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CLASSICAL SYMMETRIES: AN ELEMENTARY SURVEY *

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ABSTRACT: This is an elementary review of the invariance properties of the laws of the classical mechanics of systems of point-particles (sect. I); it is well-known that the conservation laws in this theory are not deducible from the invariance principles and Newton's equations of motion but, rather, from these principles and Lagrange's equations. The extension of the Lagrangean formalism to classical linear field theories is recalled, and the famous theorem of E. Noether, which provides a rule for the construction of conserved physical variables corresponding to invariance principles, is established (sect. II). Finally, the foundations of Einstein's relativistic theory of gravitation are sketchly presented with the intention of showing that it is not possible to associate an energy-momentum tensor - but only a coordinate-dependent object or pseudo-tensor - to the gravitational field (sect. III).

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SYMMETRIES IN CLASSICAL MECHANICS OF SYSTEMS OF POINT-PARTICLES.

I.1) The Lagrange and canonical equations.

The kinetic energy of a classical system of n point-particles, A, is :

$$T = \sum_{A=1}^{n} \frac{1}{2} m_A (\dot{\underset{\sim}{x}}_A)^2 \qquad (I,1)$$

where m_A is the mass of particle A at position $\underset{\sim}{x}_A$ and the dot over a letter denotes, as usual, its time derivative. Let V be the potential energy of this system, a function which, in general, depends on the position of each particle and on time - it may also depend on the particles' velocities, which is the case of the so-called velocity-dependent potentials.

The Lagrangean is then defined as the difference between T and V :

$$L = T - V \qquad (I,2)$$

and is clearly a function of each particle's position, velocity and the time t:

$$L = L(\underset{\sim}{x}_1,\ldots\underset{\sim}{x}_n;\ \dot{\underset{\sim}{x}}_1,\ldots\dot{\underset{\sim}{x}}_n;t). \qquad (I,3)$$

The existence of such a function, independently of the special definition (I,2), will be assumed for the dynamical systems we study in classical mechanics with the objective of a later transposition or generalisation into quantum mechanics and quantum field theory.

The action S between any two instants t_1 and t_2 is the time-integral of the Lagrangean :

$$S = \int_{t_1}^{t_2} L \ dt. \qquad (I,4)$$

It depends not only on the two times t_1 and t_2 but also on the form of the function L : S is an ordinary function of t_1 and t_2 and a functional of L (this means that, for t_1 and t_2 fixed, to each function L there corresponds a numerical value for S) :

$$S = S[L \; ; \; t_1, \; t_2] \; . \tag{I,5}$$

Between the instants t_1 and t_2 there exists, a priori, an infinite set Γ of possible trajectories for the dynamical system; out of these, the system describes, in fact, that trajectory which makes S stationary, i.e., gives a minimum or maximum. This is Hamilton's principle or the postulate of minimal action or simply the action principle.

Let α denote a parameter designed to distinguish, for a given time t, the positions of each particle A in each of the possible trajectories of the set Γ:

$$\underset{\sim}{x}_A = \underset{\sim}{x}_A(t; \; \alpha) \tag{I,6}$$

As the integration in (I,4) is over t, the functional S in (I,5) will also depend on α :

$$S = S[L \; ; \; t_1, \; t_2; \; \alpha] \tag{I,5a}$$

Let a given value of α, $\alpha = 0$ say, correspond to that trajectory which makes S stationary and define the variation of x_{jA} as :

$$\delta x_{jA} = \left(\frac{\partial x_{jA}}{\partial \alpha} \right)_{\alpha = 0} d\alpha \; , \qquad j = 1, 2, 3 \; ; \; A = 1, 2, \ldots n.$$

The action principle is expressed by the requirement :

$$\delta S = 0 \; . \tag{I, 7}$$

For t_1 and t_2 fixed, one has :

$$\delta S = \int_{t_1}^{t_2} \delta L \; dt \tag{I, 8}$$

with (see fig.1)

$$\delta x_{jA}(t_1) = \delta x_{jA}(t_2) = 0 \; , \tag{I,8a}$$

and hence, according to (I,3) and (I,8)

$$\delta S = \int_{t_1}^{t_2} \sum_{A=1}^{n} \sum_{j=1}^{3} \left\{ \frac{\partial L}{\partial x_{jA}} \delta x_{jA} + \frac{\partial L}{\partial \dot{x}_{jA}} \delta \dot{x}_{jA} \right\} dt . \qquad (I,8b)$$

The interchange of the symbols δ and $\frac{d}{dt}$, and the identity :

$$\frac{\partial L}{\partial \dot{x}_{jA}} \frac{d}{dt} \delta x_{jA} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_{jA}} \delta x_{jA} \right) - \delta x_{jA} \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_{jA}} \right)$$

lead us to write (I,8b) in the following form :

$$\delta S = \int_{t_1}^{t_2} \sum_{A=1}^{n} \sum_{j=1}^{3} \left\{ \left[\frac{\partial L}{\partial x_{jA}} - \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_{jA}} \right) \right] \delta x_{jA} + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_{jA}} \delta x_{jA} \right) \right\} dt$$

A partial integration and the conditions (I,8a) make the last term of the above form of δS to vanish.

We are thus left with :

$$\delta S = \int_{t_1}^{t_2} \sum_{a} \sum_{j} \left\{ \frac{\partial L}{\partial x_{jA}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_{jA}} \right) \right\} \delta x_{jA} \, dt$$

The postulate (I,7) , valid for arbitrary variations δx_{jA}, thus leads to the well-known Lagrange equations :

$$\frac{\partial L}{\partial x_{jA}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_{jA}} = 0 \quad , \quad j = 1,2,3 \quad ; \quad A = 1,.. n \qquad (I,9)$$

The knowledge of the equations of motion of our system of particles is, in this way, connected with that of the Lagrangean.

To these equations of motion one must add the initial conditions in order to determine the orbit of the mechanical system. In classical mechanics, the <u>observables</u> are the positions and velocities of particles. The values of the 6 n quantities $x_{jA}(t_o)$, $\dot{x}_{jA}(t_o)$, at a given instant t_o – the results of the observations (or measurements) of these variables at this instant - may provide a set of such conditions.

