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Hadron Cascade by the Method of Characteristics

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Abstract

Hadron diffusion equations with energy-dependent interaction mean free paths and inelasticities are solved using the Mellin transform. Instead of using operators on the finite difference terms, the Mellin transformed equations are Taylor expanded into a first order partial differential equation in atmospheric depth t and in the transform parameter s. Then, these equations are solved by the method of characteristics. The hadron fluxes (nucleon and meson) in real space is evaluated by the method of residues. For the case of a regularized power law primary spectrum these hadron fluxes are given by simple residues and one, never before mentioned, essential singularities. A comparison of our solutions with the nucleon flux measured at sea level and with the hadron fluxes measured at $t = 840\, g/cm^2$ and at sea level are made. The agrement between them is in general very good, greater than 90%. In order to check the accuracy of our calculations, a comparison between our solution and the simulated nucleon cascades is also made.

Key-Words: Hadron Flux, Residue Evaluation, Essential Singularity, Method of Characteristics.

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1 Introduction

Cosmic ray propagation in the earth's atmosphere are governed by integro-differential equations in the atmospheric depth t and energy E. Several approaches are used to solve these equations such as analytical and numerical methods[1]. Also this propagation can be calculated using simulation techniques such as Monte Carlo[2]. In our case we think that analytical calculations are still worth pursing because they are useful for qualitative understanding, to check Monte Carlo results and to give the relations among the different fluxes of particles accurately. Hadron fluxes play a very important role in the deriving the leptons fluxes at different atmospheric depths, in the understanding of the emulsion chamber data at mountain altitudes and in the analysing some exotic events (Halo Events, Centauro Events) detected at Mt. Chacaltaya by the Brazil-Japan Emulsion Chamber Collaboration. Due to the complex structure these integro-differential equations are often solved formally in operator forms either in real space[3] or in integral transform space [4, 5, 6, 7]. However, according to these analytical theories, the calculated fluxes in real space are often evaluated with very limited considerations on interaction mean free paths, $\lambda(E)$ inelasticities coefficients $K(E)$ and energy distribution of secondary hadrons $\Phi(E, E')$. Here we consider interactions with energy dependent of these parameters and distributions. We follow the use of Mellin transform to solve the governing nucleon equation. Instead of using operators on the finite difference terms, we Taylor expand the Mellin transformed nucleon cascade equation to a first order partial differential equation in atmospheric depth and transform parameter s. This equation is solved by the method of characteristics to give the characteristics of the partial differential equation and the transform of the flux which remains in a separable form of its variables. By inverting the transform, the nucleon flux in real space is solved as a function of the incident spectrum at the top of the atmosphere. Contrary to the customary non-analytic power law flux that does not have a Mellin transform, the boundary condition at the top of the atmosphere is represented by a regularized power law which is analytic over the entire energy range. In the past, the non-analytic incident flux transform was used as the boundary condition to calculate the nucleon flux. This flux is now recalculated adequately. For the case of power spectrum incident flux, the nucleon flux is given by the residues of two simple singularities $(s - \gamma) = 0$ and $(s + 1) = 0$ and one, never before mentioned, essential singularity $(s - s_0) = 0$. The $(s + 1) = 0$ simple singularity comes from the regularization of the incident spectrum at low energies. The method is also extended to treat charged pion cascade.

Our solutions allow us to investigate the effects of primary spectrum deviations from the power law form, the problem of the differents bending points of this spectrum (knee and ankle regions) and the energy dependence of the Z-factors (broken of the scaling law for the hadronic interactions).

This paper is divided as follow. In section 2 we derive the Mellin transformed equations for the nucleon and for the charged pion cases using an energy-dependent nucleon and pion interaction and inelasticity coefficients. We also consider a growing energy dependence for the secondary pion production spectra from hadron-air interactions. In section 3 we study the method of characteristics for both nucleon and pion cases. In section 4 we perform the calculus of residues, of the equations obtained previosly, by inverting the Mellin transform, getting such as the simple as the essential singularities. In section 5 we present some numerical results and we make a comparison of our differential nucleon flux with the experimental data measured at sea level. We also make a comparison of our calculations with the hadron fluxes measured at the atmospheric depth $t = 840 g/cm^2$ (EASTOP Experiment) and at sea level (Kascade Experiment). Finally in 6, we discuss and make some comment in our main results.

2 The Hadron Diffusion in the Atmosphere

2.1 The Nucleon Case

From considerations of different fundamental physical processes, the number density flux of nucleon $N(E,t)$ per energy interval ΔE centered at energy E at a given atmospheric depth t is described by

$$\frac{\partial N(E,t)}{\partial t} = -\frac{N(E,t)}{\lambda(E)} + \int_0^1 \int_E^\infty u(\eta)\delta(E-\eta E')\frac{N(E',t)}{\lambda(E')}dE'd\eta \tag{1}$$

where $\lambda(E)$ is the energy dependent mean free path, $\eta(E') = E/E' < 1$ is the elasticity, $u(\eta)$ is the elasticity distribution. Modelling the mean free path by a power index β, Eq.(1) reads

$$\lambda(E) = \lambda_N(\frac{E}{B})^{-\beta}, \tag{2}$$

$$\frac{\partial N(E,t)}{\partial t} = -\frac{1}{\lambda_N}(\frac{E}{B})^\beta N(E,t) + \frac{1}{\lambda_N}(\frac{E}{B})^\beta \int_0^1 (\frac{1}{\eta})^{\beta+1} u(\eta) N(\frac{E}{\eta},t)d\eta \tag{3}$$

where B is the normalization energy of mean free path. Instead of introducing mapping operators to the two terms on the right side of Eq. 3 to solve it formally in real space[3], we proceed to use the Mellin transform defined by

$$\tilde{N}(s,t) = \int_0^\infty (\frac{E}{A})^s N(E,t)\, d(\frac{E}{A}) \tag{4}$$

$$N(E,t) = \frac{1}{2\pi i}\int (\frac{E}{A})^{-(s+1)} \tilde{N}(s,t)\, ds \tag{5}$$

where the energy E is normalized to some reference energy A, so that the transform does not carry dimension of energy to power s. Now, Eq.(3) in the transform space reads

$$\begin{aligned}\frac{\partial \tilde{N}(s,t)}{\partial t} &= -\frac{1}{\lambda_N}(\frac{A}{B})^\beta \tilde{N}(s+\beta,t) \\ &+ \frac{1}{\lambda_N}(\frac{A}{B})^\beta \int_0^\infty <\eta^s> (\frac{E/A}{<\eta>})^{s+\beta} N(\frac{E/A}{<\eta>},t)d(\frac{E/A}{<\eta>})\end{aligned} \tag{6}$$

