac{d^2Q(\phi)}{d\phi^2} = -m^2 & (1.88-a) \\[2em] \dfrac{1}{\operatorname{sen}\theta}\dfrac{d}{d\theta}\left(\operatorname{sen}\theta\,\dfrac{dP}{d\theta}\right) + \left[\ell(\ell+1) - \dfrac{m^2}{\operatorname{sen}^2\theta}\right]P(\theta) = 0 & (1.88-b) \\[2em] \dfrac{d^2w(r)}{d^2r} - \dfrac{\ell(\ell+1)}{r^2}\,w(r) = 0 & (1.88-c) \end{cases}$$

Here m and ℓ are constants. By demanding a well-defined behavior on the ϕ-variable, one can write the possible solutions of eq.(1.88-a) in the form for m an integer

$$Q(\phi) = Q_0\,e^{\pm\, im\phi}. \tag{1.89}$$

Equation (1.88-c) has the immediate solution

$$w(r) = A\, r^{\ell+1} + B\, r^{-\ell}. \tag{1.90}$$

The equation eq.(1.88-b) now can be solved by Power Series (Exercise) after considering the variable change $x = \cos\,\theta(-1 \leq x \leq 1)$.

Equation (1.88-b) takes the form $(p(x) = P(\cos\,\theta))$

$$\frac{d}{dx}\left[(1-x^2)\,\frac{dp}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{(1-x^2)}\right] p(x) = 0. \tag{1.91}$$

For the special case of $m = 0$, one obtains a solution in the Frobenius Form, the other involves a logarithmic term

$$p(x) = x^\gamma \left[\sum_{n=0}^{\infty} c_n\, x^n\right] \tag{1.92}$$

with

$$C_{n+2} = \left[\frac{(\gamma+n)(\gamma+n-1) - \ell(\ell+1)}{(\gamma+n-1)(\gamma+n+2)}\right] C_n \tag{1.93}$$

and $C_0 \neq 0$ $(\Leftrightarrow\ \gamma(\gamma-1) = 0)$.

Note that the power series eq.(1.92) is a polinomial (the Legendre Polinomial) if the parameter ℓ is a positive integer, with the result

$$p_\ell(x) = \frac{1}{2^\ell \ell!}\,\frac{d^\ell}{dx^\ell}\big((x^2-1)^\ell\big). \tag{1.94-a}$$

In the general case of $m \neq 0$, the only possible values of m allowed in order for the solution be defined at the points $x = \pm 1$ are (one of the basic result of Quantum Chemestry)

$$m = -\ell,\ -(\ell-1), \ldots, 0, \ldots, \ell-1, \ell \tag{1.94-b}$$

The general solution takes the form

$$p_\ell^m(x) = (-1)^m\,(1-x^2)^{n/2}\,\frac{d^m}{dx^n}\,(p_\ell(x)) \tag{1.94-c}$$

where

$$p_\ell^{-m}(x) = (-1)^m\,\frac{(\ell-m)!}{(\ell+m)!}\,p_\ell^m(x). \tag{1.94-d}$$

By the general principle of linear superposition, one should expects solutions (still formal) of the form

$$U(r,\theta,\varphi) = \sum_{\ell=0}^{\infty} \left\{ \sum_{m=-\ell}^{\ell} \left[A_{\ell m}\, r^{\ell} + B_{\ell m}\, r^{-(\ell+1)}\right] Y_{\ell m}(\theta,\varphi) \right\} \tag{1.94-e}$$

where we have introduced the spherical harmonics function

$$Y_{\ell m}(\theta,\varphi) = \left(\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!} \right)^{1/2} \; p_{\ell}^{m}(\cos\,\theta) e^{i,\phi} \tag{1.94-f}$$

Exercise – Let us consider the problem of solving the following analytical differential ordinary equation in a given rectangle $R_{x_0,y_0} = \big\{(x,y) \mid |x-x_0| < \varepsilon;\ |y-y_0| < \delta\big\}$

$$\begin{aligned} &\frac{dy}{dx} = f(x,y(x)) \\ &y(x_0) = y_0 \end{aligned} \tag{1.95}$$

where $f(x,y)$ is an analytical function in $R_{(x_0,y_0)}$. If one calculates sucessively the higher-order derivatives from the differential equation eq.(1.95) at the point $x = x_0$, one obtains the expression of its solution in terms of a power series around $x = x_0$ (it can be showed that always exists a non empty interval of convergence for it!). Namely

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)(x-x_0)^n}{n!} \tag{1.96}$$

with

$$y'(x_0) = f(x_0,y_0).\ y''(x_0) = \frac{\partial f}{\partial x}(x_0,y_0) + \frac{\partial f}{\partial y}(x_0,y_0)(f(x_0,y_0)),\ \text{etc} \tag{1.97}$$

It is worth remark that this same (somewhat formal) procedure can be directly applied for system of ordinary diferential equations line

$$\begin{aligned} &\frac{dy_1(x)}{dx} = F_1(x,y_1,\ldots,y_N) \\ &\vdots \\ &\frac{dy_N(x)}{dx} = F_N(x,y_1,\ldots,y_N) \end{aligned} \tag{1.98}$$

with prescribed initial condition

$$y_1(x_0) = y_0^{(1)}, \ldots, y_N(x_0) = y_0^{(N)}.$$

As an exercise, a toy model for a simple read/write head of a hard disk in computers is described by the differential equation

$$J\,\ddot{\theta} + c(\theta)\dot{\theta} + \bar{k}\,\theta = k_T\, i(t, [\theta])$$

where J represents the inertia of the head assembly, C denotes the viscous damping of the bearings, $\bar{k}$ the return spring constant, k_T is the motor torque constant, $\ddot{\theta}$, $\dot{\theta}$, θ are the angular acceleration, angular velocity, and position of the head, respectively: $i(t, [\theta])$ is the current input which may be a given non-linear function the out puts $\{\theta(t), \dot{\theta}(t), \ddot{\theta}(t)\}$. Show that the state-space model with $x_1(t) \equiv \theta(t)$, $x_2(t) \equiv \dot{\theta}(t)$, and $U = i(x, x_1, x_2)$ satisfies the non-linear system

$$\begin{aligned}
&\frac{d}{dt}\,x_1(t) = x_2(t) \\
&\vdots \\
&\frac{d}{dt}\,x_2(t) = -\frac{k}{J}\,x_1(t) - \frac{c}{J}\,x_2(t) + \frac{k_T}{J}\,i(x, x_1(t), x_2(t))
\end{aligned} \tag{1.99}$$

Give the following values $J = 0.01$, $c = 0.004$, $k = (0$ and $k_T = 0.05$, the reader should determine some terms of the associated solution power series for $i(x, x_1(t), x_2(t)) = e^{-(x_1(t))^2}\,\sin(x_2(t))$ around $t = 0$, and $x_1(0) = 0 = x_2(0)$.

Exercise – The Korteweg-de Vries

An important equation of hydrodynamics, is the called Korteweg-de Vries equation. Let us consider a generalization of it as given below with two "viscosity-like" coefficients V_1 and V_2 both real positive numbers. Note the time dependence of the third-order x-derivative term

$$\frac{\partial}{\partial t}\,U(r,t) = V_1\,\frac{\partial U(x,t)}{\partial x^2} + V_2\,t^{1/2}\,\frac{\partial^3 U(x,t)}{\partial x^3}. \tag{1.100}$$

A self-similar solution for eq.(1.100) can be easily determined from elementary calculus manipulations. Let us thus consider the self-similar variable ($t > 0$) with $a \in R$ denoting a fixed parameter

$$z = \frac{x}{2a\sqrt{t}} \tag{1.101-a}$$

$$w(z) = U(x,t)$$

$$\frac{\partial}{\partial t} = -\frac{1}{2}\left(t^{-1}\right)\left(z\,\frac{\partial}{\partial z}\right) \tag{1.101-b}$$

$$\frac{\partial^k}{\partial^k x} = (2a\sqrt{t})^{-k}\,\frac{\partial^k}{\partial^k z} \tag{1.101-c}$$

Our equation eq.(1.100) takes the form of an ordinary differential equation for $\dfrac{dw}{dz} = \psi(z)$

$$\overbrace{\frac{V_2}{8a^3}}^{A}\,\frac{d^2}{d^2 z}\,\psi(z) + \overbrace{\frac{V_1}{4a^2}}^{B}\,\frac{d}{dz}\,\psi(z) + \frac{1}{2}\,z\psi(z) = 0 \tag{1.103}$$

In the particular case of $V_1 \equiv 0$, one has the solution of eq.(1.103) in terms of a power series $\big(a = (V_2/4)^{1/3}\big)$ – the so called Airy functions from Difraction Theory

$$\psi(z) = \psi(0)\,Ai_{(1)}\,(-z) + \psi'(0)\,Ai_{(2)}\,(-z)$$

$$w(z) = \Big(\psi(0)\int^3 dz'\,Ai_{(1)}(-z')\Big) + \Big(\psi'(0)\int^3 dz'\,Ai_{(2)}(-z')\Big) \tag{1.104}$$

with the linearly independent solutions

$$Ai_{(1)}(z) = 1 + \sum_{n=1}^{\infty}\frac{1.4.\ldots.(3n-2)}{(3n)!}\,z^{3n} \tag{1.105}$$

$$Ai_{(2)}(z) = z + \sum_{n=1}^{\infty}\frac{2.5.\ldots.(3n-1)}{(3n+1)!}\,z^{3n+1} \tag{1.106}$$

In the general case of $V_1 \neq 0$, after the change

$$\psi(z) = \exp\left(-\frac{B}{2A}\right)\gamma(z), \tag{1.107}$$

we have the new Generalized Airy equation

$$\frac{d^2\gamma(z)}{dz^2} + \left(\frac{z}{2A} - \frac{B^2}{4A^2}\right)\gamma(z) = 0 \tag{1.108}$$

whose power series solutions are left to our readers as an exercise. At this point we call the reader attention that a manifold of solutions of eq.(1.100) can be obtained from its n-order x-derivative.