The momentum of particle A, canonically conjugate to its coordinate x_{jA}, is defined by the relation :

$$p_{jA} = \frac{\partial L}{\partial \dot{x}_{jA}} , \qquad j = 1, 2, 3 \; ; \; A = 1, \ldots n. \tag{I,10}$$

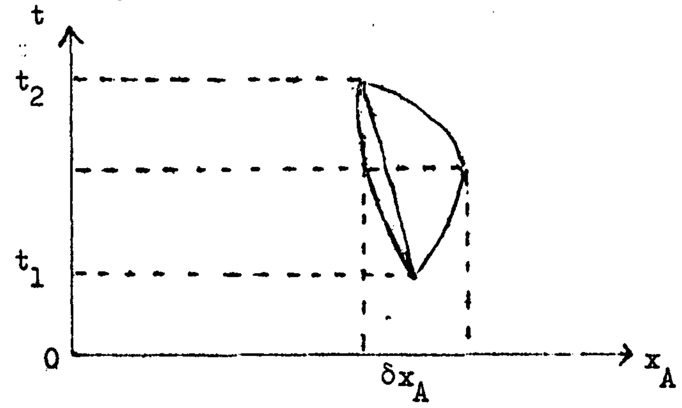


Figure 1

If these equations are solved so that one may express the velocities $\dot{x}$ in terms of the momenta, the Hamiltonian of the system, as defined by :

$$H = \sum_{j,A} p_{jA} \, \dot{x}_{jA} - L \tag{I,11}$$

will be a function of the independent variables x_{jA}, p_{jA} and t :

$$H = H\,(\underset{\sim}{x}_1, \underset{\sim}{p}_1, \ldots \underset{\sim}{x}_n, \underset{\sim}{p}_n \; ; \; t) \tag{I,11a}$$

The equations (I,9),(I,10) and (I,11) give the well-known canonical equations :

$$\dot{x}_{jA} = \frac{\partial H}{\partial p_{jA}} , \quad \dot{p}_{jA} = -\frac{\partial H}{\partial x_{jA}} \tag{I,12}$$

a system of 6 n differential equations of first-order, equivalent to the second order 3 n equations (I,9).

A typical illustration of the fact that the definition (I,2) is not always valid, is provided by a particle, the velocity of which is not small if compared to the velocity of light c; although the notion of potential energy $V(\underset{\sim}{x})$ is still assumed to be valid, a mass variation with the velocity is required by relativity theory :

$$m = \beta\, m_o \quad , \quad \beta = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} . \tag{I,13}$$

And the equation of motion :

$$\frac{d\, p_j}{dt} = - \frac{\partial V}{\partial x_j}$$

with

$$p_j = m\, v_j = m\, \dot{x}_j$$

is deduced from a Lagrangean :

$$L = T_o - V$$

where

$$T_o = m_o\, c^2 \left\{ 1 - \left(1 - \frac{v^2}{c^2}\right)^{1/2} \right\}$$

does not coincide with the particle's kinetic energy :

$$T = m_o\, c^2 \left\{ \left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1 \right\} \tag{I,14}$$

I.2) Conservation of mechanical energy.

In general, therefore, the Hamiltonian is a function of the particles' positions, momenta and time, as stated in formula (I,11a). Its total rate of change with time is thus :

$$\frac{d\, H}{dt} = \frac{\partial H}{dt} + \sum_{j,A} \left(\frac{\partial H}{\partial x_{jA}} \frac{d\, x_{jA}}{dt} + \frac{\partial H}{\partial p_{jA}} \frac{d\, p_{jA}}{dt} \right)$$

Each term of the summation on the right-hand side vanishes in virtue of (I,12):

$$\frac{d\, H}{dt} = \frac{\partial H}{\partial t} .$$

It is concluded that if the Hamiltonian does not depend on t in an explicit fashion (it depends implicitly on time through $x_{jA}(t)$ and $p_{jA}(t)$), its total rate of change with time vanishes :

$$\frac{dH}{dt} = 0$$

The Hamiltonian will then be equal to a constant E, conserved in time, the energy of the mechanical system :

$$H(\underset{\sim}{x}_1, \underset{\sim}{p}_1, \ldots \underset{\sim}{x}_n, \underset{\sim}{p}_n) = E .$$

I.3) Poisson brackets and constants of motion.

Let $F(\underset{\sim}{x}_1, \underset{\sim}{p}_1, \ldots, \underset{\sim}{x}_n, \underset{\sim}{p}_n; t)$ and $G(\underset{\sim}{x}_1, \underset{\sim}{p}_1, \ldots, \underset{\sim}{x}_n, \underset{\sim}{p}_n; t)$ be two dynamical variables of our system of particles - functions of time and of the particles' coordinates and momenta. An important quantity associated with this system is defined by means of such a pair of dynamical variables. It is the Poisson bracket of F and G , expressed as :

$$\left\{F, G\right\} = \sum_{j,A} \left(\frac{\partial F}{\partial x_{jA}} \frac{\partial}{\partial p_{jA}} - \frac{\partial F}{\partial p_{jA}} \frac{\partial G}{\partial x_{jA}} \right) \qquad (I,15)$$

This function is adequate to a more elegant form of the equations of motion, and allows a simple proof of fundamental theorems on symmetries and constants of motion, as we shall see. Moreover, it is this form of the equations of motion which allows a straightforward transition from the classical to the quantum-mechanical description of physical systems (which have a classical limit).

The total rate of change of F with time is :

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_{j,A} \left(\frac{\partial F}{\partial x_{jA}} \frac{d x_{jA}}{dt} + \frac{\partial F}{\partial p_{jA}} \frac{d p_{jA}}{dt} \right)$$

which, in view of the canonical equations (I,12) and the definition (I,15), takes the form :

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \left\{ F, H \right\} \qquad (I,16)$$

In particular, the canonical equations read :

$$\dot{x}_{jA} = \{x_{jA}, H\} \quad , \quad \dot{p}_{jA} = \{p_{jA}, H\} \qquad (I,17)$$

and the fundamental Poisson brackets, relative to the particles' coordinates and momenta, are :

$$\{x_{jA}, x_{kB}\} = \{p_{jA}, p_{kB}\} = 0 \, , \qquad (I,18)$$

$$\{x_{jA}, p_{kB}\} = \delta_{jk} \cdot \delta_{AB}$$

with

$$j,k = 1,2,3 \; ; \quad A, B = 1, 2, \ldots n.$$

A dynamical variable F is a constant of motion if its total rate of change with time vanishes :

$$\frac{dF}{dt} = 0$$

According to equation (I,16), this will be the case if F does not depend explicitely on time and its Poisson bracket with the Hamiltonian is equal to zero.

I.4) Symmetries and invariance principles.