With K as the normalization energy of elasticity, we use the following average model of elasticity to power s

$$\begin{aligned}<\eta^s> &= \frac{\int_0^1 u(\eta)\eta^s d\eta}{\int_0^1 u(\eta)d\eta} = \frac{1}{(1+\delta s)}(\frac{E'}{K})^{\kappa s} \\ &= \frac{1}{(s-s_0)}\frac{1}{\delta}(\frac{A}{K})^{\kappa s}(\frac{E'}{A})^{\kappa s} = a(s)(\frac{A}{K})^{\kappa s}(\frac{E'}{A})^{\kappa s}\end{aligned} \tag{7}$$

where $s_0 = -1/\delta$. For a uniform elasticity distribution, we have $\kappa = 0$, $\delta = 1$, and $s_0 = -1$. In particular, taking $s = 1$ gives the average elasticity

$$<\eta> = \frac{1}{(1+\delta)}(\frac{E'}{K})^\kappa \tag{8}$$

The equation of the flux transform then becomes

$$\frac{\partial \tilde{N}(s,t)}{\partial t} = -\frac{1}{\lambda_N}(\frac{A}{B})^{\beta}\tilde{N}(s+\beta,t) + \frac{1}{\lambda_N}(\frac{A}{B})^{\beta}(\frac{A}{K})^{\kappa s}a(s)\tilde{N}(s+\beta+\kappa s,t) \tag{9}$$

We note that should the energy E in the Mellin transform not be normalized to some reference energy A, then $\tilde{N}(s,t)$, $\tilde{N}\,(s+\beta,t)$, $\tilde{N}(s+\beta+\kappa s,t)$ would have different dimensions in energy which would conceal the effects of mean free path and elasticity. Here, in Eq. (6), they have the same dimension of $N(E,0)$. The mean free path factor $(A/B)^{\beta}$ and the elasticity factor $(A/K)^{\kappa s}$ are working as the weighting factors among different transforms.

We observe that the nucleon cascade equation, Eq. (1), has two competing terms on the right side. The first term is the diffusion term that drains the flux $N(E,t) \bigtriangleup E$ at E to lower energies E'. The second term is the attenuation term that fills the flux at E by higher energies E'. Since the mean free path scaled by Eq. (2) vanishes as E/B goes to infinity with $\beta > 0$, the first term would dominate the equation and the spatial gradient of the flux would be very negative at high energies. As for the elasticity $\eta = E/E' < 1$, it goes to zero at a given E as E' goes to infinity. For the average elasticity $<\eta>$ of Eq. (8) to have the same limit at a given E as E'/K becomes infinite, κ has to be negative.

2.2 The Charged Pion Case

As for the charged pion number density flux $\Pi(E,t)$ per energy interval ΔE centered at energy E at a given atmospheric depth t, it is described by

$$\begin{aligned}\frac{\partial \Pi(E,t)}{\partial t} &= -\frac{\Pi(E,t)}{\lambda_{\Pi}(E)} + \int_0^1\int_E^{\infty}(1-b)u(\eta)\delta(E-\eta E')\frac{\Pi(E',t)}{\lambda_{\Pi}(E')}dE'd\eta \\ &+\int_E^{\infty}\Phi(E',E)\frac{\Pi(E',t)}{\lambda_{\Pi}(E')}dE' + \int_E^{\infty}\Phi(E',E)\frac{N(E',t)}{\lambda(E')}dE' \end{aligned}\tag{10}$$

where $\lambda_{\Pi}(E)$ is the energy dependent pion collision mean free path, b is the charge exchange probability of the incident pion, and $\Phi(E',E)dE$ is the probability of number of pions produced at energy E in the interval dE due to incident pion or nucleon of energy E'. Modelling the mean free path by a power index β (the same of the nucleon case),

$$\lambda_{\Pi}(E) = \lambda_{\pi}(\frac{E}{B})^{-\beta}, \tag{11}$$

$$\begin{aligned}\frac{\partial \Pi(E,t)}{\partial t} &= -\frac{1}{\lambda_{\pi}}(\frac{E}{B})^{\beta}\Pi(E,t) + \frac{1}{\lambda_{\pi}}\int_0^1(1-b)(\frac{E}{B})^{\beta}(\frac{1}{\eta})^{\beta+1}u(\eta)\Pi(\frac{E}{\eta},t)d\eta \\ &+\frac{1}{\lambda_{\pi}}\int_E^{\infty}\Phi(E',E)(\frac{E'}{B})^{\beta}\Pi(E',t)dE' + \frac{1}{\lambda_N}\int_E^{\infty}\Phi(E',E)(\frac{E'}{B})^{\beta}N(E',t)dE'. \end{aligned}\tag{12}$$