Solved Exercise – Section 1.3

Exercise – The Proof of the Féjer Theorem.

We now present a proof of the useful Fejer theorem (Theorem 7) in details.

Firstly we define the following periodic real $C^\infty(R)$ functions, the called Dirichlet and Fejer's kernels respectively

a) $D_n(x) = \sum_{k=-n}^{n} e^{ikx} = \dfrac{\sin\left[\left(n+\frac{1}{2}\right)x\right]}{\sin\left(\frac{x}{2}\right)}$ and $k_n(x) = \dfrac{1}{n+1}\left(\sum_{m=0}^{n} D_m(x)\right)$ (1.108-a)

We have the validity of the relationship

b) $K_n(x) = \sum_{j=-n}^{n}\left(1 - \dfrac{|j|}{n+1}\right)e^{ijx} \geq 0$ (1.108-b)

c) $K_n(-x) = K_n(x) = \dfrac{1}{(n+1)}\left\{\dfrac{\sin\left(\frac{n}{2}\,x\right)}{\sin\left(\frac{x}{2}\right)}\right\}^2$ (1.108-c)

d) $\dfrac{1}{2n}\int_{-\pi}^{\pi} K_n(x)\,dx = 1$ (1.108-d)

e) $K_n(x) \leq \dfrac{2}{(n+1)(1-\cos\delta)}\,, \quad \delta \leq |x| < \pi$ (1.108-e)

The proof of the above statements can be made in the following way. We have the explicitly result

$$K_n(x) = \frac{1}{n+1}\left(\frac{1-\cos((n+1)x)}{1-\cos x}\right) \tag{1.109}$$

This result is obtained easily from the identity below

$$\left[(n+1)\left\{\frac{1}{n+1}\sum_{m=0}^{n} D_n(x)\right\}(e^{ix}-1)\right](e^{-ix}-1)$$
$$= (e^{-ix}-1)\left[\sum_{m=0}^{n}\left(e^{i(m+1)x}-e^{-imx}\right)\right]$$
$$= 2-e^{i(n+1)x}-e^{-i(n+1)x} \tag{1.110}$$

which leads to the searched result

$$K_n(x) = \left[\frac{\left(2-e^{i(n+1)x}-e^{-i(n+1)x}\right)}{(n+1)(e^{ix}-1)(e^{-ix}-1)}\right]$$
$$= \frac{1}{n+1}\left(\frac{1-\cos(n+1)x}{1-\cos x}\right). \tag{1.111}$$

The Césaro weighted sum (see eq.(1.75)) has an analogous integral representation (exercise)

$$\sigma_N(x,[f]) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dt\, K_n(t)\, f(x-f) \tag{1.112}$$

for any δ such that $0<\delta<\pi$

$$\sigma_N(x,[f]) - \frac{\overbrace{f(x^+)+f(x^-)}^{[f(x)]^+}}{2} = \frac{1}{2\pi}\left[\int_{-\pi}^{-\delta} dt\, K_n(t)\big(f(x-t)-[f(x)]^+\big)\right] +$$
$$+\int_{\delta}^{\pi} dt\, K_n(t)\big(f(x-t)-[f(x)]^+\big)$$
$$+\int_{-\delta}^{0} dt\, K_n(t)\big(f(x-t)-[f(x)]^+\big) + \int_{0}^{\delta} dt\, K_n(t)\big(f(x-t)-[f(x)]^+\big)$$
$$= \frac{1}{\pi}\left[\int_0^{\delta} dt\, K_n(t)\left(\frac{f(x-t)+f(x+t)}{2}-[f(x)]^+\right)\right]$$
$$+\frac{1}{\pi}\left[\int_{\delta}^{\pi} dt\, K_n(t)\left(\frac{f(x-t)+f(x+t)}{2}-[f(x)]^+\right)\right] \tag{1.113}$$

Now if our function $f(x)$ is such that for any $\varepsilon>0$, there exists $\delta>0$, such that uniformly in $x\in(-\pi,\pi)$, we have that for $|z|<\delta$ implies that $\left|\frac{(f(x+z)+f(x-z))}{2}-[f]^+(x)\right|<\varepsilon$ (if $f(x)$ is continuous this is always true since $f(x)$ is uniformly continuous on the closed interval $[-\pi,\pi]$ we can use eq.(1.108-e) to see that for a given $\varepsilon>0$: $\sup\limits_{\delta<z<\pi} K_n(z)<\varepsilon$ for $N\geq N_0(\varepsilon)$, $\big[\sin\big(\frac{z}{2}\big)\geq\frac{2\delta}{\pi}$ for $\delta<z<\pi\big]$ which by its turn leads to our "uniform convergence" estimate for $N\geq N_0(x)$

$$\begin{aligned}
\sup_{-\pi\leq x\leq\pi}|\sigma_N[x,[f]]-[f(x)]^+| &\leq \varepsilon\left(\overbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi}K_n(t)\,dt}^{1}\right) \\
&+\varepsilon\int_{\delta}^{\pi}dt\left|\frac{f(x+t)+f(x-t)}{2}-[f]^+(x)\right| \\
&\leq \varepsilon+\varepsilon\left(\int_{-\pi}^{\pi}dt\,f(t)\right)+\varepsilon\big(\pi\sup_{x\in[-\pi,\pi]}|\,[f]^+(x)|\big) \\
&=\varepsilon\,\overline{M}\overset{n\to\infty}{\longrightarrow}0
\end{aligned} \tag{1.114}$$

if all "jumps averages" at the first species discontinuities points of our function are bounded. Again note that if $f(x)\in C([-\pi,\pi])$, we have the truly convergence uniform of the Césaro trigonometrical weighted Fourier Series for $f(x)$. ■

At this point we remark that this result solves the very important inverse problem in the theory of Fourier Series: if one knows already all the Fourier coefficients $\{a_n[f],b_n[f]\}$, just write the Césaro weighted Fourier Series to recover exactly in $C([-\pi,\pi])$ the function $f(x)$ possessing such set of Fourier coefficients.

Let us exemplify such important remark in an applied setting in a more advanced mathematical context (see Chapter III for extensive details).

We search for a periodic solution of the following integral equation in the space of the continuous functions with an absolutely convergent Fourier Series. (In the complex form $\sum\limits_{n=-\infty}^{+\infty}|C_n[f]|<+\infty$, the so called space $A([0,T])$)

$$h(x)=\frac{1}{T}\int_0^T G(x,y)\,f(y)\,dy \tag{1.115}$$

Let us suppose that $G(x,y)$ is a double periodic function with period T and possessing the Complex Fourier Series in both variables

$$G(x,y) = \sum_{n,m=-\infty}^{+\infty} g_{nm}\, e^{\frac{2\pi i m}{T}x}\, e^{\frac{2\pi i n}{T}y}. \tag{1.116}$$

Additionally, we suppose that the following condition holds true

$$\sup_{m\in Z}\left\{\sum_{n=1}^{\infty}|g_{nm}|\right\} < 1. \tag{1.117}$$

After substituting eq.(1.116) and eq.(1.115), one obtains the infinite order linear system of equations involving the Complex Fourier Coefficients of the unknown function $f(x)$ (input) and given function $h(x)$ (output)

$$h_n = \sum_{m=-\infty}^{+\infty} g^{nm}\, f_m \tag{1.118-a}$$

$$f_n = h_n + \left[\sum_{n=-\infty}^{+\infty}(\delta^{mn} - g^{nm})f_m\right]. \tag{1.118-b}$$

On basis of eq.(1.117) one can solve uniquely by the iterative Picard method eq.(1.118-b) in the space $A([0,T])$ (see Chapter III for technical details) the space of those functions with absolutely convergent Fourier Series and this, determining explicitly the Fourier Coefficients of the function $f(x)$. Namely $\{f_n\}_{-\infty<n<+\infty}$. Now the usefulness of Féjer's result is apparent since one can recover numerically (in the sense of the uniform convergence) the whole function $f(x)$

$$f(x) = \lim_{N\to\infty}\sigma_N(x,[f]) \tag{1.119}$$

As a last (somewhat advanced) comments we wish to point out to our readers the following deep results on the still open subject of convergence of Fourier Series ([3]).