The notion of symmetry - in sculpture, painting and architecture, in the crystalline and biological forms, in mathematics, in physics - has been masterfully discussed in a beatiful booklet by Herman Weyl.''If I am not mistaken, writes Weyl, the word symmetry is used in our every day language in two meanings. In the one sense symmetric means something like well-proportioned, well-balanced, and symmetry denotes that sort of concordance of several parts by which they integrate into a whole. Beauty is bound up with symmetry.(...) The image of the balance provides a natural link to the second sense in which the word symmetry is used in modern times : bilateral symmetry, the symmetry of left and right, which is so conspicuous in the structure of the higher animals, especially the human body. Now this bilateral symmetry is a stricly geometric and, in contrast to the vague notion of symmetry discussed before, an absolute precise concept. A body, a spatial configuration, is symmetric to a given plane

E if it is carried into itself by reflection in E.(...) A mapping is defined whenever a rule is established by which every point p (of a spatial configuration) is associated with an image p'. Another example : a rotation around a perpendicular axis (to a plane) , say by 30°, carries each point p of space into a point p' and thus defines a mapping. A figure has rotational symmetry around an axis ℓ if it is carried into itself by all rotations around ℓ ''.(1)

One is thus led to consider special sets of mappings, of one-to-one transformations of a given space into itself, with respect to which the laws of a theory are invariant. These special sets are the so-called groups; a set S of transformations $T_1, T_2, \ldots$ is a group if, given any two elements of this set, T_i, T_j, one can define a product (or composition) $T_k = T_i\ T_j$ such that:

a) T_k belongs to the set S

b) the identity I is an element of S and is such that for any T_j: $T_j I = I T_j = T_j$

c) to each element T_j of S there is associated an inverse, T_j^{-1} , belonging to S, and defined by the equality

$$T_j^{-1}\ T_j = T_j\ T_j^{-1} = I.$$

The symmetries of a spatial configuration, of a physical system, of a set of laws of nature, are thus defined by those groups of transformations which leave invariant the given configuration, physical system, or natural laws - and they are thus appropriately called the corresponding symmetry groups.

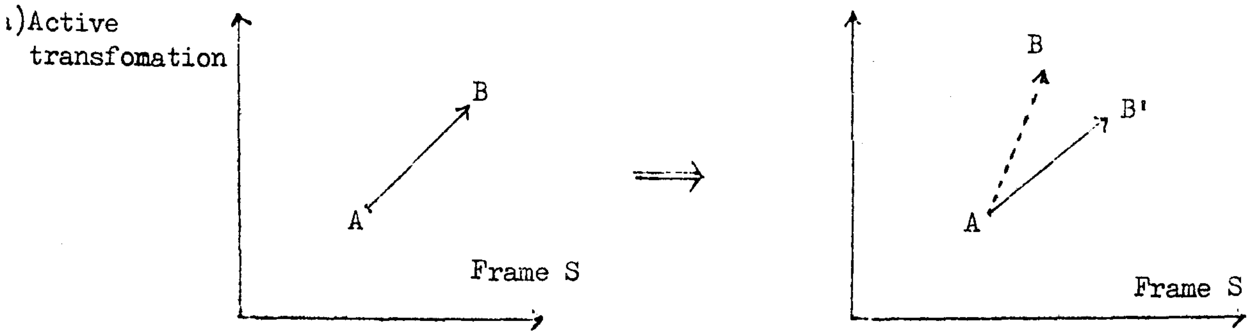
Of course, the search for such symmetries, for the invariance properties of the laws of nature is meaningful; although they do not change the forms of these laws, they have observable consequences because the initial conditions associated to the laws of nature are not unchanged under the transformations of the group.

According to Houtappel, Van Dam and Wigner (2), one can distinguish two types of definition of an invariance principle. To the adepts of the first definition, the generally valid invariance principles are only those which postulate the equivalence of the frames of reference which can physically be changed into each other. As a frame of reference consists of an observer equipped with all necessary measuring devices, this definition restricts the invariance transformations to space and time translations, to rotations and, more generally, to inhomogeneous Lorentz transformations.

The second definition of an invariance principle postulates as generally valid all those(simple) transformations which leave the laws of nature invariant, independently of whether one can or cannot physically change one observer and its apparatus into another by such transformations. The operation of time reversal or the transformation CPT which consists of the product of time reversal, space reflection and charge conjugation are examples of such transformations which cannot physically carry an observer into another (in this part of our world).

It is within the context of the first definition - that of physically equivalent reference frames - that one may distinguish, following Wigner, two equivalent points of view for carrying an invariance transformation. The active transformations change the object; the observer in his reference frame investigates the correlation between his measurements on the object before and after the transformation. According to the passive viewpoint , the transformations are correlations between the observations made by two different observers on the same object (fig.2). It is the passive type of geometric and kinematic transformations which is usually considered in classical physics, mainly, after the fundamental papers and booklets by Einstein on the special theory of relativity. Clearly, in the example pictured in fig.2, one can always, given the active transformation AB $\Rightarrow$ AB' for the observer S, imagine a second observer, S', who is in the same relation to the observations made on AB' as the first observer S is to his measurements on the original system AB.

It must, however, be mentioned that the second definition of invariance principle - which regards as valid all those simple transformations which keep the laws of nature invariant - is more general and allowed the discovery of non-geometrical (or dynamical) symmetries of certain interactions, such as the isospin, and more generally , the unitary spin symmetry of strong interactions in elementary particle physics. The gauge transformations , associated to the definition of current and charge conservation, are also non-geometrical.



b) Passive transformation

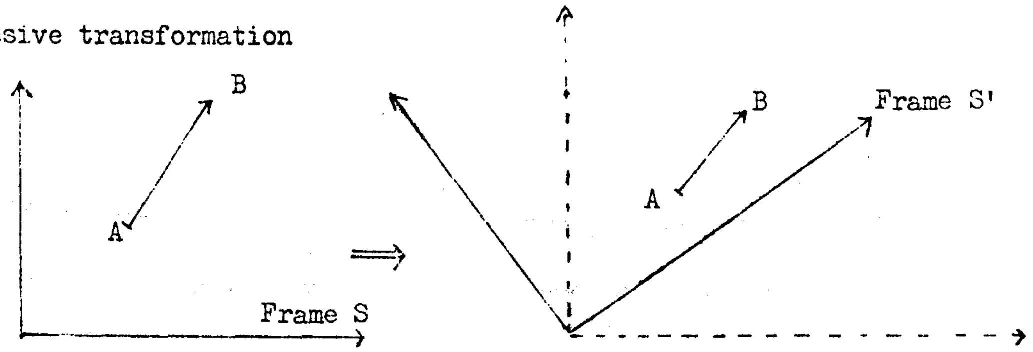


Figure 2

I.5) Symmetries and the structure of physical space in classical mechanics.

