

q -DEFORMED KINK SOLUTIONS

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Abstract

The q -deformed kink of the $\lambda\phi^4$ -model is obtained via the normalisable ground state eigenfunction of a fluctuation operator associated with the q -deformed hyperbolic functions. The kink mass, the bosonic zero-mode and the q -deformed potential in 1+1 dimensions are found.

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I. INTRODUCTION

The kink of a scalar potential in 1+1 dimensions is a static, non-singular, *classically stable* and a finite localized energy solution of the equation of motion, which can be in topologically stable sectors [1]. In a recent lecture [2], an investigation on the topological defects starting with the simplest case of domain walls was presented, and then considerations to more elaborate and realistic models were put forward.

In the present letter, one works with the algebraic technique of the SUSY QM formulated by Witten [3–5], which is associated with a second order differential equation for the q -deformed hyperbolic functions [6]. Recently, the q -deformed hyperbolic function was used to construct a new η -pseudo-Hermitian complex potential with PT symmetry [7]. Other potentials like Rosen-Morse well, Scarf, Eckart and the generalized Pöschl-Teller were constructed via shape invariance [8].

The stability equation for topological and non-topological solitons has been approached in the framework of supersymmetric quantum mechanics (SUSY QM) [9–14]. The marginal stability and the metamorphosis of Bogomol’nyi-Prasad-Sommerfield (BPS) states have been investigated, via SUSY QM, and presented a detailed analysis for a 2-dimensional $N = 2$ -Wess-Zumino model with two chiral superfields, and composite dyons in $N = 2$ -supersymmetric gauge theories [15].

In this letter, the interesting program of proposing a new potential model in 1+1 dimensions, whose essential point is associated with the translational invariance of the q -deformed kink solutions, is investigated.

II. SOLITONS IN 1+1 DIMENSIONS

Consider the Lagrangian density for a single scalar field, $\phi(x, t)$, in (1+1)-dimensions, in natural system, given by

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi), \quad (1)$$

where $V(\phi)$ is any positive semi-definite function of ϕ , which must have at least two zeroes for soliton solution to exist. It represents a well-behaved potential energy. However, as it will be shown below, we have found a new potential which is exactly solvable in the context of the classical theory in (1+1)-dimensions.

The field equation for a static classical configuration, $\phi = \phi_c(x)$, becomes

$$-\frac{d^2}{dx^2} \phi_c(x) + \frac{d}{d\phi_c} V(\phi_c) = 0, \quad \dot{\phi}_c = 0, \quad (2)$$

with the following boundary conditions: $\phi_c(x) \rightarrow \phi_{vacuum}(x)$ as $x \rightarrow \pm\infty$.

Since the potential is positive, it can be written as

$$V(\phi) = \frac{1}{2}U^2(\phi). \quad (3)$$

Thus, the total energy for the q -kink becomes

$$\begin{aligned} E &= \frac{1}{2} \int [(\phi')^2 + U^2] dx \\ &= \frac{1}{2} \int [(\phi' \mp U)^2 \pm 2U\phi'] dx. \end{aligned} \quad (4)$$

In this case, the Bogomol'nyi form of the energy, consisting of a sum of squares and surface terms, becomes

$$E \geq \left| \int dx \frac{\partial}{\partial x} U[\phi(x)] \right|, \quad (5)$$

under the well-known Bogomol'nyi condition for the kink solution,

$$\frac{d\phi}{dx} = \pm U(\phi) \quad (6)$$

where the solutions with the plus and minus signs represent two static field configurations.

