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POISSON STATISTICS AND THE DIFFUSION-EQUATION OF NUCLEONS AND CHARGED PIONS IN THE ATMOSPHERE

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ABSTRACT

In this paper we give the solution of the simplified unidirectional diffusion equations of nucleons and charged pions in the atmosphere for a general distribution of the primary component in the top of the atmosphere. We obtain further the Poisson distribution law for the nucleon's interactions with the air nuclei, as a consequence of the respective diffusion equation. The average interaction length λ and the average inelasticity K are supposed to be constant and independent of the nucleon incident energy.

1 - The differential energy spectrum of the primary cosmic ray nucleons at the atmosphere depth x (g/cm^2) may be obtained with an elegant physical reasoning used by Y. Pal and B. Peters |1| supposing the primary nucleon energy spectrum in the top of the atmosphere to be represented approximately by a power function

$$F_N(0,E) = N_o E^{-(\gamma+1)}$$

G. Brooke, P.J. Haymann, Y. Kamya and A.W. Wolfendale [2] considered the case of a primary spectrum of the general form*:

$$F_N(0,E) = G(E) \qquad (1)$$

If we suppose the average inelasticity K_N and the average interaction length λ_N to be constants, then a nucleon that makes n interactions in traversing x(g/cm^2) of the atmosphere will have its energy reduced from $E' = E/(1-K_N)^n$ to E, so that the elementary energy contribution of the primary energy spectrum to the x level differential energy spectrum is given by

$$G(E')dE' = G(E/(1-K_N)^n)dE/(1-K_N)^n \qquad (2)$$

Now assuming "a priori" that the probability of a nucleon making n interactions is given by the Poisson distribution

$$p(n) = e^{-m} m^n/n! \quad \text{with } m = \frac{x}{\lambda_N}$$

G. Brooke, et al. obtained for the total intensity at the atmosphere depth x, the following expression

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* Our G(E) corresponds to $N_p(E)$ of G. Brooke, et al.

$$F_N(x,E) = e^{-x/\lambda_N} \sum_{n=0}^{\infty} \frac{(x/\lambda_N)^n}{n!} \frac{1}{(1-K_N)^n} G(\frac{E}{(1-K_N)^n}) \qquad (3)$$

This reasoning presuposes that the function G(E) is a reasonable good function to ensure the convergence of the series.

2. Symbolic Method to Solve the Diffusion Equation for the Nucleons

Now we shall obtain the expression (3) integrating the differential equation [3] [4]

$$\frac{\partial F(x,E)}{\partial x} = - \frac{1}{\lambda} F(x,E) + \frac{1}{\lambda(1-K)} F(x,\frac{E}{1-K}) \quad * \qquad (4)$$

with the initial condition

$$F(0,E) = G(E) . \qquad (5)$$

The function G(E) is supposed to be, non negative, continuous and bounded in the interval $I = [a,\infty)$ $a \geq 0$, (for the existence of the integral spectrum we must also suppose the existence of $\int_E^\infty G(E)dE$). The mean inelasticity $\underline{K}$ and the mean interaction length $\underline{\lambda}$ are supposed to be constants.

For obtaining the solution of (4) we use the following "symbolic method":

We introduce the operation σ_K defined as

$$\sigma_K F(x,E) = \frac{1}{1-K} F(x,\frac{E}{1-K}) \qquad K < 1 \qquad (6)$$

With the aid of σ_K, the equation (4) becomes

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* For the sake of simplicity we write all the quantities without the index N.

$$\frac{\partial F(x,E)}{\partial x} = -\frac{1}{\lambda}(1-\sigma_K)F(x,E) \tag{7}$$

Solving it as $(1-\sigma_K)$ be an ordinary number we have

$$F(x,E) = e^{-\frac{x}{\lambda}(1-\sigma_K)} G(E) = e^{-\frac{x}{\lambda}} e^{\frac{x}{\lambda}\sigma_K} G(E) =$$

$$= e^{-x/\lambda} \sum_{n=0}^{\infty} \frac{(x/\lambda)^n}{n!} \overset{(n)}{\sigma_K} G(E) =$$

$$= e^{-x/\lambda} \sum_{n=0}^{\infty} \frac{(x/\lambda)^n}{n!} \frac{1}{(1-K)^n} G\left(\frac{E}{(1-K)^n}\right) \tag{8}$$

Thus we get immediately the solution (3) without any hypotheses on the collision's probability law, which results to be the law of Poisson. The preceding method is a pure heuristic mean to obtain a tentative solution that must be verified by a rigorous mathematical proof.