The equations of a classical system of point-particles are invariant under certain transformations of reference frames. The invariance under each specific group of transformations results from a basic principle, of empirical origin.

a) First of all, there is the principle of absolute time in Newtonian mechanics. It states that the time interval between two mechanical events, as measured in a reference frame, is independent of the state of motion of the frame. The principle is not generally true - it holds only for velocities which are small compared to the light velocity c. Newton, however, believed in an ''absolute, true and mechanical time, of itself, and from its own nature, (which) flows equably , without relation to anything external, and by another name is called duration'' . He, therefore, assumed that the clocks of all observers, whatever their state of relative motion, could be synchronised by means of signals, which would have to propagate with infinite speed. Newtonian mechanics is a theory of action at a distance. Actually, all known physical interactions propagate with a finite velocity and Einstein's theory of relativity allows one to show that this maximum velocity is equal to c.

b) The equation of motion of an isolated system of n point-particles, in an inertial frame, are of the form :

$$m_A \ddot{x}_{jA} = \sum_{B \neq A} F_{jB} (\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n, \dot{\underset{\sim}{x}}_1, \ldots \dot{\underset{\sim}{x}}_n ; t)$$

$$j = 1, 2, 3 \; ; \; A,B = 1, 2, \ldots n,$$

where the forces are, in general, functions of the positions and velocities and of the time at which these observables are measured.

The principle of homogeneity of the physical space, in classical mechanics states that the equations (I,19) are invariant with respect to a change of the origin of the reference frame. In other words, there exists no privilegied points in physical space, the relations between phenomena within this mechanical system must not depend on where they are determined (the absolute origin does not exist).

Let :

$$x'_{jA} = x_{jA} - a_j \quad , \quad j = 1, 2,3 \ ; \ A = 1,\ldots n \tag{I,20}$$

be the coordinates of our particles in the new reference frame, displaced by the constant vector $\underset{\sim}{a}$ from the original one. It is seen that the displacement invariance requires that the force depend only on the mutual distances between the point-particles :

$$m_A \ddot{x}_{jA} = F_{jA} = \sum_{B \neq A} F_{jBA} (\underset{\sim}{x}_{BA}, \dot{\underset{\sim}{x}}_{BA} \ ; \ t) \tag{I,19a}$$

The transformations (I,20) form the group of translations $T(\underset{\sim}{a})$, depending on the three parameters aj : the product of two translations $T(\underset{\sim}{a})$ and $T(\underset{\sim}{b})$ is a translations by $\underset{\sim}{a} + \underset{\sim}{b}$: $T(\underset{\sim}{a})\ T(\underset{\sim}{b}) = T(\underset{\sim}{a} + \underset{\sim}{b})$; the identity is $T(0)$ and the inverse of $T(\underset{\sim}{a})$ is $T(-\underset{\sim}{a})$.

c) The principle of isotropy of the physical space states that the equations of motion (I,19) are invariant with respect to any rotation of the reference frame. In other words, there are no privileged directions in space, the relations between events within our system of point-particles do not depend on the orientation of the whole system in space.

A rotation around the origin of the cartesian coordinates system associated to reference frame S, with basis $|e_1)$, $|e_2)$, $|e_3)$, changes all vectors :

$$|x_A) = \sum_{j=1}^{3} x_{jA} \ | e_j) \quad , \quad A = 1,\ldots n \tag{I,21}$$

into new vectors :

$$|x'_A) = \sum_{j=1}^{3} x_{jA} \ | e'_j) \ , \tag{I,22}$$

where $|e'_1)$, $|e'_2)$, $|e'_3)$ are the basis-vectors of the new coordinate system S'.

Expressed in terms of the basis of the original system S, the new vectors $|x'_A)$ will have new coordinates x'_{jA} :

$$|x'_A) = \sum_j x'_{jA} \; |e_j) \qquad (I,23)$$

In correspondance with the vector space Σ spanned by all linear combinations (I,21), one defines a <u>dual space</u>, a vector of which will be denoted by the symbol $(x|$. The space Σ will have euclidian structure if, to each vector $|b)$ of R and $(a|$ of its dual, one associates a number denoted by $(a|b)$, the scalar product of the two vectors with the properties :

1) $(a|a) \geqslant 0$ and $(a|a) = 0$ implies $|a) = |0)$ where $|0)$ is the null vector;
2) $(a|b) = (b|a)^*$, where in general Σ comprises vectors with complete numbers as coordinates;
3) α being a complex number one must have $(a|\alpha b) = \alpha (a|b)$;
4) $((a_1+a_2)|b) = (a_1|b) + (a_2|b)$

If the two given coordinate systems S and S' are cartesian, their bases will be orthonormal, i.e.

$$(e_k|e_\ell) = (e'_k|e'_\ell) = \delta_{k\ell}, \qquad k,\ell = 1, 2, 3 \qquad (I,24)$$

Therefore the comparison of equations (I,22) and (I,23) gives rise, in view of (I,24), to the well-known homogeneous linear relations between the coordinates x_{jA} and x'_{jA} :

$$x'_{jA} = \sum_k a_{jk} \, x_{kA} \qquad (I,25)$$

corresponding to the given rotation . And this is determined by the 9 numbers a_{jk} :

$$a_{jk} = (e_j|e'_k)$$

the cosinus of the angle between the k-axis of the system S' with the j-axis of system S.

The length of a vector $|x)$ is the non-negative number $((x|x))^{1/2}$ where, in cartesian coordinate systems :

$$(x|y) = \sum_j x_j^* y_j$$

These homogeneous linear transformations (I,25) which conserve the length of all vectors or, more generally, the scalar product :

$$(x|y)' = (x'|y') \qquad (I,26)$$

are called <u>unitary</u> transformations. In the particular case of real vector spaces, these transformations are called <u>orthogonal</u>. The latter ones keep invariant the bilinear form :

$$\sum_j x'_{jA} y'_{jA} = \sum_j x_{jA} y_{jA}$$

whence

$$\sum_j a_{jk} a_{j\ell} = \delta_{k\ell} \qquad (I,27)$$

(For the unitary transformations one has : $\Sigma a^*_{jk} a_{j\ell} = \delta_{k\ell}$).

