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BOUND STATES FOR SPIN 1/2 PARTICLES IN GENERAL RELATIVITY

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BOUND STATES FOR SPIN 1/2 PARTICLES IN GENERAL RELATIVITY *

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INTRODUCTION

In this paper we treat the problem of bound states for a spin 1/2 particle interacting with gravitational and electromagnetic fields. The gravitational field is described by tetrads, and the Hamiltonian which is obtained from Dirac's equation is made Hermitian by means of a non-unitary transformation in the Hilbert space of the vectors of state [1]. It is shown that with this Hamiltonian solutions presenting the behaviour of bound states are found. These solutions possess a behaviour which is free of unconsistencies, both at large values of the distance from the gravitating mass and at small distances.

The gravitational field presently considered is that of a static charged pointlike mass, usually called by Nordstrom field.

Regarding the notation, we indicate the tensor degree of freedom by greek letters which run from 1 to four. Spatial degree of freedom are indicated by latin letters. Indices into a round bracket denote local tetrad indices, they have a variation which is indicated by the same symbols as those of usual tensor indices. Finally, the special relativistic metric $g_{\mu\nu}$ has signature + 2.

1. TETRADS AND THE THREE-DIMENSIONAL GEOMETRY

It is well known that in order to introduce the gravitational interaction on the spin 1/2 particles is necessary to describe the gravitational field by tetrads $h_{\mu(\nu)}$, and their reciprocal $h^{\mu(\nu)}$. Presently we are concerned with this interaction as described by the Hamiltonian for the coupled system, so that initially we have to obtain a suitable three-dimensional decomposition of the tetrad field. We do that as follows

$$h_{\mu(\nu)} = (h_{r(s)}, V_{(\nu)}, Y_r), \; V_{(\nu)} = h_{o(\nu)}, \; Y_r = h_{r(o)} ,$$

$$h^{\mu(\nu)} = (h^{r(s)}, W^{\nu}, Z^{(r)}), \; W^{\nu} = h^{\nu(o)}, \; Z^{(r)} = h^{o(r)} .$$

Imposing the condition that the unit time-like normal to the hyperplane x^o = constant is given in terms of tetrads as

$$\ell_\mu = h_\mu^{(o)} = \frac{\delta_\mu^o}{\sqrt{-g^{oo}}}, \; \ell^\mu = g^{\mu\nu}\,\ell_\nu = h^{\mu(o)} \qquad (1)$$

we can eliminate both the Y_r and the $Z^{(r)}$,

$$Y_r = 0, \; Z^{(r)} = 0 \qquad (2)$$

and obtain

$$h^{o}{}_{(o)} = \sqrt{-g^{oo}} \qquad (3)$$

It may be shown that only the thirteen components $h^{r(s)}$ and W^{ν} are independent [2], since

$$h_{r(1)} = D^{-1}\,\epsilon_{rjk}\, h^{j(2)}\, h^{k(3)}\ ,$$

$$h_{r(2)} = D^{-1}\,\epsilon_{rjk}\, h^{j(3)}\, h^{k(1)}\ ,$$

$$h_{r(3)} = D^{-1}\,\epsilon_{rjk}\, h^{j(1)}\, h^{k(2)}\ ,$$

$$D = \epsilon_{rjk}\, h^{r(1)}\, h^{j(2)}\, h^{k(3)}\ ,$$

$$V_{(r)} = -h_{s(r)}\, W^{s}(W^{o})^{-1},\ V_{(o)} = (W^{o})^{-1}$$

the variables $h^{r(s)}$ and $h_{r(s)}$ are uniquely associated to the geometry on the three-dimensional hyperplane x^{o} = constant. They represent therefore physically admissible variables [3]. The same property does not hold for W^{ν} or $V_{(\nu)}$ which depend on the continuation of the coordinate system, outside the hyperplane. This structure is clear from the above equations, where we see that the total number of variables split into two parts, one involving only relations among the $h^{r(s)}$ and $h_{r(s)}$, the other among the W^{ν} and $V_{(\nu)}$.