3. Convergence of the Solution

The solution (8) is given as a product of $e^{-x/\lambda}$ by a power series $u_1+u_2 \ldots u_n + \ldots$ whose general term is

$$u_n(x,E) = \frac{(x/\lambda)^n}{n!} \frac{1}{(1-K)^n} G(E/(1-K)^n)$$

Denote by (T) the set defined by $a \leq E \leq b$ and $0 \leq x \leq X$ with $a>0$, $X>0$ and $b>a$.

Under the assumption that $G(E)$ is continuous non negative and bounded in the interval $I = [a,\infty)$. We have

$$G(E) \leq M \quad \text{for } E \in (I), \tag{9}$$

where M is some positive constant.

Hence we can write

$$|u_n(x,E)| \leq \frac{MX^n}{n!\ \lambda^n(1-K)^n} \tag{10}$$

for any point $(x,E)\varepsilon(T)$.

The continuity of $u_n(x,E)$ in (T), the inequality (10) and the uniform convergence of the exponential

$$\exp\left[\ X/\lambda(1-K)\ \right] \qquad K<1$$

in (T) ensures the uniform convergence of the series $u_n(x,E)$ in the set (T). Consequently its n-th partial sum $y_n(x,E)$ tends uniformly to a function $y(x,E)$ continuous in (T).

Since $0<X$ and $b>a$ are arbitrary the conclusion is valid for every set $0\leq x\leq X$, $a\leq E\leq b$.

The continuity of the functions $u_n(x,E)$ in (T) merits a brief comment.

The continuity of a function $f(x,y)$ of two variables separately in each variable is not sufficient to ensure its continuity as a function of (x,y), but in the case of our functions $u_n(x,E)$ the continuity respect to (x,E) is garanteed by the special form of $u_n(x,E) = f_n(x)g_n(E)$ where $f_n(x)$ and $g_n(E)$ are both continuous functions of its arguments in (T).

Now it rests only to do what J. Hadamard called the synthesis of the solution, that is to proof that the function

$$F(x,E) = e^{-x/\lambda} \lim_{n=\infty} y_n(x,E) = e^{-x/\lambda}\, y(x,E) =$$

$$= e^{-x/\lambda} \sum_{n=0}^{\infty} \frac{(x/\lambda)^n}{n!}\, G\left(\frac{E}{(1-K)^n}\right) \frac{1}{(1-K)^n} \qquad K<1 \tag{11}$$

satisfies the equation (4), but this is easily verified differentiating $F(x,E)$. The differentiation of the series (11) term by term is licit because, for every fixed value of E in (T) $F(x,E)$ is an entire function of x. In fact $y(x,E)$ is given by a power series in the real variable $x/\lambda(1-K)$ whose general term has the coefficient

$$a_n = \frac{1}{n!} G\left(\frac{E}{(1-K)^n}\right)$$

But for any fixed E we have $|a_n| < \frac{M}{n!}$ so that the radius of convergence of the complex series $\sum_0^\infty a_n Z^n$ is infinite, say

$$\frac{1}{\rho} = \limsup_{n=\infty} \sqrt[n]{|a_n|} = 0$$

and consequently the interval of convergence of the real power series is $(-\infty, +\infty)$ for any fixed E.

In the next number it will be shown that the solution (8) given by our symbolic method can be obtained by the successive approximations method of Picard.

4. Successive Approximations

To simplify the work of performing the successive approximations first we put

$$F(x,E) = e^{-x/\lambda}\, y(x,E) \tag{12}$$

With this substitution the equation (4) and the initial condition (5) become

$$\left.\begin{aligned} \frac{\partial y(x,E)}{\partial x} &= \frac{1}{\lambda(1-K)}\, y\left(x,\frac{E}{1-K}\right) \qquad K < 1 \\ y(0,E) &= G(E) \end{aligned}\right\} \tag{13}$$

The differential equation and the initial condition (13) are equivalent to the following single integral equation

$$y(x,E) = G(E) + \frac{1}{\lambda} \int_0^x \sigma_K \, y(t,E)\,dt \tag{14}$$

To solve this equation we select the function G(E) as an initial approximate determination for y(x,E) and make the following successive approximations

$$\left.\begin{aligned} y_0(x,E) &= G(E) \qquad\qquad K < 1 \\ y_n(x,E) &= G(E) + \frac{1}{\lambda}\int_0^x \sigma_K \, y_{n-1}(t,E)\,dt \end{aligned}\right\} \tag{15}$$