1 – These exists a continuous function whose Fourier Series diverges at a point.

2 – (The Lusin conjecture - L. Carleson) – The Fourier Series of a function in $L^2([-\pi,\pi])$ (roughly $f(x) \in L^2([-\pi,\pi])$ means that $\int_{-\pi}^{\pi}|f(x)|^2\,dx < +\infty$ (see Chapter 3 for the

Lebesgue Integral)), then the Fourier series converge pointwise with exception of a set of Lebesgue zero measure.

Exercise – The Weierstrass's non differentiable function is defined by the trigonometrical series

$$f(x) = \sum_{n=0}^{\infty} b^n \cos(a^n \pi x) \tag{1.120}$$

with $0 < b < 1$ and $ab > 1 + \frac{3}{2}\pi$ with a being an odd positive integer. Let us show that for any $x \in R$ (note that the function $f(x)$ is a continuous function since it is the limit of a uniformly convergente sequence of continuous functions by the Weierstrass criterion of uniform convergence since $\left|\sum_{n=0}^{N} b^n \cos(a^n \pi x)\right| \le \sum_{n=0}^{N} b^n = \frac{1 - b^{N+1}}{1-b}$); the function is non-differentiable.

The Newton differential coefficient can be written as of as

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \overbrace{\sum_{n=0}^{m-1} b^n \left\{ \frac{\cos(a^n\pi(x+h)) - \cos(a^n \pi x)}{h} \right\}}^{S_m} \\
&= \overbrace{\sum_{n=m}^{\infty} b^n \left\{ \frac{\cos(a^n\pi(x+h)) - \cos(a^n \pi x)}{h} \right\}}^{R_m}
\end{aligned} \tag{1.121}$$

Let us now apply the mean value theorem as below written (with $0 < \theta < 1$):

$$|\cos(a^n\pi(x+h)) - \cos(a^n\pi x)| = |a^n \pi h \operatorname{sen}\{a^n\pi(x+\theta h)\}| \le a^n \pi |h|. \tag{1.122}$$

Clearly

$$|S_m| \le \sum_{n=0}^{m-1} \pi\, a^n b^n = \pi \left(\frac{(ab)^m - 1}{ab - a} \right) < \frac{\pi\, a^m b^m}{ab - 1} \cdot \tag{1.123}$$

By the other side we can always write the term $a^m x = \alpha_m + \xi_m$, with α_m an integer and $-\frac{1}{2} \le \xi_m < \frac{1}{2} \cdot$ Let $h = (1 - \xi_m)/a^m$. We can see that $0 < h \le (1 - \xi_m)/a^m \le$

$\dfrac{1-\left(-\frac{1}{2}\right)}{a^m} = \dfrac{3}{2a^m}\cdot$ We have also the result

$$\begin{aligned}
&\cos\{a^m\pi(x+h)\} = \cos\{a^{n-m}\pi a^m(x+h)\}\\
&= \cos\{a^{n-m}\pi\big[(\alpha_m+\xi_m)+(1-\xi_m)\big]\} = \cos\{a^{n-m}\pi(\alpha_m+1)\}\\
&= (-1)^{a^{m-n}(\alpha_m+1)} = (-1)^{2p(\alpha_m+1)}\,(-1)^{\alpha_m+1} = (-1)^{\alpha_m+1}
\end{aligned} \tag{1.124}$$

since a^{n-m} is on odd positive integer ($a^{n-m}=2p+1$).

It follows that

$$\begin{aligned}
\cos(a^n\pi x) &= \cos(a^{n-m}\,\pi(\alpha_m+\xi_m)\\
&= \cos(a^{n-m}\,\pi\alpha_m)\cos(a^{n-m}\,\pi\xi_m) - 0\\
&= (-1)^{\alpha_m}\,\cos(a^{n-m}\,\pi\xi_m).
\end{aligned} \tag{1.125}$$

Hence

$$R_m = \frac{(-1)^{\alpha_m+1}}{h}\sum_{n=m}^{\infty}\big(b^n\{1+\cos(a^{n-m}\,\pi\xi_m)\}\big). \tag{1.126}$$

Since all the terms inside the series are positive, we just consider the first term to have the lower bound $\left(0<h<\dfrac{3}{2}\,a^{-m}\right)$

$$|R_m| > \frac{b^m}{|h|} > \frac{2}{3}\,a^m b^m. \tag{1.127}$$

Collecting the above results, one obtains the following estimate for the differential coefficient

$$\left|\frac{f(x+h)-f(x)}{h}\right| \geq |R_m| - |S_m| > \left(\frac{2}{3} - \left(\frac{\pi}{ab-1}\right)\right)a^m b^m. \tag{1.128}$$

By choosing $ab > 1 + \dfrac{3}{2}\,\pi$, and by considering $h\to 0$ and $m\to\infty$, we obtain that the above written differential coefficient takes arbitrarily large values and implying thus the non existence of the derivative of the Weierstrass function eq.(1.120) at the given point x.

Exercise – The solution of a RLC circuit under external periodic source.

One basic point in modern theory of electrical circuits is the problem of determining the functional form of an electrical current $I(t)$ as a function of the time circulating around a simple RLC circuit. The governing differential equation of such electrical flow current is given by

$$R\,I(t) + L\,\frac{dI(t)}{dt} + \frac{1}{C}\int_0^t I(\zeta)d\zeta = E(t) \tag{1.129}$$

where R, L, C are circuits parameters-real constants, the time range is the whole real line and $E(t)$ is the source voltage given by a continuously differentiable function (without any kind of discontinuities!) and periodic of period $T = 2\pi$.

Let us produce a (not unique) solution for eq.(1.129) with the same periodicity T of the source $\big(E(t+2\pi) = E(t)\big)$. Firstly we consider the usual Fourier expansion

$$E(t) = \frac{E_0}{2} + \sum_{n=1}^{\infty} a_n\big([E]\big)\cos(nt) + b_n\big([E]\big)\sin(nt). \tag{1.130}$$

We consider thus the well-defined approximent $E^{(N)}(t)$ (the truncated Expansion eq.(1.130) until order N) for eq.(1.129)

$$R\,I^{(N)}(t) + L\,\frac{dI^{(N)}(t)}{dt} + \frac{1}{C}\int_0^t I^{(N)}(\zeta)d\zeta = E^{(N)}(t). \tag{1.131}$$

Again, we note that due the hypothesis of a solution of eq.(1.129) with the same periodicity of the source ($T = 2\pi$), we have the obvious relationship

$$R\,I(0) + L\,\frac{dI(0)}{dt} + \int_0^0 d\zeta\, I(\zeta) = E(0) = R\,I(2\pi) + L\,\frac{dI(2\pi)}{dt} + \frac{\pi}{C}\left[\frac{1}{\pi}\int_0^{2\pi} I(\zeta)d\zeta\right] \tag{1.132}$$

which leads us to the constraint

$$\frac{1}{\pi}\int_0^{2\pi} I(\zeta)\,d\zeta = 0 \tag{1.133}$$

By supposing now the structural form

$$I^{(N)}(t) = \frac{a_0}{2} + \sum_{n=1}^{N}\Big(a_n([I])\cos(nt) + b_n\big([I]\big)\,\mathrm{sen}(nt)\Big) \tag{1.134}$$

one obtains the result

$$\frac{dI^{(N)}(t)}{dt} = \sum_{n=1}^{N} \big(- n a_n\big([I]\big)\,\mathrm{sen}(nt) + nB_n\big([I]\big)\cos(nt)\big) \tag{1.135-a}$$

$$\int_0^t I(\zeta)d\zeta = \sum_{n=1}^{N} \left(\frac{a_n([I])}{n}\right)\mathrm{sen}(nt) + b([I])\left(\frac{(1-cos(nt))}{n}\right); \tag{1.136-b}$$

After inserting the above truncated expansions into our master equation eq.(1.129), one obtains the algebraic system and constraints

$$\frac{E_0}{2} = \frac{1}{C}\left(\sum_{n=1}^{\infty}\frac{b_n(x)}{n}\right) \tag{1.137-a}$$

$$R\,a_n\big([I]\big) + nL\,b_n\big([I]\big) - \frac{b_n([I])}{nC} = a_n\big([E]\big) \tag{1.137-b}$$

$$R\,b_n\big([I]\big) + (-nL)\,a_n\big([I]\big) + \frac{a_n([I])}{nC} = b_n\big([E]\big) \tag{1.137-c}$$

producing thus the following solution

$$\begin{bmatrix} a_n([I]) \\ b_n([I]) \end{bmatrix} = \left\{\left(\frac{1}{[R^2+(nL)^2-(nC)^{-2}]}\right)\times\begin{bmatrix} R & -nL+\frac{1}{nC} \\ nL-\frac{1}{nC} & R \end{bmatrix}\right\}\begin{bmatrix} a_n([E]) \\ b_n([E]) \end{bmatrix} \tag{1.137-d}$$

By taking now the limit of $N \to \infty$ an eq(1.134)/eq(1.137-d), one obtains a continuously differentiable solution $I(t)$ solely under the (necesssary) condition that the source should be a continuously differentiable function.