III. STABILITY EQUATION

The classical stability of the soliton solution is investigated by considering small perturbations around it,

$$\phi(x, t) = \phi_c(x) + \eta(x, t), \quad (7)$$

where we expand the fluctuations in terms of the normal modes,

$$\eta(x, t) = \sum_n \epsilon_n \eta_n(x) e^{i\omega_n t}, \quad (8)$$

with the ϵ_n 's chosen so that $\eta(x, t)$ is real. A localized classical configuration is said to be dynamically stable if the fluctuation does not destroy it. The equation of motion becomes a Schrödinger-like equation, viz.,

$$O_F \eta_n(x) = \omega_n^2 \eta_n(x), \quad O_F = -\frac{d^2}{dx^2} + V''(\phi_c), \quad (9)$$

where O_F is the fluctuation operator. According to (3), one obtains the supersymmetric form [12,13]

$$V''(\phi_c) = U'^2(\phi_c) + U(\phi_c)U''(\phi_c), \quad (10)$$

where the primes stand for a second derivative with respect to the argument.

If the normal modes of (9) satisfy $\omega_n^2 \geq 0$, the stability of the Schrödinger-like equation is ensured. Now, we are able to implement a method that provides a new potential from the potential term that appears in the fluctuation operator.

Next, we consider the following generalized potential as corresponding to the potential part of the fluctuation operator:

$$V''(\phi_c) = V(x; q) = m^2(2 - 3q) \operatorname{sech}_q^2 \left(\frac{m}{\sqrt{2}} x \right), \quad (11)$$

where $q > 0$ and we are using the q -deformed hyperbolic functions which were introduced by Arai [6]:

$$\begin{aligned} \cosh_q(x) &= \frac{e^x + qe^{-x}}{2} \\ \sinh_q(x) &= \frac{e^x - qe^{-x}}{2} \\ \tanh_q(x) &= \frac{\sinh_q(x)}{\cosh_q(x)} \\ \operatorname{sech}_q(x) &= \frac{1}{\cosh_q(x)} \end{aligned} \quad (12)$$

where $x \in \mathbf{R}$. Thus

$$\begin{aligned} \frac{d}{dx} \cosh_q(x) &= \sinh_q(x) \\ \frac{d}{dx} \sinh_q(x) &= \cosh_q(x) \\ \frac{d}{dx} \tanh_q(x) &= q \operatorname{sech}_q^2(x) \\ \frac{d}{dx} \operatorname{sech}_q(x) &= -\tanh_q(x) \operatorname{sech}_q(x) \\ \tanh_q^2(x) + q \operatorname{sech}_q^2(x) &= 1. \end{aligned} \quad (13)$$

The q -deformed potential term provides a fluctuation operator, so that their eigenvalues satisfy the condition $\omega_n^2 \geq 0$, and the ground state associated to the zero mode ($\omega_0^2 = 0$) is given by

$$\eta^{(0)}(x; q) = N \operatorname{sech}_q^2 \left(\frac{m}{\sqrt{2}} x \right), \quad (14)$$

where N is the normalization constant. Thus, the stability of the Schrödinger-like equation is ensured.

The potential model we are now going to study presents translational invariance, then, the bosonic zero-mode eigenfunction of the stability equation is related with the kink by

$$\phi_q(x) = \int_0^x \eta^{(0)}(y; q) dy, \quad (15)$$

so that, a priori, we may find the static classical configuration by a first integration. Therefore, the potential model

$$V(\phi; q) = \frac{1}{2} \left(\frac{d}{dx} \phi(x; q) \right)^2 \quad (16)$$

yields a class of q -deformed scalar potentials, $V(\phi) = V(\phi; q)$, which have exact solutions.

Expressing the position coordinate in terms of the kink, i.e. $x = x(\phi_k)$, then, from (14) and (15) we obtain the q -deformed kink

$$\phi(x; q) = \frac{m}{\lambda q^2} \tanh_q \left(\frac{m}{\sqrt{2}} x \right). \quad (17)$$

The explicit form of the q -kink for few values of q is shown in Fig. 1.