Let us represent the nine numbers a_{jk} by a matrix R

$$R = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad (I,28)$$

The vector $|x_A)$ which has coordinates x_{jA} in a basis of the coordinate frame S will be represented by the one column matrix :

$$|x_A) = \begin{pmatrix} x_{1A} \\ x_{2A} \\ x_{3A} \end{pmatrix} \quad , A = 1,\ldots n$$

if one chooses the following representations for the basis :

$$|e_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad ; \quad |e_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad ; \quad |e_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The rule of multiplication of two matrices,

$$(R_1 R_2)_{k\ell} = \sum_n (R_1)_{kn} (R_2)_{n\ell}$$

leads us to write the equations (I,25) under the form :

$$|x'_A) = R\,|x_A) \qquad (I,29)$$

The reader will be able to prove the relationship :

$$\sum_j |e_j)(e_j| = 1 \ . \qquad (I,30)$$

It is clear from the representations above that in the case of a complex vector space, the passage to its dual space is performed by the operation of transposition and complex conjugation. In a real vector space, the dual vectors are the transposed (column $\rightarrow$ line) of the given ones. The fact that, in this case, the coordinates of $|x)$ and $(x|$, in the same basis, are the same real numbers, is usually translated by stating that the space and its dual coincide.

The relations (I,27) and (I,28) allow us to write for the orthogonal matrices :

$$R^T R = R R^T = I \qquad (I,31)$$

where R^T means the transposed of R and I is the unit matrix. For unitary matrices U one has :

$$U^+ U = U U^+ = I$$

where $U^+ = U^{*T}$, the hermitian conjugate of U, is obtained by transposition and complex conjugation of U. It follows from (I,26), (I,29) and (I,31) that:

$$(x'_A| = (x_A|\,R^T \qquad (I,32)$$

As the determinant of a matrix is the same as that of its transposed, one concludes from (I,31) :

$$\det(R) = \pm 1 \ . \qquad (I,33)$$

When transformations R have determinant +1

$$\det(R) = +1.$$

They constitute the so-called group of proper rotations - they can be continously generated from the identity.
When det(R) = -1, the transformations are called improper - they contain the spatial reflection I_S (with respect to the origin)

$$x'_k = -x_k \quad , \qquad k = 1, 2, 3.$$

An improper transformation can be expressed as the product of I_S and a rotation.

Clearly, the equations (I,19) are invariant with respect to the group of rotations if the forces transform like vectors :

$$F'_{jBA}(x'_{BA}, \dot{x}'_{BA}; t) = \sum_k a_{jk} F_{kBA}(x_{BA}, \dot{x}_{BA}; t) \qquad (I,34)$$

d) The principle of classical (newtonian or Galilei) relativity states that the mechanical laws of an isolated system of n - point particles established in a inertial frame of reference, S, are the same for another frame S', in rectilinear and uniform motion with respect to S. In other words, it is impossible, by means of mechanical observations within the system of point-particles, to detect the rectilinear uniform motion of an inertial reference frames.

The transformations between the coordinates of a point-particle A as referred to two such frames S and S' are :

$$x'_{jA} = x_{jA} - v_j t \quad , \quad j = 1, 2, 3; \; A = 1,\ldots n, \qquad (I,34a)$$

where v_j are the components of the constant velocity of S' with respect to S. They form a group which depends on the three parameters v_j, the homogeneous Galilei - group , $G(\underset{\sim}{v})$.

It is clear that the equations of motion (I,19a) will be invariant if the forces F_{jA} are Galilei-covariant vector functions of the instantaneous mutual distances and velocities of the particles; these forces have the form (2) :

$$F_{jA} = \sum_{B} (x_{jBA} f_{BA} + \dot{x}_{jBA} g_{BA}) + \sum_{BC} (\underset{\sim}{x}_{BA} \wedge \underset{\sim}{x}_{CA})_j f_{BCA} + (\underset{\sim}{\dot{x}}_{BA} \wedge \underset{\sim}{\dot{x}}_{CA})_j f_{BCA} +$$

$$+ (\dot{x}_{BA} \wedge \dot{x}_{CA}) g_{BCA} + (\underset{\sim}{x}_{BA} \wedge \underset{\sim}{\dot{x}}_{CA}) h_{BCA}$$

where the f's, g's and h are Galilei-invariant functions of the coordinates and velocities.

I.6) Canonical transformations and the Hamilton - Jacobi equation.

The connection between symmetries and constant of motion or conservation laws does not follow from Newton's equations of motion, but rather from the Lagrange's form of these equations. This will be seen in paragr. I,10. Until then we need to review some needed concepts.

Hamilton's principle, (I,4),(I,7) can be written, in view of the equation (I,11):

$$\delta \int_{t_1}^{t_2} \left\{ \sum_{jA} p_{jA} \, dx_{jA} - H \, dt \right\} = 0 \qquad (I,35)$$

We now introduce a new reference frame, S', with respect to which the coordinates and momenta of our system of point-particles, x'_{jA}, p'_{jA}, will be functions of the analogous variables as measured in the old frame S :

$$\begin{aligned} x'_{jA} &= x'_{jA}(\underset{\sim}{x}_1, \underset{\sim}{p}_1, \ldots, \underset{\sim}{x}_n, \underset{\sim}{p}_n \; ; \; t) \\ p'_{jA} &= p'_{jA}(\underset{\sim}{x}_1, \underset{\sim}{p}_1, \ldots, \underset{\sim}{x}_n, \underset{\sim}{p}_n \; ; \; t) \\ j &= 1,2,3 \; ; \; A = 1,2,\ldots n. \end{aligned} \qquad (I,36)$$

It will be assumed that the mapping functions (I,36) are continuous, differentiable and have a non-vanishing Jacobian :

$$\det \begin{pmatrix} \dfrac{\partial x'_{11}}{\partial x_{11}} & \cdots & \dfrac{\partial x'_{11}}{\partial p_{3n}} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial p'_{3n}}{\partial x_{11}} & \cdots & \dfrac{\partial p'_{3n}}{\partial p_{3n}} \end{pmatrix} \neq 0 \qquad (I,36a)$$

so that there will exist an inverse mapping :

$$x_{jA} = x_{jA}(\underset{\sim}{x}'_1 , \underset{\sim}{p}'_1, \dots, \underset{\sim}{x}'_n, \underset{\sim}{p}'_n ; t)$$
$$p_{jA} = p_{jA}(\underset{\sim}{x}'_1 , \underset{\sim}{p}'_1, \dots, \underset{\sim}{x}'_n , \underset{\sim}{p}'_n ; t) \qquad (I,36b)$$

The group of canonical (or contact) transformations is the group of mappings (I,36),(I,36a) which leave the laws of the mechanics of systems of point-particles,(I,9) or (I,12), invariant.