Note that if $E(t)$ has a discontinuity at a given point (time) $T = a$ (with $\lim_{h>0} E(a+h) - \lim_{h>0} E(a-b) = [E(a)]^+$ we have the following relationship among the complex Fourier coefficients between the function $E(t)$ and its derivative $\dot{E}(t)$ (see eq(1.75))

$$a_n[\dot{E}]) = \frac{1}{2\pi}\int_0^{2n} \dot{E}(t)\,e^{int}\,dt = \frac{1}{2\pi}\big(E(2n)-E(0)\big) - \frac{1}{2\pi}[E(a)]^+\,e^{ina} + in\,a_n([E]). \tag{1.137-e}$$

Exercise – An useful result from elementary complex calculus (Residues Theorem) is the following (prove!) for $0 \le x \le 2\pi$ (see next Section 1.5)

$$\sum_{n=-\infty}^{+\infty} \overbrace{\left(\frac{P_\ell(n)}{Q_s(n)}\right)}^{R(n)} e^{inx} = -\left\{\sum_{\{\text{Poles of } R(z)\}} \mathrm{Res}\left[R(z)\,\frac{2\pi i e^{izx}}{(e^{2\pi i z}-1)}\right]\right\} \tag{1.137-f}$$

where $P_\ell(x)$ is a polinomial of degree ℓ and $Q_s(x)$ another polinomial of order $s \geq \ell + 2$. For instance

$$\sum_{n=-\infty}^{+\infty} \left(\frac{1}{n^2 + w^2} \right) e^{inx} = -\left\{ \left[\frac{1}{2iw} \frac{2\pi i}{(e^{-2\pi w} - 1)} e^{-xw} \right] \right.$$

$$\left. + \left[-\frac{1}{2iw} \frac{2\pi i}{(e^{+2\pi w} - 1)} e^{+xw} \right] \right\} = \left(-\frac{\pi}{w} \frac{e^{-xw}}{1 - e^{-2\pi w}} \right) + \left(-\frac{\pi}{w} \frac{e^{xw}}{e^{2\pi w} - 1} \right) \qquad \text{(1.137-g)}$$

Exercise – Let us consider an analytical function $G(z)$ on the domain $\mathrm{Re}(z) \geq 0$ (the right-half plane) such that

a) $\displaystyle\int_0^\infty |G(x)|\, dx$ is convergent

b) $\displaystyle\lim_{y \to \infty} |G(x \pm iy)|\, e^{-2\pi y(u+1)} = 0$ for $0 \leq u \leq 2\pi$

and $y \in R$; $a \leq x \leq b$, for any a and b.

c) $\displaystyle\int_0^\infty |G(x \pm iy)|\, e^{-2\pi y(1+u)}\, dy \to 0$ for $x \to \infty$ and $0 \leq u \leq 2\pi$.

Then we have the formula (prove it!)

$$\sum_{n=0}^{\infty} G(n)\, e^{2\pi inx} = \frac{1}{2} G(0) + \left(\int_0^\infty G(\xi)\, e^{2\pi i\xi}\, d\xi \right)$$

$$+ i \left\{ \int_0^\infty \frac{G(iy)e^{-2\pi x} - G(-iy)e^{+2\pi x}}{(e^{2\pi y} - 1)}\, dy \right\} \qquad (1.138)$$

1.5 – Basics of Calculus for Complex Variable Functions ([4]-[5]).

The Arithmetic of Complex Numbers $\mathbb{C}$ is better defined as a field of two-uples satisfying the following algebric operations

a) $(x, y) \oplus (x', y') = (x + x', y + y')$

b) $(x, y) \otimes (x', y') = \big((xx' - yy'), (xy' + yx') \big)$

c) $(x, y) = x(1, 0) + y(0, 1) \stackrel{\text{def}}{\equiv} x + iy$

d) $\mathbb{C}$ is isomorphic to the Matrix Field of the 2×2 matrixes of the form $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$. In this isomorphic sense the imaginary unit $i=\sqrt{-1}$ corresponds to the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Let us now firstly define an analytical function of a domain Ω of the plane (a connected open set of R^2). Let us consider an infinitely differentiable vector field $F\colon \Omega \to \Omega$, $F(x,y) = (u(x,y), v(x,y))$. Associated to this planar vector field we can assign a function of complex variable $f(z,\bar{z}) = u(x,y) + iv(x,y) = f(x+iy, x-iy)$. One calls a complex $C^\infty(\Omega)$ planar vector field as an analytic function in Ω if it does not depends on the complex conjugated variable $\bar{z}$ (the Cauchy-Riemann equations)

$$\frac{\partial}{\partial \bar{z}} f(z,\bar{z}) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right)(u(x,y) + iv(x,y)) = 0 \tag{1.139}$$

which restricts to be called analytical functions in Ω solely those planar vector fields satisfying identically in Ω the following set of equations (called Riemann Equations)

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned} \tag{1.140}$$

or in polar coordinates

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r}\,\frac{\partial v}{\partial \theta} \\ \frac{1}{r}\,\frac{\partial u}{\partial v} &= -\frac{\partial v}{\partial r} \end{aligned} \tag{1.141}$$

A great number of examples of complex variable analytic functions can be obtained from real variable analytical functions through power series development.

Namely: if

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n, \quad \text{for} \quad -R + x_0 < x < R + x_0\,,$$

then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-x_0)^n, \quad \text{for} \quad \overbrace{-R < |z-x_0| < R}^{B_R(x_0)}$$

defines an analytical function in the (complex plane) open disk $B_R(x_0)$. This can be easily verified by noting the sequence of polinomial functions in the domain $B_R(x_0)$: $f_N(z) = \sum_{n=0}^{N} a_n(x+iy-x_0)^n = \sum_{n=0}^{N} a_n(\rho e^{i\theta}-x_0)^n$, converges uniformly in $B_R(x_0)$ and satisfy the set of Riemann Equations eq.(1.40)/eq.(1.41) there.

The line integrals of complex variable functions are straightforward defined from the usual line integrals of planar vector fields, with the restriction that all curves are homotopical (can be deformed continuously) to the circle S^1 (Jordan Curves)

$$\begin{aligned}\int_C f(z)\, dz &= \int_a^b f(z(t))\, \frac{dz}{dt}\, dt \\ &= \left(\int_a^b \left[u(x(t), y(t))\, \frac{dx}{dt} - v(x(t), y(t))\, \frac{dy}{dt} \right] \right) \\ &+ i \left(\int_a^b \left[u(x(t), y(t))\, \frac{dy}{dt} + v(x(t), y(t))\, \frac{dx}{dt} \right] \right) \end{aligned} \quad (1.142)$$

where $\mathcal{C}$ is a piecewise continuously differentiable path in Complex Plane in an obvious mathematical frame statement: $\mathcal{C} = \{z \in \mathbb{C} \mid z = z(t) = x(t) + iy(t),\ a \le t \le b;\ x(t),\ y(t)$ belonging to $C^1_{\text{piece}}\big([a,b]\big)\}$

It is a straightforward consequence of the Green theorem that if $f(z)$ is an analytical function in $\Omega \subset \mathbb{C}$ then for any closed line $\mathcal{C}$ inside Ω, one has immediately that

$$\oint_C f(z)dz = 0. \quad (1.143)$$

The converse is much more difficult to prove ([2]).

At this point we call the reader attention that the Complex Plane $\mathbb{C}$ can be always "compactified" to the three-dimensional sphere $S^2 = \{(\xi, \eta, z) \in R^3 \mid \xi^2 + \eta^2 + \big(z - \frac{1}{2}\big)^2 = \frac{1}{4}$ by means of the application where analytical functions are mapped in a certain class of planar vector fields in S^2. (Exercise: Re-write the Riemann Equations in S^3)

$$
\begin{aligned}
\xi &= \frac{x}{((x^2+y^2)+1)} \\
\eta &= \frac{y}{((x^2+y^2)+1)} \\
z &= \frac{1}{2}\left[\frac{x^2+y^2-1}{x^2+y^2+1}-1\right] \\
\big(F_1(\xi,\eta,z), F_2(\xi,\eta,z)\big) &= \left(u\left(\frac{\xi}{1-z},\frac{\eta}{1-z}\right), v\left(\frac{\xi}{1-z},\frac{\eta}{1-z}\right)\right)
\end{aligned} \tag{1.144}
$$

One important integral representation in the subject is the generalized Carleman-Cauchy integral representation.

Let us suppose that the real and imaginary parts of a full complex variable function $f(z,\bar{z}) = u(x,y) + iv(x,y)$ satisfie the generalized Riemann Equations in a given domain Ω

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} + a(x,b)u + b(x,y)v + F(x,y) \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} + c(x,b)u + b(x,y)v + G(x,y)
\end{aligned} \tag{1.145}
$$

where (a,b,c,d) are solely continuous functions in Ω. (Note that in the case of $a = b = c = d = 0$, the above written set eqs.(1.140-1.142) reduces to the Riemann set eq.(1.140).