We model the pion production by[3]

$$\Phi(E',E) = D(\frac{E'}{A_{\Pi}})^{\alpha}[1-(\frac{E'}{A_{\Pi}})^{\alpha'}\frac{E}{E'}]^{d}\frac{1}{E'} \tag{13}$$

where A_{Π} is the pion production normalization energy, $D = (d+1)/3$, $d = 4$, and $\alpha, \alpha' \ll 1$. The charged pion flux, Eq.(12), in the transform space reads

$$(\frac{A}{B})^{-\beta}\frac{\partial \tilde{\Pi}(s,t)}{\partial t} = -\frac{1}{\lambda_{\pi}}\tilde{\Pi}(s+\beta,t) + \frac{1}{\lambda_{\pi}}(1-b)(\frac{A}{K})^{\kappa s}a(s)\tilde{\Pi}(s+\beta+\kappa s,t) +$$

$$\frac{1}{\lambda_\pi}(\frac{A}{A_\Pi})^{\alpha-\alpha' s}\phi(s)\tilde{\Pi}(s+\beta+\alpha-\alpha' s,t)+$$
$$\frac{1}{\lambda_N}(\frac{A}{A_\Pi})^{\alpha-\alpha' s}\phi(s)\tilde{N}(s+\beta+\alpha-\alpha' s,t), \tag{14}$$
$$\phi(s) = D\int_0^1 y^{s-1}(1-y)^d dy, \tag{15}$$
$$y = (\frac{E'}{A_\Pi})^{\alpha'}\frac{E}{E'}. \tag{16}$$

3 Method of Characteristics

To solve Eq.(9)and Eq.(14), we note that both β and κ are much less than s, so that one way to solve this equation is by iterations. Some researchers define two operators in the transform space to represent the two finite difference terms on the right side of Eq.(9) and four to Eq.(14), and to solve it formally by operators [3]. Following the property $\beta,\kappa \ll s$, we choose to Taylor expand the two terms about $\tilde{N}(s,t)$ and four terms about $\tilde{\Pi}(s,t)$ to get a first order differential equations, that read

$$\lambda_N\frac{\partial\tilde{N}(s,t)}{\partial t}+(\frac{A}{B})^\beta[\beta-A(s)(\beta+\kappa s)]\frac{\partial\tilde{N}(s,t)}{\partial s} = -(\frac{A}{B})^\beta[1-A(s)]\tilde{N}(s,t) \tag{17}$$

where $$A(s)=(\frac{A}{K})^{\kappa s}\,a(s)=(\frac{A}{K})^{\kappa s}\frac{1}{\delta}\frac{1}{(s-s_0)}. \tag{18}$$

This partial differential equation is equivalent to the following set of ordinary differential equations with $(s,\, t,\, \tilde{N})$ as the coordinates of a point in the functional space parameterized to ξ [8]

$$\frac{dt}{\lambda_N}=\frac{ds}{(A/B)^\beta[\beta-A(s)(\beta+\kappa s)]}=-\frac{d\tilde{N}}{(A/B)^\beta[1-A(s)]\tilde{N}}=d\xi \tag{19}$$

This method of characteristics in solving first order partial differential equations was used in superradiant free electron lasers [9, 10]. Solving for the equality between dt and ds,

$$\frac{dt}{\lambda_N}=\frac{ds}{(A/B)^\beta[\beta-A(s)(\beta+\kappa s)]} \tag{20}$$

we get a trajectory between the variables t and s through the parameter ξ, $t=t(s,\beta,\kappa)$, which is the characteristics of the partial differential equation, Eq.(17). To get the transform of the flux, we could solve the equality of $d\tilde{N}$ with $d\xi$, or with ds, or with dt. Since the boundary condition of $\tilde{N}$ is given in terms of s at $t=0$, we choose to solve with dt to get

$$\frac{dt}{\lambda_N}=-\frac{d\tilde{N}}{(A/B)^\beta[1-A(s)]\tilde{N}}$$
$$\ln(\frac{\tilde{N}(s,t)}{\tilde{N}(s,0)})=-\mu(s)\frac{t}{\lambda_N} \tag{21}$$

where $\mu(s)=(\frac{A}{B})^\beta[1-A(s)]$.

The factor $(A/B)^\beta$ in Eqs.(20, 21) represents the relative weight of mean free path to elasticity effect.

For the pion case, Taylor expanding Eq.(14) and denoting $A(s) = (A/K)^{\kappa s} a(s)$, with $\kappa = \alpha' - \alpha$, the partial differential equation reads

$$\lambda_\pi (\frac{A}{B})^{-\beta} \frac{\partial \tilde{\Pi}(s,t)}{\partial t} + \{\beta - (1-b)A(s)(\beta + \kappa s) - (\frac{A}{A_\Pi})^{\alpha - \alpha' s}(\beta + \alpha - \alpha' s)\phi(s)\} \frac{\partial \tilde{\Pi}(s,t)}{\partial s} =$$
$$-\{[1 - (1-b)A(s) - (\frac{A}{A_\Pi})^{\alpha-\alpha' s}\phi(s)]\tilde{\Pi}(s,t) - \frac{\lambda_\pi}{\lambda_N}(\frac{A}{A_\Pi})^{\alpha-\alpha' s}\phi(s)\tilde{N}(s,t)\}. \quad (22)$$

The corresponding set of ordinary differential equations is

$$(\frac{A}{B})^{-\beta}\frac{ds}{dt} = \frac{1}{\lambda_\pi}[\beta - (1-b)A(s)(\beta + \kappa s) - (\frac{A}{A_\Pi})^{\alpha-\alpha' s}(\beta + \alpha - \alpha' s)\phi(s)], \quad (23)$$

$$\begin{aligned}(\frac{A}{B})^{-\beta}\frac{d\tilde{\Pi}}{dt} &= -\frac{1}{\lambda_\pi}[1 - (1-b)A(s) - (\frac{A}{A_\Pi})^{\alpha-\alpha' s}\phi(s)]\tilde{\Pi} + \frac{1}{\lambda_N}(\frac{A}{A_\Pi})^{\alpha-\alpha' s}\phi(s)\tilde{N} \\ &= -\frac{\Pi(s)}{\lambda_\pi}\tilde{\Pi} + \frac{Q(s)}{\lambda_N}\tilde{N}. \quad (24)\end{aligned}$$