For simplicity, let us represent by $X = (x,p)$ a point of the 6n-dimensional phase-space, the coordinates of which are x_{jA}, p_{jA} with $j = 1, 2, 3$; $A = 1,\dots n$.

Let $F(X,t)$ be a dynamical variable, a function of the observables x, p measured at a given time. This function will be mapped into another one,$F'(X',t)$ by the canonical transformations (I,36); the latter will be symbolised by the application of the operator $\mathcal{C}$ of the corresponding group :

$$X' = \mathcal{C} X \quad , \quad X = \mathcal{C}^{-1} X' \quad . \qquad (I,37)$$

The group of canonical transformations $\mathcal{C}$ defined in the phase-space induces a set of transformations T in the space of functions $F(X,t)$:

$$F'(X,t) = T\, F(X,t) \qquad (I,38)$$

The transformed functions of the transformed orbits, $F'(X',t)$ will be defined as those which have the same values which the original functions assume for the original orbits :

$$F'(X',t) = F(X,t) \qquad (I,39)$$

In this way one will be able to associate the product of two mappings T_1T_2 in the space of functions F to the product $\mathcal{C}_1\ \mathcal{C}_2$ in the phase-space. In fact, it follows from (I,37),(I,38),(I,39) that(omitting t) :

$$T_2\, F(X) = F'(X) = F(\mathcal{C}_2^{-1} X) \qquad (I,40)$$

One then substitutes $\mathcal{C}_1^{-1} X$ for X; this will induce a transformation which reads, according to (I,39) :

$$T_1 F'(X) = F(\mathcal{C}_1^{-1} X)$$

and hence, in view of (I,40):

$$T_1 T_2 F(X) = F(\mathcal{C}_2^{-1} \mathcal{C}_1^{-1} X) \quad F([\mathcal{C}_1 \mathcal{C}_2]^{-1} X) \qquad (I,41)$$

The Lagrangean and the Hamiltonian, however, does not satisfy the relation (I,39) because the definition of canonical transformations requires the invariance of the variation of the functional (I,5a):

$$\delta S [L; t_1, t_2; \alpha] = \delta S [L'; t_1, t_2; \alpha]$$

where $L'(x', \dot{x}', t)$ is the new Lagrangean. This is satisfied by requiring that the two functionals differ by an ordinary function of t_1 and t_2.

$$S [L ; t_1, t_2; \alpha] = S [L'; t_1, t_2; \alpha] + f(t_1, t_2) . \qquad (I,42)$$

This may be satisfied by the differential condition :

$$L\, dt = L'\, dt + dW \qquad (I,43)$$

where $W = W(\underset{\sim}{x}_1, \ldots \underset{\sim}{x}_n; \underset{\sim}{x}'_1, \ldots \underset{\sim}{x}'_n; t)$ and dW is the exact differential of W :

$$dW = \frac{\partial W}{dt}\, dt + \sum_{jA} \left(\frac{\partial W}{\partial x_{jA}}\, dx_{jA} + \frac{\partial W}{\partial x'_{jA}}\, dx'_{jA} \right).$$

One sees that, in this case :

$$f(t_1, t_2) = W(\underset{\sim}{x}_1(t_2) \ldots ; t_2) - W(\underset{\sim}{x}_1(t_1), \ldots ; t_1)$$

and (in view of (I,8a)) : $\delta f = 0$.

From the relations (I,43) and (I,11) it follows that :

$$p_{jA} = \frac{\partial W}{\partial x_{jA}} \quad , \quad p'_{jA} = - \frac{\partial W}{\partial x'_{jA}} \quad , \quad H' = H + \frac{\partial W}{\partial t} \qquad (I,44)$$

where $H'(\underset{\sim}{x}'_1, \ldots, p'_1, \ldots, t)$ is the new Hamiltonian .

These equations determine a canonical transformation. From the first of equations (I,44) there results a functional relationship among the x_{jA}, x'_{jA}

and p_{jA} :

$$x'_{jA} = x'_{jA}(x_{11}, p_{11}, \dots x_{3n}, p_{3n} ; t) \qquad j = 1, 2, 3; \; A = 1, 2, \dots n;$$

The latter equations together with the second (I,44) give rise to the following ones :

$$p'_{jA} = p'_{jA}(x_{11}, p_{11}, \dots, x_{3n}, p_{3n}; t).$$

The Hamilton-Jacobi equation allows the determination of the function W - the generating function of the canonical transformation in question. It is obtained by the requirement that H' = 0 :

$$H(x_{11}, \dots x_{3n}, \frac{\partial W}{\partial x_{11}}, \dots \frac{\partial W}{\partial x_{3n}} ; t) + \frac{\partial W}{dt} = 0 \qquad (I,45)$$

so that the new coordinates and momenta are constants of motion :

$$\dot{x}'_{jA} = 0 \quad , \quad \dot{p}'_{jA} = 0$$

The integral $W(\underset{\sim}{x}_1, \dots \underset{\sim}{x}_n; \underset{\sim}{x}'_1, \dots \underset{\sim}{x}'_n; t)$ of the equation (I,45) will contain 3n constants x'_{jA} and another set of 3n constants given by the second equations (I,44).

I.7) Infinitesimal canonical transformations.

In general, a mapping of a space Σ into another, Σ', is continuous at the point $x_o \in \Sigma$ if an arbitrary neighborhood $N(f(x_o),\varepsilon)$, of radius ε, of the point $f(x_o)$ contains a neighborhood $N(x_o,\delta)$ of the point x_o, the transformed (image) of which is contained in $N(f(x_o),\varepsilon)$. In other words, if $f(x) \rightarrow f(x_o)$ when $x \rightarrow x_o$ (3). A continuous transformation is a continuous function of its parameters. Thus, a rotation in 3-dimensional real space is a continuous function of three independent parameters a_{jk} (see (I,27),(I,28)), a Galilei transformation is a continuous function of the three parameters v_j (I,34 a).

Clearly an infinitesimal transformation maps every point x_o of the set in which it is defined into another one arbitrarily near x_o.