Through the use of complex derivative operators (Chain rule!)

$$
\begin{aligned}
\frac{\partial}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right) \\
\frac{\partial}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right)
\end{aligned} \tag{1.146}
$$

one can re-write the Green theorem in the complex form

$$
\frac{1}{2i}\oint_C f(z,\bar{z})\,dz = \iint_{R(C)}\left(\frac{\partial f(z,\bar{z})}{\partial \bar{z}}\right) dxdy, \tag{1.147}
$$

where the domain $R(C)$ is the interior (open) set bounded by the closed line C, denoted by the (homological) formula $\partial R = C$ (see [6]).

Let us now consider $\partial B_\varepsilon(z)$, a small circunference around a given fixed point $z \in \Omega$ and entirely contained in this domain Ω. We easily see that (Green's theorem)

$$\frac{1}{2i}\oint_C f(\zeta,\bar{\zeta})d\zeta - \frac{1}{2i}\oint_{\partial B_\varepsilon(z)} f(\zeta,\bar{\zeta})d\zeta = \iint_{R(C)\setminus B_\varepsilon(z)} \left(\frac{\partial f}{\partial\bar{\zeta}}\right)\frac{dxdy}{(\zeta - z)}. \tag{1.148}$$

By noting that

$$\begin{aligned}
\lim_{\varepsilon\to 0}\frac{1}{2i\pi}\oint_{\partial B_\varepsilon(z)} f(\zeta,\bar{\zeta})d\zeta &= \lim_{\varepsilon\to 0^+}\left\{\frac{1}{2\pi i}\left[\int_0^{2\pi}\frac{f(z+\varepsilon e^{i\theta},\bar{z}+\varepsilon e^{-i\theta})}{\varepsilon\, e^{i\theta}}\,\varepsilon i e^{i\theta}\,d\theta\right]\right\} \\
&\overset{\text{(mean value theorem)}}{=} f(z,\bar{z})
\end{aligned} \tag{1.149}$$

one has the following result

$$f(z,\bar{z}) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta,\bar{\zeta})d\zeta}{\zeta - z} - \frac{1}{\pi}\iint_{R(C)}\frac{\partial f(\zeta,\bar{\zeta})}{d\bar{\zeta}}\,\frac{dxdy}{\zeta - z}\cdot \tag{1.150}$$

In our case eq.(1.145), we just note that in complex variables notation

$$\begin{aligned}
\frac{\partial f(z,\bar{z})}{\partial\bar{z}} &= A(z,\bar{z})f(z,\bar{z}) + B(z,\bar{z}), \overline{f(z,\bar{z})} + G(z,\bar{z}) \\
A &= \frac{1}{4}\,(a+d+ic-ib)(z,\bar{z}) \\
B &= \frac{1}{4}\,(a-c+ic+ib)(z,\bar{z}) \\
G &= F_1 + iF_2
\end{aligned} \tag{1.151}$$

which leads to the basic Cauchy (Carleman) complex variable integral representation for a complex valued function $f(z,\bar{z})$ satisfying the Riemann-Carleman eq.(1.151)

$$\begin{aligned}
f(z,\bar{z}) = \frac{1}{2\pi i}\left(\oint_C \frac{f(\zeta)}{\zeta - z}\,d\zeta\right) &- \frac{1}{\pi}\left(\iint_{R(C)}\left(\frac{A(\zeta,\bar{\zeta})f(\zeta,\bar{\zeta}) + B(\zeta,\bar{\zeta})\overline{f(\zeta,\bar{\zeta})}}{\zeta - z}\right)dxdy\right) \\
&- \frac{1}{\pi}\left\{\iint_{R(C)}\frac{G(\zeta,\bar{\zeta})}{\zeta - z}\,dxdy\right\}
\end{aligned} \tag{1.152}$$

In the case of analytical functions in Ω with $A = B = G = 0 + i0 \equiv 0$, one obtains the elementary Cauchy formula

$$\frac{d^k f(z)}{dz^k} = \frac{k!}{(2\pi i)}\oint_C \frac{f(\zeta)}{(\zeta - z)^{k+1}}\,d\zeta\cdot \tag{1.153}$$

On basis of the above written integral formula eq.(1.153), one has the important K. Weierstrass-P. Laurent theorem:

Theorem 1. Let Ω be an region of the following annulus form $\Omega = \{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}$. Then any analytic function $f(z)$ in Ω can be written uniquely in the following form

$$f(z) = \sum_{n=-\infty}^{+\infty} \big(A_n(z - z_0)^n\big) = h(z - z_0) + m\big((z - z_0)^{-1}\big) \tag{1.154-a}$$

where

$$h\big((z - z_0)\big) = \sum_{n=0}^{\infty} A_n(z - z_0)^n \tag{1.154-b}$$

$$m\big((z - z_0))^{-1}\big) = \sum_{n=1}^{\infty} A_{-n}(z - z_0)^{-n} \tag{1.154-c}$$

with $\{\mathcal{C}_R = z \mid |z - z_0| = R;\ R_1 < R < R_2\}$

$$A_n = \frac{1}{2\pi i}\left(\oint_{\mathcal{C}_R} \frac{f(z)}{(z - z_0)^{n+1}}\, dz\right). \tag{1.154-d}$$

Note that this result remains true for any region bounded by a closed line (loop) homological to this annulus situation.

The coefficient A_{-1} is called the residue of the function $f(z)$ at the point $z = z_0$ and denoted by

$$A_{-1} \stackrel{\text{def}}{=} \operatorname{Res}[f(z), z = z_0]. \tag{1.154-e}$$

The proof of this important result is based on the Cauchy integral representation eq.(1.150) for any closed contour $\mathcal{C} = \Gamma_1 \cup \Gamma_2$ lying in the region Ω

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i}\oint_{\Gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta \\
&= \left(\frac{1}{2\pi i}\oint_{\substack{\Gamma_1 \\ |\zeta - z_0| < |z - z_0|}} f(\zeta)\left[-\sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}\right]\right) \\
&\quad - \left(\frac{1}{2\pi i}\oint_{\Gamma_2} f(\zeta)\left[-\sum_{n=-\infty}^{-1} \frac{\big((z - z))^n\big)}{(\zeta - z_0)^{n+1}}\right]\right) \\
&\equiv h\big((z - z_0)\big) + m\big((z - z_0)^{-1}\big)
\end{aligned} \tag{1.154-f}$$

At this point let us introduce some terminology. We say that a given point $z = z_0 \in \Omega$ where the analyticity of a given function fails, however there is a integer m such that $\lim\limits_{z\to z_0} (z - z_0)^m f(z) < \infty$, is an pole of order m of the function $f(z)$ in Ω. The residue of such function at the pole $z - z_0$ is this explicitly given by

$$A_{-1} = \text{Res}\,[f(\zeta), z = z_0] = \frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{d^{m-1}z}\big((z - z_0)^m f(z)\big). \tag{1.155}$$

An important relation with Fourier Series can be done now.

Let $f(z)$ be an analytic function in the annulus $1 - \varepsilon < |z| < 1 + \varepsilon$, for $\varepsilon > 0$. As a consequence of eqs.(1.154), one has the P. Laurent expansion there

$$f(z) = \sum_{n=-\infty}^{+\infty} C_n\, z^n \tag{1.156}$$

with

$$C_n = \frac{1}{2\pi i}\int_{|z|=1}\frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}\,d\theta. \tag{1.157}$$

If we consider $z = e^{it}\big(z \in S^1 = \{z \mid |z| = 1\}\big)$, eq.(3.156) becomes the complex Fourier Series of the function $\varphi(t) = f(e^{it})$. Namely

$$\varphi(t) = f(e^{it}) = \sum_{n=-\infty}^{+\infty} C_n\, e^{int} \tag{1.158}$$

For instance, let $a \in \mathbb{C}$ such that $|a| < 1$. We consider now the following function with poles in $z_1 = a$ and $z_2 = 1/a$

$$f(z) = \frac{1 - z^2}{2i\{z^2 - \big(a + \frac{1}{a}\big)z + 1\}}\,. \tag{1.159-a}$$

We have now

$$f(z) = \frac{1}{2i}\left\{-1 + \frac{1}{1 - \frac{z}{a}} + \frac{1}{1 - az}\right\} \tag{1.159-b}$$

$$= \frac{1}{2i}\left\{\sum_{n=1}^{\infty} a^n\left(z^n - \frac{1}{z^n}\right)\right\} \tag{1.159-c}$$

It yields too

$$f(e^{it}) = \varphi(t) = \frac{a\sin(t)}{1 - 2a\,\cos(t) + a^2} = \sum_{n=1}^{\infty} a^n\,\sin(nt) \tag{1.159-d}$$

It is worth remark that Laurent Series are mathematical objects equivalent to Fourier Series of complex variable. In general a complex variable Fourier Series ($z = x + iy$)

$$f(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\,\cos\,nz + b_n\,\text{sen}\,nz) \tag{1.160-a}$$

is entirely equivalent to a Laurent Series after the variable change $e^{iz} = w$. Namely for $r < |w| < R$

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n\,w^n. \tag{1.160-b}$$

Here for $n = 1, 2, \ldots$

$$C_0 = \frac{a_0}{2}\,, \quad C_n = \frac{a_n - ib_n}{2}\,, \quad C_{-n} = \frac{a_n + ib_n}{2}\,. \tag{1.161}$$

Note that the complex variable Fourier Series converges in the strip $\ell n\, r < -y < \ell n R$ and $-\infty < x < \infty$.