As for the boundary condition, the incident flux at $t = 0$ is often taken to have a power scaling in energy E of the form

$$N(E,0) = N_0(\frac{E}{G})^{-(\gamma+1)} = N_0(\frac{E/B}{G/B})^{-(\gamma+1)} \quad (25)$$

where G is the normalization energy of the incident flux. The coefficient N_0 corresponds to the flux at energy $E = G$. We note that this incident flux is singular at $E = 0$. Taking the Mellin transform, we have

$$\tilde{N}(s,0) = N_0(\frac{A}{G})^{-(\gamma+1)}[(\frac{E_M}{A})^{s-\gamma} - (\frac{E_m}{A})^{s-\gamma}]\frac{1}{(s-\gamma)} \quad (26)$$

where E_m goes to zero and E_M goes to infinity. We note that this transform is not analytic due to the divergence either at the lower limit or at the upper limit because of the singularity in the incident flux. Truncating the spectrum at E_m and E_M is not desirable since it generates discontinuities on derivative with respect to energy. Traditionally, in the circulating literatures,

$$\tilde{N}(s,0) = N_0(\frac{A}{G})^{-(\gamma+1)}\frac{1}{(s-\gamma)} \quad (27)$$

is often taken as the transform simply because it renders the correct $N(E,0)$ when substituted into Eq.(5). Nevertheless, this does not warrant that $\tilde{N}(s,0)$ of Eq.(27) is the transform of $N(E,0)$ of Eq.(25) according to Eq.(4).

In order to overcome the nonexistence of the transform, we take the incident flux at $t = 0$ be

$$N(E,0) = N_0[1 + (\frac{E}{G})]^{-(\gamma+1)} = N_0[1 + (\frac{E/A}{G/A})]^{-(\gamma+1)}) \quad (28)$$

which levels off to N_0 at $E = 0$ but goes to the power scaling at high energies with $E \gg G$. With this flux, the transform is analytic and it reads [11]

$$\tilde{N}(s,0) = N_0(\frac{A}{G})^{-(s+1)}B(s+1, \gamma - s) = N_0(\frac{A}{G})^{-(s+1)}\frac{\Gamma(s+1)\Gamma(\gamma - s)}{\Gamma(\gamma+1)} \quad (29)$$

where $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the beta function.

4 The Calculus of Residues

Taking the inversion, the nucleon flux in real space is expressed as

$$
\begin{aligned}
N(E,t) &= \frac{1}{2\pi i}\int (\frac{E}{A})^{-(s+1)}\tilde{N}(s,t)\,ds = \frac{1}{2\pi i}\int (\frac{E}{A})^{-(s+1)}\tilde{N}(s,0)\,e^{-\mu(s)t/\lambda_N}\,ds \\
&= \frac{N_0}{2\pi i}(\frac{E}{G})^{-(\gamma+1)}\int (\frac{E}{G})^{\gamma-s}\frac{\Gamma(s+1)\Gamma(\gamma-s)}{\Gamma(\gamma+1)}\,e^{-\mu(s)t/\lambda_N}\,d\,s
\end{aligned}
\tag{30}
$$

The flux is written in such a way that it contains explicitly the factor $(E/G)^{-(\gamma+1)}$ in front of the integral. This factor gives the $(\gamma+1)$ slope on a logarithmic plot. The integral gives the amplitude of the residues. The integration contour is a straight line parallel to the vertical imaginary axis cutting the horizontal real axis at $Re(s) > 0$ such that all the singularities lie to the left of the contour. Customarily, there are two ways to evaluate this integral. The first one is the saddle point method[12] by writing the whole integrant into an analytic function $F(s)$ to the exponent reading $\int e^{F(s)}\,ds$. Making use of the liberty in positioning the integration contour, $Re(s)$ is chosen such that $F(s)$ is stationary at $(Re(s),0)$ to evaluate approximately the integral on this saddle point. The second one is the residue method where the simple pole $s=\gamma$ is picked up by using the Cauchy theorem. Unfortunately, there is the singularity $s=s_0$ in $A(s)$. The function $F(s)$ is, therefore, not analytic. This introduces an essential pole to the residue method and makes these evaluations incomplete.

For this fundamental reason, we choose to evaluate Eq.(30) by the method of residues including the essential one. We note that the Gamma function $\Gamma(z)$ is analytic over the half plane of $Re(z) > 0$ but has simple poles at $z=-\ell$ where ℓ is a positive integer. According to Eq.(15), there are two groups of simple singularities. The first group is at $s=\gamma+\ell$, and the second group is at $s=-1-\ell$ that comes from our regularization of the incident flux at $t=0$. These two residues are given by

$$
\begin{aligned}
Res(\gamma) &= N_0(\frac{E}{G})^{-(\gamma+1)}\{\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}(\frac{E}{G})^{-\ell}\frac{\Gamma(\gamma+\ell+1)}{\Gamma(\gamma+1)}e^{-\mu(\gamma+\ell)t/\lambda_N}\} \\
&= N_0(\frac{E}{G})^{-(\gamma+1)}F_1(E,t)
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
Res(-1) &= N_0(\frac{E}{G})^{-(\gamma+1)}\{(\frac{E}{G})^{(\gamma+1)}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}(\frac{E}{G})^{+\ell}\frac{\Gamma(\gamma+\ell+1)}{\Gamma(\gamma+1)}e^{-\mu(-\ell-1)t/\lambda_N}\} \\
&= N_0(\frac{E}{G})^{-(\gamma+1)}F_2(E,t)
\end{aligned}
\tag{32}
$$

We notice that the series in Eq.(31) is a convergent series in energy when $(E/G) > 1$ due to the $(E/G)^{-\ell}$ factor under which the residue is well defined, and contrary outside this energy range. On the other hand, the residue in Eq.(32) is well defined when $(E/G) < 1$ due to the $(E/G)^{+\ell}$ and $(E/G)^{\gamma+1}$ factors. The nucleon flux over the entire energy range is consisted of $Res(-1)$ for $(E/G) < 1$ and $Res(\gamma)$ for $(E/G) > 1$.

