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CONCERNING HOLOMORPHY TYPES FOR BANACH SPACES *

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1. INTRODUCTION

Let E be a complex Banach space and H(E) be the locally convex space of all entire complex-valued functions on E. Given a convolution operator O in H(E), that is a continuous linear mapping of H(E) into itself commuting with translations in H(E) by elements of E, we are interested in proving that every solution f in H(E) of Of = 0 may be approximated by finite sums of solutions in H(E) of the same equation which are exponential-polynomials. Technical difficulties that thus arise on the dual space E' lead us to introduce the locally convex space $H_N(E)$ of

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all nuclearly entire functions on E. There are two natural candidates for the definition of $H_N(E)$; it is not known so far whether they do coincide. One of these candidates is a particular case of $H_\theta(U;F)$ as defined in [5] when U = E, F = C and the holomorphy type θ is the nuclear type. The other candidate was introduced in [6]. The indicated approximation theorem is known to hold true for it, as indicated in [6]; see also [4] for the so-called nuclearly bounded case. Known properties and open problems concerning the locally convex spaces H(E) and $H_N(E)$ have been considered and indicated. To treat them in a unified way, the concept of a holomorphy type θ from E to another complex Banach space F (which was assumed to be F = C in the preceding considerations) was introduced. It is a sequence of Banach spaces $P_\theta({}^mE;F)$, each of which is a vector subspace of the Banach space $P({}^mE;F)$ of all continuous m-homogeneous polynomials from E to F, for m = 0, 1,..., certain additional technical conditions being imposed; see §2 below, as well as [5]. It is then possible to define and investigate the locally convex space $H_\theta(U;F)$ of all mappings of θ-holomorphy type from a non-void open subset U of E into F, thus subsuming the current case H(E) and the nuclear case $H_N(E)$ when U = E, F = C and the holomorphy type θ is either the current or the nuclear type. This has been done in the case of one of the candidates for the definition of $H_N(E)$; we refer to [5] and [1] for details. As to the case of the other candidate for the definition of $H_N(E)$, namely that introduced in [6], a study of

$H_\Theta(E;F)$ has been made in [2] under the restriction that $U = E$. The results thus obtained are quite interesting and include a proof of completeness of $H_\Theta(E;F)$, which is different from the one described below for the current type, when the two candidates for the definition of $H_\Theta(E;F)$ do coincide. We refer to [6] and [2] for details. Reference is made once for all to [5] for general notation and terminology.

2. HOLOMORPHY TYPES

Let E and F be complex Banach spaces and $P(^mE;F)$ be the Banach space of all continuous m-homogeneous polynomials from E to F, for $m = 0, 1, \ldots$. A holomorphy type Θ from E to F is a sequence of Banach spaces $P_\Theta(^mE;F)$, for $m = 0,1,\ldots$, the norm on each of them being denoted by $P \longmapsto \|P\|_\Theta$, such that the following conditions hold true:

(1) Each $P_\Theta(^mE;F)$ is a vector subspace of $P(^mE;F)$.

(2) $P_\Theta(^0E;F)$ coincides with $P(^0E;F)$ as a normed vector space.

(3) There is a real number $\sigma \geq 0$ for which the following is true. Given any $k = 0,1,\ldots$ and $m = 0,1,\ldots$, $k \leq m$, $x \in E$ and $P \in P_\Theta(^mE;F)$, we have

$$\hat{d}^k P(x) \in P_\Theta(^kE;F) \ ,$$

$$\left\| \frac{1}{k!} \hat{d}^k P(x) \right\|_\Theta \leq \sigma^m \cdot \|P\|_\Theta \cdot \|x\|^{m-k} \ .$$

The current holomorphy type from E to F is the holomorphy type θ for which $P_\theta(^mE;F) = P(^mE;F)$ as normed spaces, for $m = 0,1,\ldots$. On the other hand, certain questions in applications, concerning for instance convolution and partial differential operators, Fourier and Borel transforms, distributions, etc. in infinite dimensions, lead to important types of holomorphy, such as the nuclear, the integral, the Hilbert-Schmidt cases, etc.; see [6], [4], [3]. Let us review briefly the definition of the nuclear type. If E' indicates the dual Banach space to E, we shall have that $\varphi^m \cdot y \in P(^mE;F)$ for every $\varphi \in E'$, $y \in F$ and $m = 0,1,\ldots$, where $\varphi^m \cdot y$ denotes the mapping $x \in E \longmapsto [\varphi(x)]^m y \in F$. We shall represent by $P_f(^mE;F)$ the vector subspace of $P(^mE;F)$ generated by all $\varphi^m . y$ when φ runs over E' and y varies in F. It consists of those elements of $P(^mE;F)$ each of which may be represented as a finite sum $(\varphi_1)^m \cdot y_1 + \ldots + (\varphi_r)^m \cdot y_r$, where $\varphi_j \in E'$ and $y_j \in F$ for $j = 1,\ldots, r$. The Banach space $P_N(^mE;F)$ of all nuclear m-homogeneous polynomials from E to F is characterized by the following requirements:

(1) $P_N(^mE;F)$ is a vector subspace of $P(^mE;F)$.