We now assume that we have a given analytic function $f(z)$ in a region Ω, where $|f'(z)|^2 \neq 0$. Let $f(z_0) = w_0$. We thus determine its inverse function (which is analytic too!) in the region $W = f(\Omega)$ through a power series

$$z = f^{-1}(w) = \sum_{n=0}^{\infty} b_n(w - w_0)^n. \tag{1.162}$$

In order to accomplish this task let us consider the following integral for a closed path $\Gamma \subset \Omega$.

$$\begin{aligned}
I(w) &= \frac{1}{2\pi i}\oint_\Gamma \left(\frac{z\,f'(z)}{f(z) - w}\right) dz \\
&= \text{Res}\left[\frac{z\,f'(z)}{f(z) - w}\,; z - f^{-1}(w)\right] \\
&= \lim_{f(z)\to w}\left[(z - f^{-1}(w))\,\frac{z\,f'(z)}{f(z) - w}\right] \\
&= \lim_{z\to\bar{z}=f(w)}\left[\frac{(z\,f'(z))(z - \bar{z})}{f(z) - f(\bar{z})}\right] = \lim_{z\to\bar{z}}\left[\frac{z\,f'(z))(z - \bar{z})}{f'(\bar{z})(z - \bar{z})}\right] = \bar{z} = f^{-1}(w)
\end{aligned} \tag{1.163}$$

Now

$$I(w) = \frac{1}{2\pi i} \oint_\Gamma z f'(z) \left\{ \frac{1}{f(z) - w_0} \frac{1}{\left(1 - \frac{(w-w_0)}{f(z)-w_0}\right)} \right\} dz$$

$$= \sum_{n=0}^{\infty} - \left\{ \frac{1}{2\pi i n} \oint_\Gamma \left(z \frac{d}{dz} (f(z) - w_0)^{-n} \right) dz \right\} (w - w_0)^n \tag{1.164-a}$$

(by parts integration)

$$= \sum_{n=0}^{\infty} \left\{ + \frac{1}{2\pi i n} \oint_\Gamma \frac{dz}{(f(z) - w_0)^n} \right\} (w - w_0)^n \tag{1.164-b}$$

which leads to our formulae

$$b_n = \frac{1}{n} \left\{ \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{d^{n-1} z} \left[\frac{(z - z_0)^n}{(f(z) - f(z_0))^n} \right] \right\} \tag{1.164-c}$$

Not that L'Hôpital rule for evaluation of limits still remains true in the complex domain (prove it!).

Exercise – The Schwartz reflection principle.

Let be $f(z) = u(x,y) + iv(x,y)$ an analytical function in a region Ω bounded by a closed curve in the complex upper half-plane possessing a segment $a \le x \le b$ in the real axis as its piece and such it is a real function there. Then $\overline{f(\bar{z})} = u(x,-y) - iv(x,-y)$ is analytic in the reflected Ω along the segment in the complex lower half-plane, by a simple application of the Riemann equations (1.145).

Solved Exercise – On the Riemann Conjecture – The Riemann's series $\zeta(x) = \sum_{n=1}^{\infty} n^{-x}$ converges uniformly for all real numbers x greater then or equal to a any given (fixed) abscissa $\bar{x} : x > \bar{x} > 1$. It is well-known that the complex valued (meromorphic) continuation to complex values ($z = x + iy$) throughout the Complex Plane $z \in \mathbb{C}$ is obtained from standard analytic (finite-part) complex variables methods applied to the integral representation ([1])

$$\zeta(z) = \frac{i\Gamma(1-z)}{2\pi}\left(\int_C \left(\frac{(-2)^{z-1}}{e^w-1}\right) dw\right) = \frac{1}{\Gamma(z)}\left(\int_0^\infty w^{z-1}\times\left[\frac{1}{e^w-1}-\frac{1}{w}\right] dw\right)$$
$$= \frac{i\Gamma(1-z)}{2\pi}\left\{\int_C \left[(-w)^{z-2}-\frac{1}{2}(-w)^{z-1}+\sum_{n=1}^{\infty}\frac{(-1)^{n+z-2}B_n/w^{z+2n-2}}{(2n)!}\right]\right\}, \quad (1.166)$$

here B_n are the Bernoulli's numbers and C as any contour in the Complex Plane, coming from positive infinity and encircling the origin once in the positive direction.

An important relationship resulting from eq.(1.166) is the so called functional equation satisfied by the Zeta function, holding true for any $z \in \mathbb{C}$

$$\frac{\zeta(z)}{\zeta(1-z)} = 2^z \cdot \pi^{z-1} \cdot \operatorname{sen}\left(\frac{\pi z}{2}\right) \cdot \Gamma(1-z). \quad (1.167)$$

In applications to Number Theory, where this Special function plays a special role, it is a famous conjecture proposed by B. Riemann (1856) that the only non trivial zeros of the Zeta function lie in the so-called critical line Real $(z) = x = \frac{1}{2}\cdot$ Let us state our conjecture

Conjecture: In each horizontal line of the complex Plane of the form Imaginary $(z) = y = b =$ constant, the Zeta function $\zeta(z)$ *posseses at most a unique zero.*

We show now that the above written conjecture leads elementarily to a proof of the Riemann's conjecture.

Theorem (The Riemann's Conjecture). All the non-trivial zeros of the Riemann Zeta function lie on the critical line Real $(z) = x = \frac{1}{2}\cdot$

Proof: Let us consider a given non-trivial zero $\bar{z} = \bar{x} + i\bar{y}$ on the open strip $0 < x < 1$, $-\infty < y < +\infty$. It is a direct consequence of the Schwartz's reflection principle since $\zeta(x)$ is a real function in $0 < x < 1$ that $(\bar{z})^* = \bar{x} - i\bar{y}$ is another non-trivial zero of the Riemann function on the above pointed out open strip. The basic point of our proof is to show that $1-(\bar{z})^* = (1-\bar{x}) + i\bar{y}$ is another zero of $\zeta(z)$ in the same horizontal line $Im(z) = \bar{y}$. This result turns out that $(1-(\bar{z})^*) = \bar{z}$ on the basis of the validity of our

Conjecture. As a consequence we obtain straightforwardly that $1 - \bar{x} = \bar{x}$. In others words Real $(\bar{z}) = \frac{1}{2}$·

At this point, we call the reader attention, on the result that if $\bar{z}$ is a zero of the Riemann Zeta function, then $1-\bar{z}$ must be another non-trivial zero is a direct consequence of the functional eq.(1.167), since $\sin(\frac{\pi z}{2})$ and $\Gamma(1-z)$ never vanish both on the open strip $0 <$ Real$(z) < 1$.

At this point of our note, we want so state clearly that the significance of replacing the Riemann's original conjecture by our complex oriented conjecture rests on the possibility of progress in producing sound results for its proof.

Let us now sketch a "hand-wave" argument to prove our conjectue. Firstly we remark that it is well-known that the Zeta function $\zeta(x)$ does not possesses zeroes in the interval $0 < x < 1$ $\big(\zeta(x) = (1 - 2^{1-x})(\sum_{n=1}^{\infty}(-1)^n \cdot n^{-x})\big)$. As $\zeta(z)$ is an analytic function in the region $0 < x < 1$, $-b < y < b$; for any $b > 0$, one can see that the Taylor expansion holds true

$$\zeta(x+ib) = \left(\sum_{n=0}^{\infty} \frac{(-1)^n\,\zeta^{(2n)}(x)}{(2n)!}\,b^{2n}\right) + i\left(\sum_{n=0}^{\infty} \frac{(-1)^n\,\zeta^{(2n+1)}(x)}{(2n+1)!}\,b^{2n+1}\right) \tag{1.168-a}$$

Now if $x_0 + ib$ is a zero of $\zeta(x + ib)$, then necessarily we have the validity of the (functional) equations

$$\sum_{n=0}^{\infty}\left(\frac{(-1)^n\,\zeta^{(2n)}(x_0)}{(2n)!}\right) b^{2n} = 0 \tag{1.168-b}$$

$$\sum_{n=0}^{\infty}\left(\frac{(-1)^n\,\zeta^{(2n+1)}(x_0)}{(2n+1)!}\right) b^{2n+1} = 0 \tag{1.168-c}$$

Now for b very small ($b \approx 0$), the above system of funcitonal equations should be naively expected to be replaced by the "effective one" for $x_0 \in (0, 1)$:

$$\zeta(x_0) = 0 \qquad \text{(1.168-c)}$$

$$\zeta'(x_0) = 0 \qquad \text{(1.168-d)}$$

which is clearly impossible with the result that $\zeta(x)$ does not have zeroes in the interval $0 < xM1$. This would proves the Riemann Conjecture.