By expanding the exponential function in power series, the essential residue is described by

$$
\begin{aligned}
Res(s_0) = &\frac{N_0}{2\pi i}(\frac{E}{G})^{-(\gamma+1)}\{(\frac{E}{G})^{\gamma}\,e^{-(A/B)^\beta t/\lambda_N}\sum_{n=0}^{\infty}\frac{1}{n!}((\frac{A}{B})^\beta\frac{t}{\lambda_N}\frac{1}{\delta})^n \\
&\int [(\frac{G}{E})(\frac{A}{K})^{n\kappa}]^s\frac{\Gamma(s+1)\Gamma(\gamma-s)}{\Gamma(\gamma+1)}\frac{1}{(s-s_0)^n}\,ds\}
\end{aligned}
\tag{33}
$$

We define two functions $G(s) = [(G/E)(A/K)^{n\kappa}]^s = g^s$ and $V(s) = \Gamma(s+1)\Gamma(\gamma-s)/\Gamma(\gamma+1)$ where the first one originates from elasticity and the second one represents the incident power spectrum at the top of the atmosphere. Since $G(s)$ is analytic in the neighborhood of $s = s_0$, we expand it in a Laurent series about $s = s_0$ so that $Res(s_0)$ becomes

$$\begin{aligned} Res(s_0) &= \frac{N_0}{2\pi i}(\frac{E}{G})^{-(\gamma+1)} \{(\frac{E}{G})^{\gamma} e^{-(A/B)^{\beta}t/\lambda_N} \\ & \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{1}{n!}\frac{1}{m!}((\frac{A}{B})^{\beta}\frac{t}{\lambda_N}\frac{1}{\delta})^n G^{(m)}(s_0) \int \frac{V(s)}{(s-s_0)^{n-m}} ds\}. \end{aligned} \tag{34}$$

By taking $(n-m) = +1$ terms, we pick up the contributions to the essential residue so that

$$\begin{aligned} Res(s_0) &= N_0(\frac{E}{G})^{-(\gamma+1)}\{(\frac{E}{G})^{\gamma-s_0} e^{-(A/B)^{\beta}t/\lambda_N} V(s_0) \\ & [(\frac{A}{K})^{\kappa s_0}((\frac{A}{B})^{\beta}\frac{t}{\lambda_N}\frac{1}{\delta})] \sum_{n=1}^{\infty} \frac{1}{n!}\frac{1}{(n-1)!}[\frac{Z^2(n)}{4}]^{n-1}\} \end{aligned} \tag{35}$$

where $Z^2(n)/4 = (A/K)^{\kappa s_0}(A/B)^{\beta}(t/\lambda_N)(1/\delta)(\ln g)$. For $E < G$, $\ln g$ is positive, and the essential residue reads

$$\begin{aligned} Res(s_0) &= N_0(\frac{E}{G})^{-(\gamma+1)}\{(\frac{E}{G})^{\gamma-s_0} e^{-(A/B)^{\beta}t/\lambda_N} V(s_0) \\ & [(\frac{A}{K})^{\kappa s_0}((\frac{A}{B})^{\beta}\frac{t}{\lambda_N}\frac{1}{\delta})] \frac{2}{Z(n)} I_1(Z(n)) \end{aligned} \tag{36}$$

For $E > G$, $\ln g$ is negative, and the summation is over an alternating series with the flux given by

$$\begin{aligned} Res(s_0) &= N_0(\frac{E}{G})^{-(\gamma+1)}\{(\frac{E}{G})^{\gamma-s_0} e^{-(A/B)^{\beta}t/\lambda_N} V(s_0) \\ & [(\frac{A}{K})^{\kappa s_0}((\frac{A}{B})^{\beta}\frac{t}{\lambda_N}\frac{1}{\delta})] \frac{2}{Z(n)} J_1(Z(n)) \end{aligned} \tag{37}$$

where the argument $Z^2(n)/4$ is now calculated by $|(\ln g)|$ and $J_1(Z)$ is the Bessel function of order one. Due to the $(E/G)^{\gamma-s_0}$ factor where $s_0 < 0$, $Res(s_0)$ is well defined for $(E/G) < 1$. For $(E/G) > 1$, the factor $(E/G)^{\gamma-s_0}$ makes $Res(s_0)$ a rapidly increasing function of E. This, by itself, does not make $Res(s_0)$ nonanalytic in the domain $(E/G) > 1$. However, for an increasing flux in energy E, the integrated flux would be divergent at any given energy E which makes it nonanalytic. For this reason, the essential residue is defined in $(E/G) < 1$ only.

The Bessel function solutions of the essential residue make contact with the single nucleon case, and provide analytic understanding of the qualitative behavior of the essential residue. Nevertheless, these analytic solutions in closed form are derived by isolating the incident spectrum function $V(s)$ from the elasticity function $G(s)$. When taking the residue, $V(s)$ is evaluated at s_0. This amounts to a simplification that over estimates the flux. To evaluate the residue correctly, we need to absorb $V(s)$ into $G(s)$ by writing $G(s) = [(G/E)(A/K)^{n\kappa}]^s V(s) = g^s V(s)$. Expanding this redefined $G(s)$ in a Laurent series about $s = s_0$, and taking $(n-m) = +1$ terms, the essential residue and the nucleon flux read

$$\begin{aligned} Res(s_0) &= N_0(\frac{E}{G})^{-(\gamma+1)}\{(\frac{E}{G})^{\gamma} e^{-(A/B)^{\beta}t/\lambda_N} \sum_{n=1}^{\infty} \frac{1}{n!}\frac{1}{(n-1)!}((\frac{A}{B})^{\beta}\frac{t}{\lambda_N}\frac{1}{\delta})^n G^{n-1}(s_0)\} \\ &= N_0(\frac{E}{G})^{-(\gamma+1)} F_3(E,t) \end{aligned} \tag{38}$$

$$N(E,t) = \quad Res(-1) + Res(s_0) = N_0(\frac{E}{G})^{-(\gamma+1)}[F_2(E,t) + F_3(E,t)], \quad E < G \tag{39}$$

$$N(E,t) = \quad Res(\gamma) = N_0(\frac{E}{G})^{-(\gamma+1)}[F_1(E,t)], \quad E > G \tag{40}$$