In this paper we consider only static gravitational fields, for which we have $g_{oi} = 0$. This implies that

$$V_{(r)} = 0 \qquad (4)$$

$$W^{r} = 0 \qquad (5)$$

and we are left with just the $h^{s(r)}$ and W^{o} as independent variables. The above equations simplify to

$$h_{r(s)} = \frac{D^{-1}}{3!} \epsilon_{(s)(m)(n)} \epsilon_{rjk} h^{j(m)} h^{k(n)} \quad (6)$$

$$V_o = (W^o)^{-1} \quad (7)$$

where we also have, from (3),

$$W^o = -\sqrt{-g^{oo}} \,. \quad (8)$$

The three-dimensional contravariant metric tensor $e^{rs} = g^{rs} - g^{or} g^{os}/g^{oo}$, is here identical to g^{rs}, and is given in function of the tetrads as

$$e^{rs} = h^r_{(m)} h^{s(m)} = h^r_{(m)} h^s_{(m)} \,. \quad (9)$$

2. HAMILTONIAN FORM OF DIRAC'S EQUATION FOR THE COUPLED SYSTEM

According to a previous paper [1], the Schrodinger form for the Dirac equation in presence of a static gravitational field has the expression (the quantity V indicates the Coulomb potential)

$$H\psi = \frac{1}{2} c\, \alpha^{(k)} \left(\tilde{h}^{i(k)} p_i + p_i \tilde{h}^{i(k)} \right) \psi + \tilde{m}\, c^2 \beta\psi + V\psi \quad (10)$$

where the symbol $\sim$ indicates, for any quantity F,

$$\tilde{F} = F/(-W^o) \,.$$

Since the Hamiltonian formulation does not maintain the four-dimensional covariance of the theory, all relations of the present formulation possess only general three-dimensional covariance. The Hamiltonian of equation (10) is Hermitian, and may be written in a form which shows its covariant charcter with respect to the three-dimensional general coordinate transformations. For doing this, we first introduce the scalar product in three-dimensions

$$\vec{a}.\vec{b} = e^{rs}\, a_r\, b_s \ .$$

The Hermitian scalar product of $\vec{\tilde{\alpha}}$ by $\vec{p}$ is,

$$\frac{1}{2}\left(\vec{\tilde{\alpha}}\cdot\vec{p} + \vec{p}\ \vec{\tilde{\alpha}}\right) = \frac{1}{2}\left(e^{rs}\,\tilde{\alpha}_r p_s + p_s\, e^{rs}\,\tilde{\alpha}_r\right) \quad (11)$$

Using (2) and (9) we may write (11) as

$$e^{rs}\,\tilde{\alpha}_r p_s + p_s\, e^{rs}\,\tilde{\alpha}_r = \alpha^{(r)}\left(\tilde{h}^{s(r)}\ p_s + p_s\ \tilde{h}^{s(r)}\right) \quad (12)$$

(where use have been made of the usual formula of tetrad calculus, $\alpha_\nu = h_\nu^{(\alpha)}\ \alpha_{(\alpha)}$). Thus, we can present the Hamiltonian in the form manifestly covariant with respect to the three-dimensional transformations

$$H = \frac{c}{2}\left(\vec{\tilde{\alpha}}\circ\vec{p} + \vec{p}\cdot\vec{\tilde{\alpha}}\right) + \tilde{m}\ c^2\beta + V \ . \quad (13)$$

This Hamiltonian has, except for the presence of the scalar potential $h^{o}_{(o)}$ in $\vec{\alpha}$ and in the mass term, the same form as the special relativistic Hamiltonian, except that the scalar product now refers to the Riemannian three-dimensional metric e^{rs}, and since $\vec{\alpha}$ is point dependent the scalar product of $\vec{\alpha}$ by $\vec{p}$ appears symmetrized.

In what will follow we seek for solutions of (10) presenting the behaviour of bound states. For practical applications, as will be the case in what follows, is convenient to work only with the constant matrices $\alpha_{(r)}$, instead of the α_r. With this finallity we use the commutation relation

$$\left[\tilde{h}^{s(\Lambda)},\ p_s\right] = i\hbar \frac{\partial \tilde{h}^{s(\Lambda)}}{\partial x^s}$$

and write the equation (10) as

$$H\psi = \left(c\,\frac{h^s{}_{(r)}}{h^o{}_{(o)}}\,\alpha_{(r)}\, p_s - \frac{i\hbar c}{2}\,\alpha_{(r)}\,\frac{\partial}{\partial x^s}\left(\frac{h^s{}_{(r)}}{h^o{}_{(o)}}\right) + \tilde{m}c^2\beta + V \right)\psi \quad (14)$$