Let $\varepsilon_{(k)}$ be the set of (infinitesimal) parameters of an infinitesimal canonical transformation. This means that one restricts oneself to those mappings (I,36)which can be put in the form :

$$x'_{jA} = x_{jA} + \delta\, x_{jA} \quad ,$$
$$p'_{jA} = p_{jA} + \delta\, p_{jA} \qquad (I,46)$$

where
$$\delta\, x_{jA} = \sum_k \varepsilon_{(k)} \left(\frac{\partial x'_{jA}}{\partial \varepsilon_{(k)}}\right)_o ,$$
$$\delta\, p_{jA} = \sum_k \varepsilon_{(k)} \left(\frac{\partial p'_{jA}}{\partial \varepsilon_{(k)}}\right)_o \quad , \quad j = 1, 2, 3; \; A = 1,\ldots n.$$

The index o stands for all $\varepsilon_{(k)} = 0$ in the derivatives. Thus,for a dynamical variable F, δ F is proportional to ε_k whereas d F is proportional to dt.

Let δ W be the Hamilton Jacobi function for an infinitesimal transformation; one has in view of equations (I,11),(I,43),(I,46) :

$$d(\delta\, W) = - \sum_{jA} \left[p_{jA}\, d(\delta x_{jA}) + \delta\, p_{jA}\, dx_{jA} \right] + \delta\, H\, dt \qquad (I,47)$$

where $\delta\, H = H' - H$.

The definition of functions $U_{(k)}(x_{11},\ldots;\, p_{11},\ldots;\, t)$ by the relationship:

$$\delta\, W = - \sum_{jA} p_{jA}\, \delta x_{jA} + \sum_k \varepsilon_{(k)}\, U_{(k)} \qquad (I,48)$$

gives rise, in view of (I,47) and (I,46) to the equation :

$$\sum_{jA} \left[\frac{\partial U_{(k)}}{\partial x_{jA}} \dot{x}_{jA} + \frac{\partial U_{(k)}}{\partial p_{jA}} \dot{p}_{jA} \right] + \frac{\partial U_{(k)}}{\partial t} = \sum_{jA} \left[-\left(\frac{\partial p'_{jA}}{\partial \varepsilon_{(k)}}\right)_o \dot{x}_{jA} + \left(\frac{\partial x'_{jA}}{\partial \varepsilon_{(k)}}\right)_o \dot{p}_{jA} \right] + \left(\frac{\partial H}{\partial \varepsilon_{(k)}}\right)_o$$

The following are , therefore, the equations which the functions $U_{(k)}$ satisfy and define an infinitesimal canonical transformation :

$$\left(\frac{\partial\, x'_{jA}}{\partial\, \varepsilon_{(k)}}\right)_o = \frac{\partial\, U_{(k)}}{\partial\, p_{jA}} \quad ,$$
$$\left(\frac{\partial\, p'_{jA}}{\partial\, \varepsilon_{(k)}}\right)_o = - \frac{\partial\, U_{(k)}}{\partial\, x_{jA}} \quad , \qquad (I,49)$$
$$\left(\frac{\partial\, H'}{\partial\, \varepsilon_{(k)}}\right)_o = \frac{\partial\, U_{(k)}}{\partial t}$$

It was, in fact, to get equations (I,49), similar to canonical equations (I,12), that the functions $U_{(k)}$ - the generators of the canonical transformation - were introduced by means of the relationship (I,48).

I.8) Transformation of dynamical variables under the group of canonical transformations; Generators of time and space translations and of rotations.

Let $F(x,p;t)$ be a dynamical variable (where $x = (x_{11},\ldots x_{3n}), p=(p_{11},\ldots p_{3n})$) such that, for a canonical transformation (I,36) one has, according to (I,39):

$$F'(x',\ p';\ t) = F(x(x',\ p'),\ p(x',\ p');\ t)$$

In the case of an infinitesimal transformation (I,46), one can write :

$$F'(x',\ p';\ t) \cong F'(x + \delta x,\ p + \delta p;\ t)$$

or

$$F'(x',p';t) = F'(x,p;t) + \sum_{kjA} \varepsilon_{(k)} \left[\frac{\partial F'}{\partial x'_{jA}} \frac{\partial x'_{jA}}{\partial \varepsilon_{(k)}} + \frac{\partial F'}{\partial p'_{jA}} \frac{\partial p'_{jA}}{\partial \varepsilon_{(k)}} \right] \text{o}$$

up to terms in the first power of the ε's. We shall then have, in view of the equations (I,49) :

$$\delta F = \sum_{k} \varepsilon_{(k)} \left\{ F,\ U_{(k)} \right\} \qquad (I,50)$$

where

$$\begin{aligned} \delta F &= F'(x',p';t) - F'(x,p;t) \\ &= F\ (x(x',p'),\ p(x',p');t) - F'(x,p;t), \qquad (I,50a) \end{aligned}$$

$$\lim_{\varepsilon \to 0} \frac{\partial F'}{\partial x'_{jA}} = \frac{\partial F}{\partial x_{jA}}, \quad \lim_{\varepsilon \to 0} \frac{\partial F'}{\partial p'_{jA}} = \frac{\partial F}{\partial p_{jA}} \quad .$$

The equation (I,50) states that the change in form of the dynamical variables F corresponding to an infinitesimal canonical mapping of the coordinates and momenta, at a given time, is determined by the Poisson bracket of F with the genarators of this mapping.

We shall now proceed to identify the generating functions of the simplest canonical transformations.

a) The Hamiltonian is the generator of an infinitesimal time displacement (and a movement is the successive composition of such infinitesimal transformations).

In fact, the identification of the parameter ε with dt gives :

$$\delta x_{jA} = \left(\frac{\partial x'_{jA}}{\partial t}\right)_{t=0} dt$$

$$\delta p_{jA} = \left(\frac{\partial p'_{jA}}{\partial t}\right)_{t=0} dt$$

or

$$x'_{jA}(t,\varepsilon) \equiv x_{jA}(t + dt)$$

$$p'_{jA}(t,\varepsilon) = p_{jA}(t + dt)$$

The equations (I,49) show that we can, in fact, identify the generator U with the Hamiltonian :

$$U = H \quad \text{for} \quad \varepsilon = dt \qquad \text{(I,51)}$$

In this case the reader will show the validity of the equation : $\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F,H\}$ (by considering $F'(x + dx, p + \delta p; t + dt)$). (I,51a)

b) The total momentum of a system of point-particles is the generator of an infinitesimal space translation.