From Eqs. (16) and (17), we find a q -deformed ϕ^4 -potential model with spontaneously broken symmetry in scalar field theory, viz.,

$$V(\phi; q) = \frac{\lambda^2}{4q^4} \left(q^2 \phi^2 - \frac{m^2}{\lambda^2} \right)^2. \quad (18)$$

It represents a well-behaved potential energy. Note that the q -deformed ϕ^4 model has a discrete symmetry as $\phi \rightarrow -\phi$ but it is spontaneously broken for the vacuum state by the existence of two degenerate vacua:

$$\phi_1 = \frac{m}{q\lambda}, \quad \phi_2 = -\frac{m}{q\lambda}. \quad (19)$$

The fact that the energy is finite is ensured because the kink by the behaviour of the approaches one of the vacuum solutions at $\pm\infty$. In the q -deformed ϕ^4 model there are four topological sectors, which are represented by two spaces Γ_1 and Γ_2 containing the q -deformed vacuum solutions ϕ_1 and ϕ_2 and two spaces Γ_3 and Γ_4 containing the kink and the antikink solutions.

The energy density of the q -kink is given by

$$\epsilon(x) = \frac{1}{2} [(\phi')^2 + U^2] = \frac{m^4}{2q^2\lambda^2} \operatorname{sech}_q^4 \left(\frac{m}{\sqrt{2}} x \right). \quad (20)$$

Therefore, the kink mass or the so-called classical mass of the pseudoparticle is given by

$$M_{cl} = \int_{-\infty}^{+\infty} \epsilon(x) dx = \frac{2}{3} \left(\frac{m^3 \sqrt{2}}{\lambda^2 q^4} \right) \quad (21)$$

which is dependent of q . Note that when $q = 1$ the undeformed case is re-obtained. In figure 2, we plot the energy density given by Eq.(20), for few values of q .

The conserved topological current becomes:

$$\mathcal{J}_\mu = \frac{1}{2} \epsilon_{\mu\nu} \partial^\nu \tilde{\phi}_q, \quad \tilde{\phi}_q = \frac{m}{q\lambda} \phi_q, \quad \partial^\mu \mathcal{J}_\mu = 0, \quad (22)$$

where $\epsilon_{\mu\nu}$ is the antisymmetric tensor in two dimensions $\epsilon_{01} = -\epsilon_{10} = 1$ and is zero for the case with repeated index. The kink number or conserved topological charge is given by

$$Q = \int_{-\infty}^{+\infty} \mathcal{J}_0 dx = \frac{1}{2} \left[\lim_{x \rightarrow +\infty} \tilde{\phi}_q(x) - \lim_{x \rightarrow -\infty} \tilde{\phi}_q(x) \right], \quad (23)$$

which does not generate symmetries of the Lagrangian density and, therefore, Q is not a Noether charge. However, this charge is absolutely conserved, $\frac{d}{dt} Q = 0$, so that the q -kink represents stable particlelike states. Thus, the q -kink states can not decay by quantum tunneling into the vacuum.

From the q -deformed potential, one then obtains the supersymmetric form

$$V_-(x; q) = W_q^2(x) + W_q', \quad (24)$$

where the prime mean a first derivative with respect to the argument, and $W_q(x) = -U_q'(\phi_k)$ is the q -deformed superpotential associated to the q -kink solution. Thus, the bosonic and fermionic sector fluctuation operator are respectively given by

$$\begin{aligned} O_{F-} &= -\frac{d^2}{dx^2} + W_q^2 - W_q' \\ O_{F+} &= -\frac{d^2}{dx^2} + W_q^2 + W_q', \end{aligned} \quad (25)$$

where $W_q(x) = -2m \tanh_q \left(\frac{m}{\sqrt{2}} x \right)$. These fluctuation operators are also called the supersymmetric partners, which are isospectral up to the ground state. The shape invariance condition of the pair of SUSY partner potential will be investigated in a forthcoming paper.