This equation shows that in presence of a gravitational field we have to replace the usual Dirac matrices $\alpha_{(r)}$ by the combination $\tilde{h}^s_{(r)}\,\alpha_{(r)}$, where the $\alpha_{(r)}$ are still the same constant matrices of special relativity. Besides this, we also have the extra middle term of the right side of (14) and the presence of the potential $h^o{}_{(o)}$ in the mass term. Using these replacements into the correspondent Hamiltonian for the motion of a spin 1/2 particle in the Coulomb field of a point source in special relativity [4], we find

$$H = c\,\tilde{\alpha}_r\, p_r + \frac{i\hbar c}{r}\,\tilde{\alpha}_r\,\beta P + \beta\tilde{m}\,c^2 + V - \frac{i\hbar c}{2}\,\alpha_{(r)}\,\frac{\partial}{\partial x^s}\left(\frac{h^s_{(r)}}{h^o_{(o)}}\right) \quad (15)$$

where P is the operator, $\hbar P = \beta(\vec{\sigma}\cdot\vec{L}+\hbar)$ [5] and $\tilde{\alpha}_r$ is [6]

$$\tilde{\alpha}_r = \frac{h^i_{(k)}}{h^o{}_{(o)}}\begin{pmatrix} 0 & \dfrac{\sigma_{(k)}\,x_i}{r} \\ \dfrac{\sigma_{(k)}x_i}{r} & 0 \end{pmatrix} \quad (16)$$

The Hamiltonian (15) will be further simplified by imposing that

$$h^r{}_{(s)} = f^r\,\delta^r_s \quad (17)$$

(in this relation we are not summing over r). As we will see in the next section, the gravitational potentials presently considered satisfy the conditions (17). Using (17) we may put the matrix $\tilde{\alpha}_r$ of (16) in the form

$$\tilde{\alpha}_r = \frac{1}{h^o_{(o)}} \begin{pmatrix} 0 & \frac{\vec{\Sigma}\cdot\vec{r}}{r} \\ \frac{\vec{\Sigma}\cdot\vec{r}}{r} & 0 \end{pmatrix} \tag{18}$$

where the scalar product has the usual Euclidian form, and $\vec{\Sigma}$ has components

$$\vec{\Sigma} = (f^1\,\sigma_1,\ f^2\,\sigma_2,\ f^3\,\sigma_3)$$

3. SOLUTION OF THE DIRAC EQUATION FOR THE FIELD OF A STATIC CHARGED POINTLIKE SOURCE

We consider the field acting on the electron, as originating from a point charge at rest. The metric in spherical polar coordinates corresponding to this source is [7]

$$ds^2 = -e^{\nu}(dx^o)^2 + e^{-\nu}\,dr^2 + r^2(d\theta^2 + \mathrm{sen}^2\theta\,d\varphi^2) \tag{19-1}$$

with

$$e^{\nu} = 1 - \frac{2GM}{c^2 r} + \frac{G\,e^2}{c^4\,r^2} \tag{19-2}$$

where M is the mass of the source particle and G the gravitational constant. In the case where the metric is diagonal, it is possible to obtain directly the tetrad components from the formulas [1]

$$-W^{\mu} = h^{\mu}_{(o)} = \sqrt{-g^{oo}}\ \delta^{\mu}_{o} = \frac{1}{\sqrt{-g_{oo}}}\ \delta^{\mu}_{o}\ , \tag{20}$$

$$h^{\mu}_{(r)} = \sqrt{g^{ii}}\ \delta^{\mu}_{(r)} = \frac{1}{\sqrt{g_{ii}}}\ \delta^{\mu}_{(r)}\ , \tag{21}$$

here g_{ii} indicates g_{11}, g_{22} or g_{33}. The relation (20) shows that W^r vanishes and that W^o has the value given by (8). Similarly, the eq. (21) shows that $Z_{(r)}$ vanishes and that

$$f^i = \sqrt{g^{ii}} = \sqrt{\frac{1}{g_{ii}}} \,, \tag{22}$$

for each one of the three values of i. Using the above $g_{\mu\nu}$ we find

$$f^1 = f^r = e^{\nu/2} \,, \tag{23-1}$$

$$f^2 = f^\theta = 1/r \tag{23-2}$$

$$f^3 = f^\varphi = 1/r \sin\theta \tag{23-3}$$

with this choice for the f^i we can write the last term which stands on the right hand side of (15), in spherical coordinates as