(2) $P_N(^mE;F)$ is a Banach space with respect to a norm $P \longmapsto \|P\|_N$ called the nuclear norm, to be distinguished from the current norm $P \longmapsto \|P\|$ on $P(^mE;F)$. Moreover we have $\|P\| \leq \|P\|_N$ if $P \in P_N(^mE;F)$.

(3) $P_f(^mE;F)$ is contained and dense in $P_N(^mE;F)$ with

respect to the nuclear norm. For each $P \in P_f(^mE;F)$, its nuclear norm $\|P\|_N$ is the infimum of $\|\varphi_1\|^m \cdot \|y_1\| + \ldots + \|\varphi_r\|^m \cdot \|y_r\|$ for all possible representations $P = (\varphi_1)^m \cdot y_1 + \ldots + (\varphi_r)^m \cdot y_r$, where $\varphi_j \in E'$ and $y_j \in F$ for $j = 1,\ldots,r$.

3. THE LOCALLY CONVEX SPACE $H_\theta(U;F)$

If U is a non-void open subset of E, we shall denote by H(U;F) the vector space of all holomorphic mappings from U to F. Each $f \in H(U;F)$ has its differential $\hat{d}^m f(x) \in P(^mE;F)$ at $x \in U$ of order $m = 0,1,\ldots$. If θ is a holomorphy type from E to F, a given $f \in H(U;F)$ is said to be of θ-holomorphy type on U if, for every $x \in U$, we have that

$$\hat{d}^m f(x) \in P_\theta(^mE;F) \quad \text{for } m = 0,1,\ldots$$

and that the sequence

$$\left(\left\| \frac{1}{m!} \hat{d}^m f(x) \right\|_\theta\right)^{1/m} \quad \text{for } m = 1,2,\ldots$$

is bounded (in which case it results that this sequence is uniformly bounded for x variable in a sufficiently small neighborhood in U of every compact subset of U). We shall denote by $H_\theta(U;F)$ the vector subspace of H(U;F) of all such f of θ-holomorphy type on U.

A seminorm p on $H_\theta(U;F)$ is said to be θ-ported by a compact subset K of U if the following equivalent conditions hold:

(1) Given any real number $\epsilon > 0$, we can find a real number $c(\epsilon) > 0$ such that

$$p(f) \leqslant c(\epsilon) \cdot \sum_{m=0}^{\infty} \epsilon^m \cdot \sup_{x \in K} \left\| \frac{1}{m!} \hat{d}^m f(x) \right\|_\theta$$

for every $f \in H_\theta(U;F)$.

(2) Given any real number $\epsilon > 0$ and any open subset V of U containing K, we can find a real number $c(\epsilon,V) > 0$ such that

$$p(f) \leq c(\epsilon,V) \cdot \sum_{m=0}^{\infty} \epsilon^m \cdot \sup_{x \in V} \left\| \frac{1}{m!} \hat{d}^m f(x) \right\|_\theta$$

for every $f \in H_\theta(U;F)$.

The natural topology $T_{\omega,\theta}$ on $H_\theta(U;F)$ is defined by the semi norms on this vector space each of which is θ-ported by some compact subset of U.

4. THE LOCALLY CONVEX SPACE $H_\theta(K;F)$

Let now K be a compact subset of E. If U_1 and U_2 are non-void open subsets of E containing K, we say that $f_1 \in H_\theta(U_1;F)$ and $f_2 \in H_\theta(U_2;F)$ are equivalent modulo K if f_1 and f_2 coincide in some open subset of E contained in $U_1 \cap U_2$ containing K. This defines an equivalence relation on the union of all $H_\theta(U;F)$ for U a variable non-void open subset of E containing the fixed compact subset K of E. Each equivalence class is called a germ from K to F of θ-holomorphy type. Let $H_\theta(K;F)$ be the quotient space of the aforementioned union modulo the indicated equivalence relation. Then $H_\theta(K;F)$ is a vector space in a natural way so that each natural mapping $H_\theta(U;F) \longrightarrow H_\theta(K;F)$ be linear, where U is a non-void open subset of E containing K. Notice that $H_\theta(K;F)$ is the directed union of the image of $H_\theta(U;F)$ by $H_\theta(U;F) \longrightarrow H_\theta(K;F)$ for U a variable non-void open subset of E containing K.