From (15) we obtain successively

$$\left.\begin{aligned} y_1(x,E) &= G(E) + \frac{x}{\lambda}\sigma_K \, G(E) \qquad\qquad K < 1 \\ y_2(x,E) &= G(E) + \frac{x}{\lambda}\sigma_K \, G(E) + \frac{1}{2}\frac{x^2}{\lambda^2}\sigma_K^{(2)} \, G(E) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n(2,E) &= \sum_{\nu=0}^{n} \frac{1}{\nu!}\frac{x}{\lambda^\nu}\sigma_K^{(\nu)} \, G(E) \\ &= \sum_{\nu=0}^{n} \frac{1}{\nu!}\frac{x^\nu}{\lambda^\nu}\frac{1}{(1-K)^\nu} G\left(\frac{E}{(1-K)^\nu}\right) \end{aligned}\right\} \tag{16}$$

Convergence of the Succession $y_n(x,E)$

Under the assumption that G(E) is non negative and bounded in the interval $I = [a_1,\infty)$ we have $G(E) \leq M$ for $E \in I$, where M is some positive constant. The series $S = \sum_0^\infty u_\nu(x,E)$ (whose nth partial sum is $y_n(x,E)$ converges uniformly with respect to x and E in any set (T) such that: $0 \leq x \leq X$, and $a \leq E \leq b$, $a \geq 0$. This can be easily seen because we have

$$|u_{\nu}(x,E)| < \frac{M}{\nu!} \frac{X^{\nu}}{\lambda^{\nu}(1-K)^{\nu}} \qquad K < 1$$

This inequality and the uniform convergence of the exponential exp $X/(\lambda(1-K))$ in the set (T) ensures the uniform convergence of the series S in (T), and consequently its partial sum $y_n(x,E)$ tends uniformly to a function $y(x,E)$ in (T). Now it is an easy matter to show that $y(x,E)$ satisfies the equation (11). Considering the approximations (15) we have successively

$$\begin{aligned}
y(x,E) &= \lim_{n=\infty} y_n(x,E) = G(E)) + \frac{1}{\lambda} \lim_{n=\infty} \int_0^x \sigma_K\, y_{n-1}(t,E)dt \\
&= G(E) + \frac{1}{\lambda(1-K)} \lim_{n=\infty} \int_0^x y_{n-1}(t,\tfrac{E}{1-K})dt \\
&= G(E) + \frac{1}{\lambda(1-K)} \int_0^x \lim_{n=\infty} y_{n-1}(t,\tfrac{E}{1-K})dt = \qquad (17) \\
&= G(E) + \frac{1}{\lambda(1-K)} \int_0^x y(t,\tfrac{E}{1-K})dt \\
&= G(E) + \frac{1}{\lambda} \int_0^x \sigma_K\, y(t,E)dt
\end{aligned}$$

Hence

$$y(x,E) = \lim_{n=\infty} y_n(x,E) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \frac{x^{\nu}}{\lambda^{\nu}(1-K)^{\nu}} G(\frac{E}{(1-K)^{\nu}}) \qquad (18)$$

is a solution of (14).

Note that $y(x,E)$ continuous and bounded in any set (T): $0 \leq x \leq X$, $X > 0$, $a \leq E \leq b$ $a > 0$, $b > a$.

5. Uniqueness of the Solution y(x,E)

Proof. Suppose that the equation (14) has another solution Z(x,E), continuous and bounded in every set (T): $0 \leq x \leq X$, $X > 0$, $a \leq E \leq b$, and satisfying the same initial condition

$$Z(0,E) = G(E) \quad \text{for } E \in 1$$

In this case the difference $u(x,E) = y(x,E) - Z(x,E)$ should satisfy the homogeneous equation

$$u(x,E) = \frac{1}{\lambda} \int_0^x \sigma_K \, u(t,E)dt \tag{19}$$

Now substituting iteratively the function u(t,E) under the sign of integration and using the relations (15) we have successively

$$u(x,E) = \frac{1}{\lambda} \int_0^x \sigma_K \, u(t,E)dt =$$

$$= \frac{1}{\lambda^2} \int_0^x dt_1 \int_0^{t_1} \sigma_K^{(2)} \, u(t_2,E)dt_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$= \frac{1}{\lambda^n} \int_0^x dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} \sigma_K^{(n)} \, u(t_n,E)dt_n$$