$$\frac{i\hbar c}{2} \alpha_{(k)} \frac{\partial \tilde{h}^i_{(k)}}{\partial x^2} = i\hbar c \alpha_r G_1 + A(\theta,\varphi) \tag{24-1}$$

with

$$G_1 = \frac{G}{c^2 r^2}\left(M - \frac{e^2}{c^2 r}\right) \tag{24-2}$$

The part depending on the angles, the $A(\theta,\varphi)$, is summed to the term $\hbar P$ which stands on the right side of (15), thus yielding a new operator P in presence of the gravitational field. The eq. (15) then takes the form (all calculations are done up to linear terms in the constant G)

$$H = c\tilde{\alpha}_r p_r + \frac{i\hbar c}{r} \tilde{\alpha}_r \beta k + \tilde{\beta} mc^2 + V - i\hbar c \alpha_r G_1 \tag{25}$$

and the Schrodinger equation for this Hamiltonian is

$$\left(c\tilde{\alpha}_r p_r + \frac{i\hbar c}{2}\tilde{\alpha}_r \beta k + \tilde{\beta}\, mc^2 + V - i\hbar c\alpha_r\, G_1 + E \right) \psi = 0 \qquad (26)$$

using the relations

$$\tilde{\alpha}_r = \left(1 - \frac{GM}{c^2 r} + \frac{G\, e^2}{2c^4 r^2} \right) \alpha_r \qquad (27\text{-}1)$$

$$\alpha_r = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad (27\text{-}2)$$

$$p_r = -i\hbar \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) \qquad (27\text{-}3)$$

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (27\text{-}4)$$

$$\tilde{\beta} = \left(1 - \frac{GM}{c^2 r} + \frac{Ge^2}{2c^4 r^2} \right) \beta \qquad (27\text{-}5)$$

$$V = -\frac{e^2}{r} \qquad (27\text{-}6)$$

$$\psi = \frac{1}{r} \begin{pmatrix} F_1(r) \\ F_2(r) \end{pmatrix} \qquad (27\text{-}7)$$

we can write (26) as two separate equations

$$\left(-\xi_2 - \xi_1\, kr^{-1} - \xi_1 \frac{\partial}{\partial r} \right) F_2 + \left(mc^2 \xi_1 + E + V \right) \frac{F_1}{\hbar c} = 0 \qquad (28\text{-}1)$$

$$\left(-\xi_2 - \xi_1\, kr^{-1} + \xi_1 \frac{\partial}{\partial r} \right) F_1 - \left(mc^2 \xi_1 - E - V \right) \frac{F_2}{\hbar c} = 0 \qquad (28\text{-}2)$$

where ξ_1 and ξ_2 are a short for

$$\xi_1 = 1 - \frac{GM}{c^2 r} + \frac{G\, e^2}{2c^4 r^2}$$

$$\xi_2 = \frac{G_1}{2}$$

introducing the dimensionaless variable ρ, $(E \leqslant mc^2)$

$$\rho = \lambda r$$

$$\lambda = \sqrt{\lambda_1 \lambda_2}$$

$$\hbar c\, \lambda_1 = E + mc^2$$

$$\hbar c\, \lambda_2 = mc^2 - E$$

we obtain from (28),

$$\left(-\frac{\xi_2}{\lambda} - \frac{\xi_1 k}{\rho} - \xi_1 \frac{d}{d\rho}\right) F_2 + \left(\frac{\lambda_1}{\lambda} + \frac{V}{\hbar c\lambda} + \frac{\xi_3}{\hbar c\lambda}\right) F_1 = 0 \quad (29\text{-}1)$$

$$\left(\frac{\xi_2}{\lambda} - \frac{\xi_1 k}{\rho} + \xi_1 \frac{d}{d\rho}\right) F_1 - \left(\frac{\lambda_2}{\lambda} - \frac{V}{\hbar c\lambda} + \frac{\xi_3}{\hbar c\lambda}\right) F_2 = 0 \quad (29\text{-}2)$$