In conclusion, we can say that, starting from a potential $V(x; q)$ in terms of the q -deformed hyperbolic functions in the stability equation, we construct the q -deformed topological kink associated to the q -deformed ϕ^4 potential model. We shown that the q -kink mass is dependent of q . We stress that a very rich spectrum of the states (the q -kink and the quantum excitations about them), which was totally unexpected in this

model has emerged because of the existence of soliton solutions. Finally, it is important to pointed out that one can extend our approach to 3+1 dimensions. Indeed, the present work opens a new route for future investigations on domain walls [2] from q -deformation of potential model in terms of coupled scalar fields. For instance, let us point out that our approach can be applied from two [15,16] and three [17] coupled scalar fields in higher dimensions, where in both cases the q -soliton solutions only depend on z but not on x and y [18].

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FIGURES

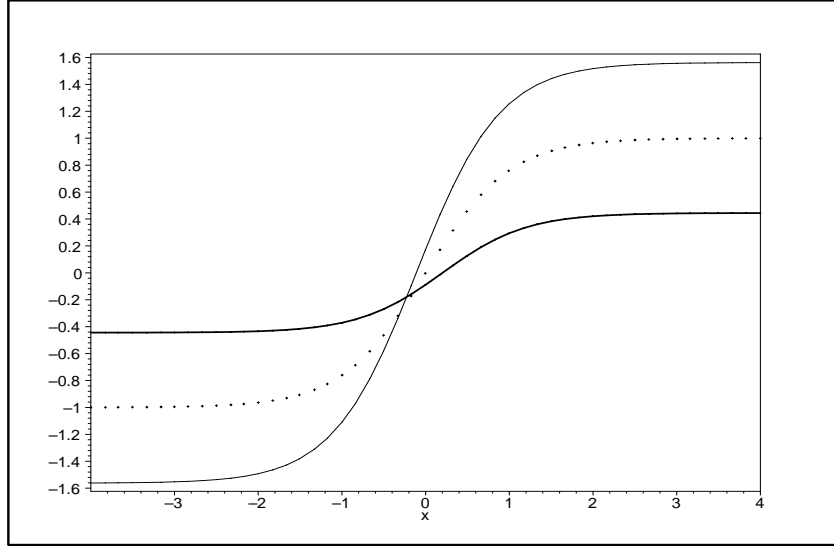


FIG. 1. The q -deformed kink profile, with $q = 0.8$ (thickness=1), $q = 1.0$ (dotted curve), and $q = 3.0$ (thickness=3), respectively, for $m = \lambda = \sqrt{2}$.

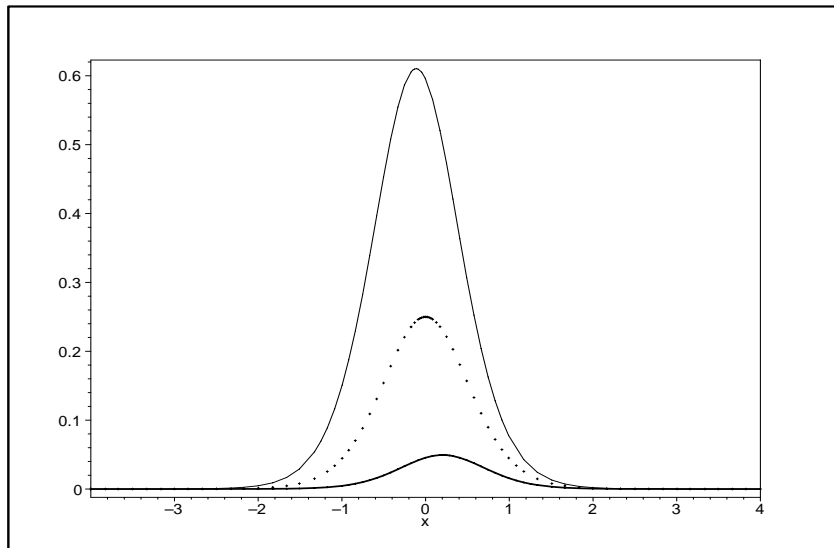


FIG. 2. The energy density given by Eq.(20), with $q = 0.8$ (thickness=1), $q = 1.0$ (dotted curve), and $q = 3.0$ (thickness=3), respectively, for $m = \lambda = \sqrt{2}$.