The full, mathematical formalization of this "hand-wave" argument is left to our readers-worthing US$1,000,000!.

Exercise – Bürmann-Lagrange Series.

Let $f(z)$ and $w(z)$ complex valued analytic functions in a region Ω of the complex plane and $w(z)$ an injective function in Ω $\big(|w'(z)|^2 \neq 0$ for $z \in \Omega\big)$. Then one can expand $f(z)$ in powers series of the given function $w(z)$ around a zero z_0 of $w(z)$:

$$f(z) = \sum_{n=0}^{\infty} d_n (w(z))^n, \quad |z - z_0| \leq R. \tag{1.169-a}$$

Here the coefficients d_n possesse the following complex integral representation

$$\begin{aligned} d_n &= \frac{1}{2\pi i} \oint_C \frac{f(\zeta)w'(\zeta)}{w^{n+1}(\zeta)}\, d\zeta \\ &= \frac{1}{n!} \lim_{z \to a} \left\{ \frac{d^n}{dz^n} \left(\frac{f(z)w'(z)(z-a)^{n+1}}{w^{n+1}(z)} \right) \right\} (n = 0, 1, \dots) \end{aligned} \tag{1.169-b}$$

and the closed line C is any Jordan curve such $|w(z)/w(\zeta)| \leq q < 1$ and $|z - a| \leq \delta$. Note that the convergence radius is given by eq.(1.41) (generalized for the complex case).

Solution: Consider the complex integral

$$I(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)w'(\zeta)}{w(z) - w(z)}\, d\zeta. \tag{1.169-c}$$

Exercise – (Rouché theorem).

Let $f(z)$ and $g(z)$ be analytical functions in a region $R(C)$; bounded by a curve $\mathcal{C}$ and such that $f(z)$ and $g(z)$ when restricts to $\mathcal{C}$ are continuous functions and satisfy there the inequality $|f(z)| > |g(z)|$. Then the functions $f(z)$ and $f(z)+g(z)$ have the same number of zeros in $R(C)$.

Proof: Consider the logarithmic integral

$$N = \frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)}\, dz = \frac{1}{2\pi}\, \Delta_C(\arg\, f(\zeta)) \tag{1.170-a}$$

and show that under the given hypothesis that

$$\Delta_C \arg(f(z) + g(z)) = \Delta_C \arg\, f(z). \quad \blacksquare \tag{1.170-b}$$

As a consequence we have a straightforward proof of the Algebra fundamental theorem for complex polinomials in the complex plane.

Classical references.

Standard classical books on the subject are

[1] E.C. Titchmarsh, The Theory of Functions, Oxford University Press, 1968.

[2] S. Lang, Analysis I, Addison-Wesley Publishing Company, 1968.

[3] W. Ruldin, Principles of Mathematical Analysis.

A more advanced book on advanced calculus is

[4] Lynn H. Loomis and Shlomo Sternberg, Advanced Calculus, Addison-Wesley Publishing Company, (1961).

A complete text book on the subject of Advanced Complex Variable Calculus is

[5] M. Lavrentiev et B. Chabat, Méthodes de la Théorie des Fonctions d'une Variable Complexe, Édition Mir, Moscow, 1972.

[6] – A very nice and mandatory book on geometrical advanced calculus is the book by Paul Bamberg and Shlomo Sternberg – A course in Mathematics for students of Physics, vol 1 and vol 2, Cambridge University Press, 1991.

[7] – An useful book of introductory mathematical analysis is

– Andrew Browder: Mathematical Analysis, An Introduction (Undergraduate Texts in Mathematics), Springer-Verlag, 1996.

APPENDIX A

An Elementary Comment on the Zeros of The Zeta Function (on the Riemann's Conjecture) – Version II

1 Introduction – "Elementary may be deep" ([1])

The Riemann's series $\zeta(x) = \sum_{n=1}^{\infty} n^{-x}$ converges uniformly for all real numbers x greater then or equal to a any given (fixed) abscissa $\bar{x}$: $x > \bar{x} > 1$. It is well-known that the complex valued (meromorphic) continuation to complex values ($z = x+iy$) throughout the Complex Plane $z \in \mathbb{C}$ is obtained from standard analytic (finite-part) complex variables methods applied to the integral representation ([1]))

$$\begin{aligned}\zeta(z) &= \frac{i\Gamma(1-z)}{2\pi}\left(\int_C \left(\frac{(-w)^{z-1}}{e^w - 1}\right) dw\right) = \frac{1}{\Gamma(z)}\left(\int_0^\infty w^{z-1} \times \left[\frac{1}{e^w-1} - \frac{1}{w}\right] dw\right) \\ &= \frac{i\Gamma(1-z)}{2\pi}\left\{\int_C \left[(-w)^{z-2} - \frac{1}{2}(-w)^{z-1} + \sum_{n=1}^{\infty} \frac{(-1)^{n+z-2} B_n / w^{z+2n-2}}{(2n)!}\right]\right\}, \qquad \text{(1-A)}\end{aligned}$$

here B_n are the Bernoulli's numbers and C is any contour in the Complex Plane, coming from positive infinity and encircling the origin once in the positive direction.

An important relationship resulting from eq.(1) is the so called functional equation satisfied by the Zeta function, holding true for any $z \in \mathbb{C}$

$$\frac{\zeta(z)}{\zeta(1-z)} = 2^z \cdot \pi^{z-1} \cdot \operatorname{sen}\left(\frac{\pi z}{2}\right) \cdot \Gamma(1-z) \qquad \text{(2-A)}$$

In applications to Number Theory, where this Special function plays a special role, it is a famous conjecture proposed by B. Riemann (1856) that the only non trivial zeros of the Zeta function lie in the so-called critical line Real $(z) = x = \frac{1}{2}$.

In the next section we intend to propose an equivalent conjecture, hoped to be more suitable for handling the Riemann's problem by the standard methods of Classical Complex Analysis ([2]), besides of proving a historical clue for the reason that led B. Riemann to propose his conjecture ([2] – second reference).

2 On the Equivalent Conjecture 1

Let us state our conjecture

Conjecture: In each horizontal line of the complex Plane of the form Imaginary $(z) = y = b =$ constant, the Zeta function $\zeta(z)$ *posseses at most a unique zero.*

We show now that the above written conjecture leads elementarly to a proof of the Riemann's conjecture.

Theorem. (The Riemann's Conjecture) *All the non-trivial zeros of the Riemann Zeta function lie on the critical line Real* $(z) = x = \frac{1}{2}$.

Proof: Let us consider a given non-trivial zero $\bar{z} = \bar{x} + i\bar{y}$ on the open strip $0 < x < 1$, $-\infty < y < +\infty$. It is a direct consequence of the Schwartz's reflection principle since $\zeta(x)$ is a real function in $0 < x < 1$ that $(\bar{z})^* = \bar{x} - i\bar{y}$ is another non-trivial zero of the Riemann function on the above pointed out open strip. The basic point of our proof is to show that $1 - (\bar{z})^* = (1 - \bar{x}) + i\bar{y}$ is another zero of $\zeta(z)$ in the same horizontal line $Im(z) = \bar{y}$. This result turns out that $(1 - (\bar{z})^*) = \bar{z}$ on the basis of the validity of our Conjecture. As a consequence we obtain straightforwardly that $1 - \bar{x} = \bar{x}$. In others words Real $(\bar{z}) = \frac{1}{2}$.

At this point, we call the reader attention, on the result that if $\bar{z}$ is a zero of the Riemann Zeta function, then $1-\bar{z}$ must be another non-trivial zero is a direct consequence of the functional eq.(2), since $\sin\left(\frac{\pi z}{2}\right)$ and $\Gamma(1 - z)$ never vanishe both on the open strip $0 < \text{Real}(z) < 1$.

At this point of our note, we want to state clearly that the significance of replacing the Riemann's original conjecture by our complex oriented conjecture rests on the possibility of progress in producing sound results for its proof, which is not claimed in our elementary note. However, we intend to point out directions (arguments) in its favor.

Let us thus consider the combined zeta function of Riemann (see second reference - [2]). As defined below

$$\xi(z) = \Gamma(\frac{z}{2})(z-1)\pi^{-\frac{3}{2}}\zeta(z). \tag{3-A}$$

Since $\xi(z)$ is an entire function on the complex plane it has the well known power series around the real point $z = \frac{1}{2}$ with an arbitrary convergence radius R

$$\xi(z) = \sum_{n=0}^{\infty} a_{2n}(z-\frac{1}{2})^{2n}; \quad |z-\frac{1}{2}| < R \tag{4-A}$$

where the power series coeficients are given explicitly by the Riemann's integral formula

$$a_{2n} = 4\sum_{p=0}^{\infty}\Big\{\int_1^{\infty}\Big[(p^4\pi^2 x - \frac{3}{2}p^2\pi)x^{\frac{1}{2}}e^{-p^2\pi x}\,x^{-\frac{1}{4}}\frac{(\frac{1}{2}lgx)^{2n}}{(2n)!}\Big]dx\Big\}. \tag{5-A}$$

By noting that all the above coeficients are strictly positive real numbers, one see that in the interval $0 < x < 1$, there is no zeros for $\xi(x)$. As a consequence $\zeta(x)$ does not have any zeros in $0 < x < 1$.