Now consider a fixed set (T). Since y(t,E) is bounded and continuous in (T) and Z(t,E) is also bounded an continuous in (T) (by hypothesis), we can write:

$$|u(x,E)| \leq \frac{1}{\lambda^n} \int_0^x dt_1 \ldots \int_0^{t_{n-1}} \left|\sigma_K^{(n)} \, u(t_n,E)\right| dt_n$$

$$= \frac{1}{\lambda^n(1-K)^n} \int_0^x dt_1 \ldots \int_0^{t_{n-1}} |u(t_n, \frac{E}{(1-K)^n})| dt_n$$

$$\leq \frac{N(X)}{\lambda^n(1-K)^n} \frac{X^n}{n!} \qquad X > 0 \qquad K < 1$$

where $N(X)$ is the maximum of $u(t,E)$ in (T).

Letting $n \to \infty$, (with $X > 0$ fixed) we obtain $|u(x,E)| \to 0$ whence $y(t,E) = Z(t,E)$ for $(t,E) \varepsilon T$.

Since T is arbitrary we have

$$y(t,E) = Z(t,E)$$

Q.E.D.

6. The Inhomogeneous Equation

The symbolic method can be successfully applied to the inhomogeneous equation

$$\frac{\partial F(x,E)}{\partial x} = -\frac{1}{\lambda} F(x,E) + \frac{1}{\lambda(1-K)} F(x,\frac{E}{1-K}) + P(x,E) \qquad (20)$$

with the initial condition

$$F(0,E) = H(E)$$

where $0 \leq K < 1$ and $P(x,E)$ is a known function that is supposed to be continuous and bounded in the set (T) defined in (3), and the function $H(E)$ satisfies the same conditions admitted for $G(E)$ in (1).

Introducing the operator σ_K, the equation (20) becomes

$$\frac{\partial F(x,E)}{\partial x} = -\frac{1}{\lambda} (1-\sigma_K) F(x,E) + P(x,E) \qquad (21)$$

with

$$F(0,E) = H(E)$$

Solving it, as $(1-\sigma_K)$ be an ordinary number we get immediately the solution

$$F(x,E) = e^{-\frac{1}{\lambda}(1-\sigma_K)x}\left[H(E) + \int_0^x e^{\frac{1}{\lambda}(1-\sigma_K)\eta} P(\eta,E)d\eta\right] = \quad (22)$$

$$= F_1(x,E) + F_2(x,E)$$

To verify that (22) is effectively the solution of (20)we observe that

1) for $x = 0$ $F(0,E) = H(E)$
2) $F_1(x,E)$ as we have seen previously is the solution of the homogeneous equation
3) $F_2(x,E)$ is a particular solution of the inhomogeneous equation (20) as can be easily verified differentiating the function

$$F_2(x,E) = \int_0^x e^{-\frac{1}{\lambda}(1-\sigma_K)(x-\eta)} P(\eta,E)d\eta =$$

$$= \int_0^x e^{-\frac{x-\eta}{\lambda}} \sum_{n=0}^{\infty} \frac{(x-\eta)^n}{n!} \frac{1}{(1-K)^n} P(\eta,\frac{E}{(1-K)^n})d\eta$$

If we put $F(x,E) = F_\pi(x,E)$, $\lambda = \lambda_\pi$ and $K = K_\pi$, $H(E) = 0$ in (22) we obtain the solution of the one-dimension diffusion equation for the intensity $F_\pi(x,E)$ of charged pions in the atmosphere when we disregard the pions of the second generation and $\pi \rightarrow \mu$ decay (for pions of energy greater than 1 TeV in the laboratory system). In this case $P(x,E)$ is the production spectrum [3] [4] [5] of pions by nucleons P_π^{NN} (x,E) at the atmospheric depth $x(g/cm^2)$ in the range of energy E, E+dE.

When the pions of the second generation are taken into account, the solution of the one dimensional diffusion equation for the intensity $F_\pi(x,E)$ of charged pions in the atmosphere can be obtained by a method of successive approximations using the above results to construct the first one.

Since the proof of convergence of the process is more involved the problem will be treated later in another paper.

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