where

$$\xi_3(\rho) = mc^2\, \xi_1(\rho) - mc^2$$

Taking solutions of the equation (29) in the form of power series of ρ,

$$F_1(\rho) = e^{-\rho} \sum_{\nu=0}^{\infty} a_\nu\, \rho^{\nu+s}$$

$$F_2(\rho) = e^{-\rho} \sum_{\nu=0}^{\infty} b_\nu\, \rho^{\nu+s}$$

we find, by equating to zero the coefficient of $\rho^{\nu+s-1}$,

$$b_{\nu-1} + \frac{\lambda_1}{\lambda} a_{\nu-1} - (k+\lambda GM+\nu+s)b_\nu - \left(\frac{e^2}{\hbar c} + GMm\right) a_\nu +$$

$$+ \lambda GM(k+\nu+s)b_{\nu+1} + \frac{\lambda^2 Ge^2}{2\hbar^2 c^4} b_{\nu+1} + \frac{\lambda Ge^2 m}{2\hbar c} a_{\nu+1} + \frac{\lambda^2 Ge^2}{\hbar^2\ c^4}\left(1-\frac{k}{2}-\frac{\nu+s+2}{2}\right) b_{\nu+2} = 0$$

$$- a_{\nu-1} - \frac{\lambda_2}{\lambda} b_{\nu-1} + (\lambda GM-k+\nu+s)a_\nu + \left(GMm - \frac{e^2}{\hbar c}\right) b_\nu + \lambda GM\ (k-\nu-s)a_{\nu+1} -$$

$$- \frac{\lambda^2 Ge^2}{2\hbar^2 c^4} a_{\nu+1} - \frac{\lambda Ge^2 m}{2\hbar c} b_{\nu+1} + \frac{\lambda^2 Ge^2}{2\hbar c}\left(\frac{\nu+s+2}{2} - \frac{k}{2} - 1\right) a_{\nu+2} = 0$$

(from here on we shall take $\hbar = c = 1$). Taking $\nu = 0$ in these equations, we find

$$b_{-1} + \frac{\lambda_1}{\lambda} a_{-1} - (k+\lambda GM+s)b_o - (e^2+GMm)a_o +$$

$$+ \lambda GM(k+s)b_1 + \frac{\lambda^2 Ge^2}{2} b_1 + \frac{\lambda Ge^2 m}{2} a_1 - \lambda^2\ Ge^2\left(-1+\frac{k}{2}+\frac{s+2}{2}\right) b_2 = 0$$

$$- a_{-1} - \frac{\lambda_2}{\lambda} b_{-1} + (\lambda GM - k+s)a_o + (GMm - e^2)b_o + \lambda GM(k-s)a_1 -$$

$$- \frac{\lambda^2 Ge^2}{2} a_1 - \frac{\lambda Ge^2 m}{2} b_1 + \lambda^2 Ge^2\left(\frac{s+2}{2} - \frac{k}{2} - 1\right) a_2 = 0$$

Choosing as a particular case of the above equations

$$- b_o(\lambda GM + k+s) + (-e^2 - GMm)a_o = 0$$

$$b_o(GMm - e^2) + (\lambda GM - k + s)\ a_o = 0 \qquad (30)$$

Which implies in the condition

$$s^2 + 2\lambda GMs + (\lambda^2 G^2 M^2 - k^2 + e^4 - G^2M^2m^2) = 0 \qquad (31)$$

this equation gives a explicit value for the constant s. We will neglect the negative sign in front of the radical since it conducts to a wave function diverging at the origin. We obtain

$$S = -\lambda GM + (k^2 - e^4 + G^2M^2m^2)^{\frac{1}{2}} \quad . \qquad (32)$$

In the limit where G goes to zero this gives the value for s obtained for the motion in the Coulomb field of the charge.[4]

In order that the wave function (27-7) be finite at the origin is necessary that s be of the form,

$$s = a/2$$

where a is a constant taking on value equal or large than two,

$$a \geqslant 2 \; .$$

This represents a condition on the possible values of the mass M which generates part of the gravitational interaction on the particle of spin 1/2. It can be shown that the behaviour of this wave function for large values of the distance is free of divergences. Besides this, the wave function can be normalized by suitable choice of the coefficients.

* * *

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2. We might as well use as independent components the $h_{r(s)}$ and $V_{(\gamma)}$, but by presently this will not be convenient.

3. P. A. M. Dirac, Recent developments in general relativity, page 191, Pergamon Press (1962).

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5. This scalar product has the usual Euclidian form.

6. This letter r indicates the "radial part" and should not be confunded with a sufix r.

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