Let us now analyze the important case of our conjecture, i.e. the zeros of the function $\xi(x+ib)$, for a given $b \in R$ fixed. In this case we have the obvious power expansion for the real part of the above horizontal, strip combined zeta-function

$$\text{Real}(\xi(x+ib)) = \sum_{m=0}^{\infty}\frac{(x-\frac{1}{2})^{2m}}{(2m)!}\Big\{\sum_{n=2m}^{\infty} a_{2n}(-1)^{n-m}b^{2n-2m}\frac{(2n)!}{(2n-2m)!}\Big\}. \tag{6-A}$$

Let us denote the coeficients of the above written power series by $\hat{B}_m$

$$\widehat{B}_{2m} \equiv \sum_{n=2m}^{\infty}\Big[(a_{2n})(-1)^{n-m}\Big((b^2)^{n-m}\frac{(2n)!}{(2(n-m)!)}\Big)\Big\}. \tag{7-A}$$

If all the above numbers are non-negative, one could follow the same argument exposed above to show that the only possible root of $\text{Real}(\xi(x+ib))$ is localized on the horizontal line $x = \frac{1}{2}$.

As a result of our note, we have replaced a difficult statement on the precise localization of zeroes of an entire function by a somewhat pure arithmetical statement on the non-negativity of the infinite sums $\hat{B}_m$, for each $m \in Z^+$.

References for this Appendix A

[1] Kato, T. "Analysis Workshop", Univ. of California at Berkeley (1976), USA.

[2] Any standard classical book on the subject as for instance

– E.C. Titchmarsh - The Theory of Functions, Oxford University Press - Second Edition, (1968).

– H.M. Edwards - "Riemann's Zeta Function", Pure and Applied Mathematics; Academic Press, (1974), New York.

– A. Ivic - "The Riemann Zeta-Function"; John Wiley & Sons, (1986), New York.

– A.A. Karatsuba and S.M. Doramin - "The Riemann Zeta-Function", Gruyter Expositions in math, Berlin, (1992).

APPENDIX B

On the Cisotti integral representation for the Schwarz-Christoffel polygonal-conformal transformation and a new proof of the Riemann theorem on Conformal Transformations.

Let us give the detailed proof of the Conformal Transformations ideas. Let us start by the Cisotti formulae.

Firstly let us consider a bounded simply connected region Ω, with its Jordan curve boundary Γ given by the map of S^1 by an analytical function in the disc, namelly $\Gamma = \{f(e^{i\theta});\, 0 \leq \theta < 2\pi\}$. It is a straightforward result of elementary calculus that the angle with the Γ-tangent at the point $\omega = f(e^{i\theta})$, makes with the x-axis, is the following function of the θ-argument of the z-variable:

$$v(\theta) = \frac{|f(e^{i\theta})| \cos\left[\arg(f(e^{i\theta}))\right] + \left(\frac{d\theta}{d\theta'}\right)\left(\frac{d}{d\theta}|f(e^{i\theta})|\right) \sin\left[\arg(f(e^{i\theta}))\right]}{(-|f(e^{i\theta})|) \sin\left[\arg(f(e^{i\theta}))\right] + \left(\frac{d\theta}{d\theta'}\right)\left(\frac{d}{d\theta}|f(e^{i\theta}))\cos\left[\arg(f(e^{i\theta}))\right]} \qquad \text{(B-1)}$$

Here the argument θ' of the complex number $f(e^{i\theta})$ in Γ related to the argument θ in S^1 is given by the formula

$$\theta' = \arg(f(e^{i\theta})). \qquad \text{(B-2)}$$

It is now a simple result that the following holomorphic function in the unity disc $\{z \mid |z| < 1\}$

$$g(z) = -i\,\ell n\left[-i(1-z)^2\,\frac{df}{dz}\right] \qquad \text{(B-3)}$$

is given by the Dirichlet integral representation in terms of its real part boundary value $v(\theta)$. Namelly:

$$g(z) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, v(\theta)\,\frac{e^{i\theta}+z}{e^{i\theta}-z}\,. \qquad \text{(B-4)}$$

As a consequence of eq.(B-3) and the unicity of holomorphic continuations inside bounded domains in $\mathbb{C}$, we have the Cisotti integral representation for the function $f(z)$ inside the unity disk

$$f(z) = \left(i \int_{z_0}^{z} dz' \, \frac{\exp(ig(z'))}{(1-z')^2} \right) + f(z_0) \tag{B-5}$$

Let us now consider Γ being a polygonal closed line with n-vertexs in the complex plane with exterior angles denoted by ψ_k and the associated function

$$v(\theta) = \sum_{k=1}^{n} \psi_k \big(u(\theta - \theta_{k-1}) - u(\theta - \theta_k) \big). \tag{B-6}$$

Here $u(x)$ is the Hecviside function ($u(x) = 1$ if $x > 0$. $u(x) = 0$ if $x < 0$).

By using now the standard representation (with $z = re^{i\theta}$)

$$\begin{aligned} \frac{e^{i\overline{\theta}+z}}{e^{i\overline{\theta}-z}} &= \left(\frac{1-r^2}{1-2r\cos(\overline{\theta}-\theta)+r^2} \right) \\ &\quad + i \left(\frac{2r\,\mathrm{sen}(\theta-\overline{\theta})}{1-2r\cos(\theta-\overline{\theta})+r^2} \right), \end{aligned} \tag{B-7}$$

we can evaluate exactly the Cisotti formulae mapping of the disk with marked points $\{e^{i\overline{\theta}_k}\}_{1\le k\le N}$ into the Polygonal domain with vertexs $V_k = f(e^{i\overline{\theta}_k})$.

$$\begin{aligned} \overbrace{\left(\frac{e^{i\theta}d}{dr} - \frac{ie^{-i\theta}}{r}\right)}^{\frac{d}{dz}f(z)} f(re^{i\theta}) &= \frac{1}{(1-re^{i\theta})^2} \left\{ \left| \frac{e^{i\overline{\theta}_k} - re^{i\theta}}{e^{i\overline{\theta}_{k-1}}} - re^{i\theta} \right|^{\frac{\psi_k}{\pi}} \right\} \\ \times \exp \Bigg\{ -\frac{1}{\pi} \sum_{h=1}^{N} \Bigg[\arctan &\left(\frac{r+1}{r-1} \tan\left(\frac{\overline{\theta}_k - \theta}{2} \right) \right) (\psi_{k+1} - \psi_k) \Bigg] \Bigg\}. \end{aligned} \tag{B-8}$$

It is worth remark that we have the topological relationships below among the Polygonal internal angles α_k and the external angles ψ_k by the Gauss theorem:

$$\begin{cases} \pi - (\psi_k - \psi_{k-1}) = \alpha_k \\ \left(\displaystyle\sum_{h=1}^{N} \left(\frac{\alpha_k}{\pi} - 1 \right) \right) = -2 \end{cases} \tag{B-9}$$

At in this point one can reverse the usual analysis to produce a somewhat more constructive Cavalieri-Arquimedes proof of the Riemann theorem by just considering a sequence of Polygons curves $\omega_n(e^{i\theta})$ uniformly bounded in the unit disc $|\omega_n(\rho e^{i\theta})| \leq \sup_{0\leq\theta<2\pi} |\omega_n(e^{i\theta})| \leq M$ since by the Jordan Curve theorem all the images of the unit disk by the Cisotti-Schwarz-Christoffel applications are in the bounded region $\overline{\Omega}$. The Riemann conformal transformation of the unit disk onto the region Ω is straightforwardly given by a simple application of the well-known compacity criterium in the space of the holomorphic mappings in the unit disk. Namely: there is a subsequence $\{\omega_{n_k}(z)\}_{k\in\mathbb{Z}}$, uniformly converging to a certain holomorphic function in the unit disk with a piecewise continuously differentiable boundary value on the circle and applying conformally the unit disk into the region Ω, bounded by a Jordan Curve C.

As an important application of the Conformal Transformation Methods for device approximate numerical schemes for the Dirichlet problem with general boundaries is the Lavrentiev approximate formulae for the conformal transformation of a circle perturbed by a "small" curve $\sigma(t) = \varepsilon \left[\sum_{n=-\infty}^{+\infty} A_n e^{int} \right]$ with ε denoting a small parameter, to a given unity circle ([1])

$$\omega_{\mathrm{Lav}}(z) \cong z \left\{ 1 + \frac{\varepsilon}{2\pi} \int_0^{2\pi} \sigma(t)\, \frac{e^{it} + z}{e^{it} - z}\, dt \right\} + O(\varepsilon^2). \tag{B-10}$$

As a consequence one may formally consider $\sigma(t)$ as a random object with the Fourier coefficients obeying probability distributions and thus throught the Poisson explicitly formulae, try device approximate solutions for the Dirichlet problem with weakly random (Fractal) boundaries.

We left to our readers to proceed further in these matters.