If U is a non-void open subset of E and $\epsilon > 0$, we denote by $H_{\theta,\epsilon}(U;F)$ the vector subspace of $H_\theta(U;F)$ consisting of those f such that

$$\|f\|_{\theta,\epsilon} = \sum_{m=0}^{\infty} \epsilon^m \cdot \sup_{x\in U} \left\| \frac{1}{m!} \hat{d}^m f(x) \right\|_\theta$$

is finite. Then $H_{\theta,\epsilon}(U;F)$ is a Banach space with respect to the norm $f \longmapsto \|f\|_{\theta,\epsilon}$. Notice that, if K is a compact subset of E, then $H_\theta(K;F)$ is still the directed union of the image of $H_{\theta,\epsilon}(U;F)$ by $H_\theta(U;F) \longrightarrow H(K;F)$ for U a variable non-void open subset of E containing K and $\epsilon > 0$ variable too.

The natural topology $T_{\omega,\theta}$ on $H_\theta(K;F)$ is obtained by considering $H_\theta(K;F)$ as the locally convex inductive limit of the Banach space $H_{\theta,\epsilon}(U;F)$ with respect to the natural linear mapping $H_{\theta,\epsilon}(U;F) \longrightarrow H_\theta(K;F)$ as the non-void open subset U of E containing K shrinks to K and as $\epsilon > 0$ decreases. We then write

$$H_\theta(K;F) = \varinjlim_{\substack{U\supset K \\ \epsilon>0}} H_{\theta,\epsilon}(U;F)$$

5. INTERPLAY BETWEEN $H_\theta(U;F)$ AND $H_\theta(K;F)$

It is also true, for a fixed compact subset K of E, that the topology $T_{\omega,\theta}$ on $H_\theta(K;F)$ may also be obtained by considering

$H_\theta(K;F)$ as the locally convex inductive limit of $H_\theta(U;F)$ endowed with its topology $T_{\omega,0}$ with respect to the natural linear mapping $H_\theta(U;F) \longrightarrow H_\theta(K;F)$ as the non-void open subset U of E containing K shrinks to K. We then write

$$H_\theta(K;F) = \lim_{\substack{\longrightarrow \\ U \supset K}} H_\theta(U;F) \cdot$$

The following question then comes up. We want to know whether it is true that, given the non-void open subset U of E, the topology $T_{\omega,\theta}$ on $H_\theta(U;F)$ may also be obtained by considering $H_\theta(U;F)$ as the projective limit of $H_\theta(K;F)$ endowed with its topology $T_{\omega,\theta}$ with respect to the natural linear mapping $H_\theta(U;F) \longrightarrow H_\theta(K;F)$ as the compact subset K of E contained in U grows to U. We then write

$$H_\theta(U;F) = \lim_{\substack{\longleftarrow \\ K \subset U}} H_\theta(K;F) \cdot$$

A partial answer to this question is known as follows.

If K and U are respectively a compact and a non-void open subset of E, we say that K is a θ-Runge compact sub-set in U if K is contained in U and the image of $H_\theta(U;F)$ in $H_\theta(K;F)$ under the natural linear mapping $H_\theta(U;F) \longrightarrow H_\theta(K;F)$ is dense in $H_\theta(K;F)$. Then, if every compact subset of U is contained in another compact subset of U which is θ-Runge in U, it is true that $H_\theta(U;F)$ is the required projective limit. A simple instance in which every compact subset of U is contained in another compact subset of U which is θ-Runge in U is that when U is ξ-equilibrated with respect to someone of its points ξ, that is $(1-\lambda)\xi + \lambda x \in U$ whenever $x \in U$, $\lambda \in C$ and $|\lambda| \leq 1$. This is the case when $U = E$ and, therefore, the preceding considerations apply to $H_\theta(E;F)$. In the finite dimensional situation, the following facts are pertinent to complex analysis in several complex variables. If K and U are respectively a compact and a non-void open subset of C^n, then K is said to be a Runge compact subset in U if the image of H(U;C) in H(K;C) under the natural linear mapping $H(U;C) \rightarrow H(K;C)$ is dense in H(K;C); in this case we restrict ourselves to the current holomorphy type on C^n for C-valued functions. It is known that every compact subset of U is contained in some Runge compact subset of U at least in the following noteworthy cases: either U is ξ-equilibrated with respect to someone of its points ξ, or else U is a domain of holomorphy; neither of these two cases contains the other one as a particular instance. Apparently it is a question as yet unanswered in the literature whether every compact subset of a non-void open subset U of C^n is contained in some other compact subset of U which is Runge in U.

The above considerations have an application as follows. It can be shown that $H_\theta(K;F)$ is complete in the sense of Cauchy under $T_{\omega,\theta}$ and that $H_\theta(U;F)$ is complete in the sense of Cauchy under $T_{\omega,\theta}$ too, in case it is the required projective limit, this being the case if U satisfies the indicated condition for compact subsets which are θ-Runge in U. Completeness of $H_\theta(U;F)$ for an arbitrary open subset U of E remains unanswered if E is infinite dimensional.

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