For the calculus of the charged pion residues we will use the same procedure as the nucleon case. The pion production function with $d = 4$ can be integrated to give

$$\phi(s) = D \sum_{n=0}^{4} (-1)^n C_n^4 \frac{1}{(s+n)}. \tag{41}$$

With the boundary condition $\tilde{\Pi}(s,0) = 0$, and using Eq.(21) for $\tilde{N}(s,t)$, the solution of Eq.(24) is given by

$$\begin{aligned}
\tilde{\Pi}(s,t) &= \frac{Q(s)/\lambda_N}{[\Pi(s)/\lambda_\pi - (1-A(s))/\lambda_N]}\{e^{-(A/B)^\beta(1-A(s))t/\lambda_N} - e^{-(A/B)^\beta \Pi(s)t/\lambda_\pi}\}\tilde{N}(s,0) \\
&= Z(s)(s-s_0)\{e^{-(A/B)^\beta(1-A(s))t/\lambda_N} - e^{-(A/B)^\beta \Pi(s)t/\lambda_\pi}\}\tilde{N}(s,0).
\end{aligned} \tag{42}$$

We have written $(Q(s)/\lambda_N)/[\Pi(s)/\lambda_\pi - (1-A(s))/\lambda_N] = Z(s)(s-s_0)$ so that the $(s-s_0)$ dependence, where $s_0 \neq -1$, appears explicitly, and $Z(s)$ is an analytic function of s that contains no singularities. The charged pion flux in real space is given by the inverse transform

$$\begin{aligned}
\Pi(E,t) &= \frac{1}{2\pi i}\int (\frac{E}{A})^{-(s+1)}\tilde{\Pi}(s,t)\, ds \\
&= \frac{N_0}{2\pi i}(\frac{E}{G})^{-(\gamma+1)}\int (\frac{E}{G})^{\gamma-s} Z(s)\frac{\Gamma(s+1)\Gamma(\gamma-s)}{\Gamma(\gamma+1)}(s-s_0)e^{-(A/B)^\beta[1-A(s)]t/\lambda_N}\, ds \\
&\quad -\frac{N_0}{2\pi i}(\frac{E}{G})^{-(\gamma+1)}\int (\frac{E}{G})^{\gamma-s} Z(s)\frac{\Gamma(s+1)\Gamma(\gamma-s)}{\Gamma(\gamma+1)}(s-s_0) \\
&\quad e^{(A/B)^\beta (A/A_\Pi)^{\alpha-\alpha' s}\phi(s)t/\lambda_\pi} e^{-(A/B)^\beta[1-(1-b)A(s)]t/\lambda_\pi}\, ds.
\end{aligned} \tag{43}$$

The first integral has two simple poles $s = -1-\ell$ and $s = \gamma + \ell$ from the incident regularized power spectrum, and an essential pole $s = s_0$ on the exponent. The second integral contains the same $s = s_0$ essential pole plus five more essential poles $s = -n$ with $n = 0, 1, 2, 3, 4$, and the $s = \gamma + \ell$ simple pole. The $s = -1-\ell$ simple pole is practically absorbed by the $s = -n$ essential poles. Evaluating the residues, the charged pion fluxes for $E < G$ and $E > G$ are respectively given by

$$\begin{aligned}
\Pi(E,t) &= Res(-1) + Res(s_0) + Res(-n) \\
&= N_0(\frac{E}{G})^{-(\gamma+1)}[\Pi_2(E,t) + \Pi_{30}(E,t) + \Pi_{3n}(E,t)] \quad E < G \ ,
\end{aligned} \tag{44}$$

$$\Pi(E,t) = Res(\gamma) = N_0(\frac{E}{G})^{-(\gamma+1)}[\Pi_1(E,t)] \ \ E > G. \tag{45}$$

The functions in Eqs.(44,45) are defined as follows

$$\Pi_1(E,t) = \sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}(\frac{E}{G})^{-\ell} Z(\gamma+\ell)\frac{\Gamma(\gamma+\ell+1)}{\Gamma(\gamma+1)}(\gamma+\ell-s_0)$$

$$[e^{-(A/B)^\beta(1-A(\gamma+\ell))t/\lambda_N} - e^{-(A/B)^\beta P(\gamma+\ell)t/\lambda_\pi}], \quad (46)$$

$$\Pi_2(E,t) = (\frac{E}{G})^{(\gamma+1)} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} (\frac{E}{G})^{+\ell} Z(-\ell-1) \frac{\Gamma(\gamma+\ell+1)}{\Gamma(\gamma+1)} (-\ell-1-s_0)$$

$$[e^{-(A/B)^\beta(1-A(-\ell-1))t/\lambda_N} - e^{-(A/B)^\beta P(-\ell-1)t/\lambda_\pi}]. \quad (47)$$

As for the essential residue $s = s_0$, recalling $G(s) = [(G/E)(A/K)^{n\kappa}]^s V(s) = g^s V(s)$, we define two functions $H(s) = Z(s)G(s)$ and $K(s) = Y(s)Z(s)G(s)$ where $Y(s) = e^{(A/B)^\beta Q(s)t/\lambda_\pi}$. With these functions, we get

$$\Pi_{30}(E,t) = (\frac{E}{G})^\gamma e^{-(A/B)^\beta t/\lambda_N} \sum_{n=2}^{\infty} \frac{1}{n!} \frac{1}{(n-2)!} ((\frac{A}{B})^\beta \frac{t}{\lambda_N} \frac{1}{\delta})^n H^{n-2}(s_0)$$

$$-(\frac{E}{G})^\gamma e^{-(A/B)^\beta t/\lambda_\pi} \sum_{n=2}^{\infty} \frac{1}{n!} \frac{1}{(n-2)!} ((\frac{A}{B})^\beta (1-b) \frac{t}{\lambda_\pi} \frac{1}{\delta})^n K^{n-2}(s_0) \ . \quad (48)$$

Other essential residues $s = -n$ can be obtained in a similar manner by defining the corresponding $Y(s)$.

5 Numerical Results

For the case of a power spectrum incident flux, we notice that the traditional calculation with the incorrect transform of the boundary condition, Eq.(27), gives

$$N^*(E,t) = N_0^*(\frac{E}{G})^{-(\gamma+1)} e^{-\mu(\gamma)t/\lambda_N} = N_0^*(\frac{E}{G})^{-(\gamma+1)} F^*(E,t). \quad (49)$$

We have used superscript $^{(*)}$ to distinguish the corresponding parameters of Eqs.(35) calculated with Eq.(29). It is noted numerically that $F(E,t)$ goes asymptotically to $F^*(E,t)$ at high energies compatible to Eq.(28) which goes to Eq.(25) at high energy limit. Nevertheless, $F(E,t)$ and $F^*(E,t)$ are measured in units of N_0 and N_0^* respectively, and these two units are very different. We recall that Eq.(25) diverges at $E = 0$ and $N_0^* = N^*(G,0)$ is the flux at $E = G$. The normalization energy G in the divergent incident power spectrum, Eq.(25), can always be chosen arbitrarily because the coefficient N_0^* can be defined with respect to G according to $N_0^* = N^*(G,0)$. On the other hand, Eq.(28) is regular at $E = 0$ with $N_0 = N(0,0)$. In this spectrum, G can no longer be arbitrary since it defines the energy where the spectrum begins to level off at low energies. At $E = G$, we have $2^{-(\gamma+1)}N_0 = N(G,0)$.

To do numerical calculations of residues and consequently differential nucleon fluxes at sea level, we take $B = 20\,GeV$, $K = 1000\,GeV$ and $G = 1\,GeV$. We also take $k = -0.04$ and $\delta = 1.5$, which means a growing nucleon inelasticity coefficients between 0.53 and 0.65, with the energy ranges from $100\,GeV$ to $40\,TeV$. In order to compare this calculation with experimental data we use for the nucleon mean free path in the atmosphere $\lambda_N = 96.40\,g/cm^2$ and $\beta = 0.027$ [13] which are parametrized in the Eq.(2), and for the power index of the primary sprectrum we used $\gamma = 1.62$ [14]. Figure 1 shows the comparison of our calculations with the differential nucleon flux measured at sea level[15, 16], $t = 1030\,g/cm^2$. We see that when the mean inelasticity $1- < \eta >$ (Eq. (8)) is in the range mentioned above our solutions fit very well the experimental data.

In order to calculate the differential hadron flux we need to take in addition some parameters which describe the pion production in this energy range. Taking into account the isospin symmetry we use then $b = 1/3$, which appears in the Eq.(10). We use the values $d = 4$ and $D = (d+1)/3 = 5/3$ for the parametrization of the energy distribution of secondary pions from hadron-air interactions, Eq.(13). Considering the same energy dependence for nucleon and pion interaction lengths we use $\lambda_\Pi/\lambda_N = 1.4$ [17]. We call the attention that we are using the same growing with the energy for both nucleon and pion inelasticity coefficients. In Figure 2, our calculations for the hadronic fluxes are compared with the experimental ones obtained from the (a) EASTOP [18] and (b) KASKADE [19] collaborations at $t = 840\, g/cm^2$ and $t = 1030\, g/cm^2$, respectively. We see that our calculations reproduce quite well the experimental data when these coefficients are taken between 0.53 and 0.65 as in the nucleon case. This results confirm an estimative made in a recent work [20] where the mean nucleon inelasticity coefficient was taken constant, but showing a crescent trend.

6 Conclusions

To conclude, we have reconsidered the use of analytic methods to establish the physics of cosmic ray cascades. We have introduced the method of characteristics to solve the partial differential equation of the transformed flux with energy dependent mean free path and elasticity distribution. Special attention is given to the analyticity of the incident spectrum at the top of the atmosphere which enters as the boundary condition. According to this method, the flux transform remains in a separable form of its variables. The nucleon flux in real space is evaluated by the method of residues. For the case of a regularized power law incident spectrum, the flux is given by the simple residues $(s-\gamma) = 0$ and $(s+1) = 0$, and the essential residue $(s - s_0) = 0$. Interpretation of the characteristics of the partial differential equation of the flux transform is proposed in a manner analogous to the functional analysis of Landau and Rumer [21]. This method is also extented to evaluate charged pion flux. In this case, the differential pion flux is given by simple residues $(s-\gamma) = 0$ and $(s+1) = 0$ and the essential residues $s = s_0$ and $s = -n$ with $n = 0, 1, 2, 3, 4$. The $(s+1) = 0$ simple residue comes from the regularization of the incident primary nucleon spectrum at low energies, and the essential ones appears as a consequence of the analytical treatment used here and we would like to stress that they were never been considered. We also have shown, in the last section, that our solutions describe very well the experimental fluxes both for nucleons and hadrons. Finally, we would like to mention that this method allow us to include, besides the energy dependence of the hadron interaction lengths, the energy dependence of the inelasticity coefficients and the breaking of the scaling law for the hadronic interacions. Thus, this approach becomes a very powerful instrument for modelling cascade theory in a very broad energy range.

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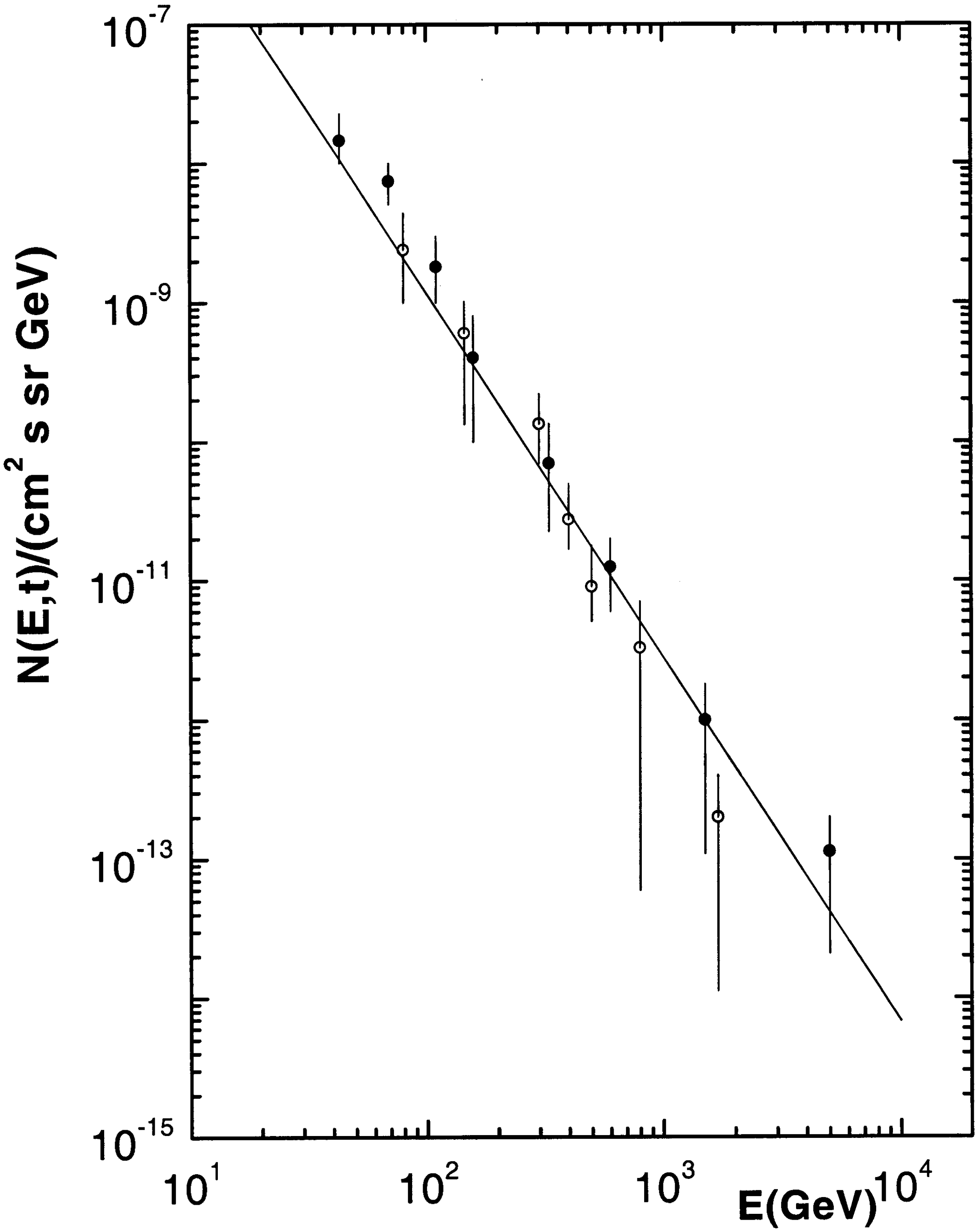

Figure 1: Our numerically calculated nucleon flux $N(E,t)$ is plotted against the energy, together with experimental data from [15] (full circle) and from [16] (open circle) at sea level, atmospheric depth $t = 1030\, g/cm^2$. See text for detailed procedures

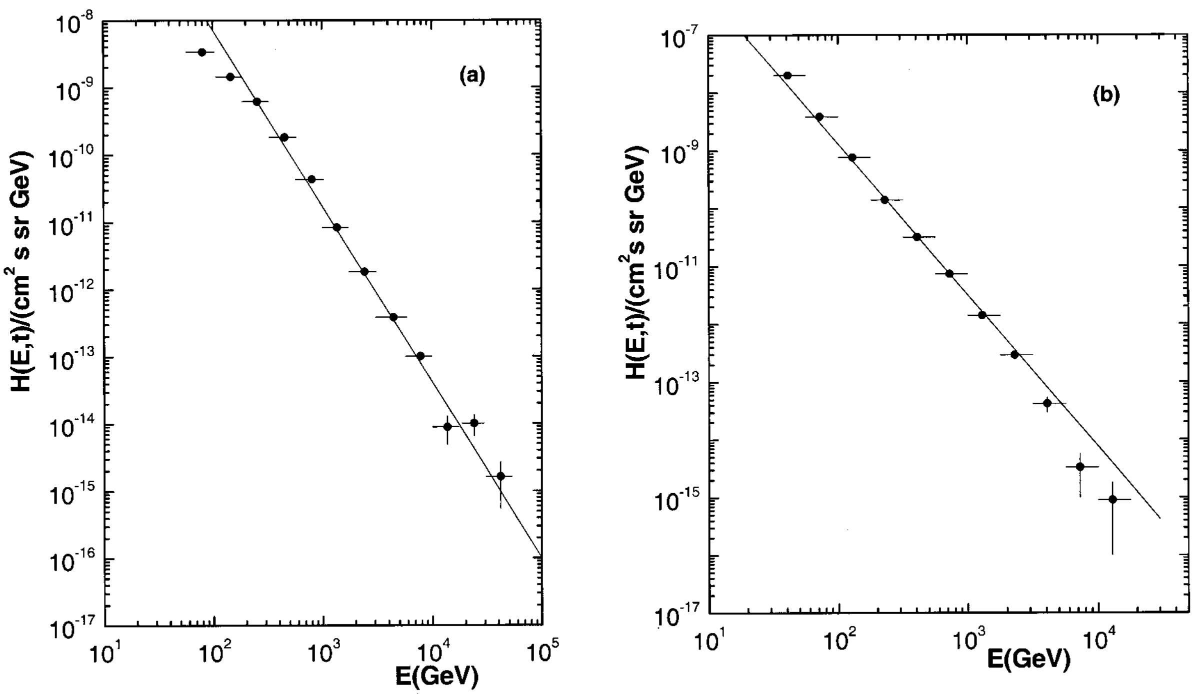

Figure 2: Our numerically calculated hadron flux $H(E,t)$ is plotted against the energy, (a) with data from EASTOP experiment [18] $t = 840\,g/cm^2$ and (b) with data from KASCADE experiment [19] $t = 1030\,g/cm^2$. See text for detailed procedures.

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