Here the parameters $\varepsilon_{(k)}$ are the three infinitesimal components of an ordinary vector :

$$x'_{jA} = x_{jA} + \varepsilon_j \quad , \qquad j = 1, 2, 3; \quad A = 1,\dots,n.$$

Therefore

$$\frac{\partial x'_{jA}}{\partial \varepsilon_k} = \delta_{jk}$$

for all particles A, whence, according to (I,49) :

$$\frac{\partial U_k}{\partial p_{jA}} = \delta_{jk} \quad ,$$

$$\frac{\partial U_k}{\partial x_{jA}} = 0$$

that is to say

$$U_k \equiv P_k = \sum_{A=1}^{n} p_{kA} \quad , \quad k = 1, 2, 3 \qquad (I,52)$$

c) <u>The generator of an infinitesimal rotation of the coordinate system around the origin is the angular momentum of our system of point-particles</u>.

For such a rotation the parameters a_{jk} in (I,25) will differ from δ_{jk} by very little :

$$a_{jk} = \delta_{jk} + \varepsilon_{jk}$$

The orthogonality condition (I,27) requires that the parameters ε_{jk} be anti-symmetric :

$$\varepsilon_{jk} = - \varepsilon_{kj} \, , \qquad k \, , \, j = i, 2, 3.$$

The three independent values of this tensor are the parameters of the group. In association with these are the generators U_{jk} , defined by a trivial extension of the equation (I,48) :

$$\delta W = - \sum_{jA} p_{jA} \, dx_{jA} + \frac{1}{2} \sum_{k\ell} \varepsilon_{k\ell} U_{k\ell} \quad ,$$

$$U_{k\ell} = - U_{\ell k}$$

Then the equations which define these mappings, corresponding to (I,49) are now :

$$\sum_n \varepsilon_{jn} x_{nA} = \frac{1}{2} \sum_{k\ell} \varepsilon_{k\ell} \frac{\partial U_{k\ell}}{\partial p_{jA}} \quad ,$$

$$\sum_n \varepsilon_{jn} p_{nA} = - \frac{1}{2} \sum_{k\ell} \varepsilon_{k\ell} \frac{\partial U_{k\ell}}{\partial x_{jA}}$$

which, in view of the equations :

$$x'_{jA} = x_{jA} + \varepsilon_{jk} x_{kA}$$

are satisfied by the generator :

$$U_{\ell k} \equiv L_{k\ell} = \sum_{A=1}^{n} (x_{kA} p_{\ell A} - x_{\ell A} p_{kA}) \quad . \qquad (I,53)$$

where $L_{k\ell}$ are the components of the orbital angular momentum of the system of point-particles.

The reader will be able to verify, in the simple example where the dynamical variable is a position coordinate :

$$F_j(x\ ,p) = x_j$$

that

$$F'_j(x',p') = F_j(x(x',p'),p(x',p')) = x'_j - \varepsilon_{jk}\, x'_k$$

and hence

$$F'_j(x,p) = x_j - \varepsilon_{jk}\, x_k\ ,$$

$$\delta\, F_j = F_j(x,p) - F'_j(x,p) = \varepsilon_{jk}\, x_k$$

which coincides with $1/2\ \varepsilon_{k\ell}\left\{F_j, U_k\right\}$ and (I,53).

I.9) Fundamental theorem on symmetries and constants of motion.

If the Hamiltonian of a mechanical system is invariant under all transformations of a canonical group G, the generators of this group are constants of motion.

Let us, in fact, oonsider equation (I,50) and set

$$F = H$$

under the assumption that :

$$\delta H = 0 \qquad \text{(I,54)}$$

or (see (I,47))

$$H' = H\ .$$

It follows from equation (I,50) that, for arbitrary values of the parameters $\varepsilon_{(k)}$:

$$\left\{H, U_{(k)}\right\} = 0 \qquad \text{(I,54a)}$$

On the other hand , the last of the equations (I,49) and the requirement (I,54) imply that the generators $U_{(k)}$ do not depend explicitely on time :

$$\frac{\partial\, U_{(k)}}{\partial t} = 0$$

Therefore, from the relation (I,51 a) :

$$\frac{d\,U_{(k)}}{dt} = \frac{\partial\,U_{(k)}}{\partial t} + \left\{U_{(k)}, H\right\}$$

one concludes that the $U_{(k)}$s are constants of motion :

$$\frac{d\,U_{(k)}}{dt} = 0 .$$

The following[2]/well-known results are, thus, corrollaries of this theorem:

- Any mechanical system of point-particles the Halmitonian of which is invariant with respect to the group of translations, admits of its total momentum (I,52) as constant of motion.

- If the invariance of the Hamiltonian is under the group of rotations, the constants of motion will be the components of the angular momentum of the system, (I,53).

A group is said to be a symmetry group of the (Hamiltonian of the) system if its generators satisfy the equation (I,54a).

The group of inhomogeneous orthogonal transformations (translations plus rotations) as a symmetry group determines, therefore, the linear and angular momenta of the system as constants of motion.

Finally, let us note the following Poisson brackets between the generators of space translations and rotations, (I,52) and (I,53):

$$\begin{aligned}
\left\{P_j, P_k\right\} &= 0, \\
\left\{P_j, L_{k\ell}\right\} &= \delta_{j\ell} P_k - \delta_{jk} P_\ell , \\
\left\{L_{ij}, L_{k\ell}\right\} &= \delta_{ik} L_{j\ell} + \delta_{j\ell} L_{ik} - \delta_{i\ell} L_{jk} - \delta_{jk} L_{i\ell}
\end{aligned}$$

If one identifies the three components of the total orbital angular momentum of a system of point-particles , L_x, L_y, L_z, with :

$$L_x = L_{23} \quad , L_y = L_{31} , L_z = L_{12}$$

the last relationship leads to

$$\begin{aligned}
\left\{L_x, L_y\right\} &= L_z , \\
\left\{L_z, L_x\right\} &= L_y \\
\left\{L_y, L_z\right\} &= L_x .
\end{aligned}$$

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II

SYMMETRIES IN CLASSICAL LINEAR FIELD THEORY

II.1) The electromagnetic laws and the Galilei relativity principle.

It is well-known that the Galilei relativity principle cannot be extended to the electromagnetic laws. This means that experimental observations do not confirm the transformation laws of the electromagnetic variables corresponding to an assumed invariance of Maxwell's equations under the Galilei group.

These equations have the following form (in the M.K.S. system) :

$$\begin{aligned} \nabla \cdot D &= \rho , \\ \nabla \cdot B &= 0 , \\ \nabla \wedge E + \frac{\partial B}{\partial t} &= 0 , \\ \nabla \wedge H - \frac{\partial D}{\partial t} &= \vec{j} \end{aligned} \qquad \text{(II,1)}$$

where ρ and $\vec{j}$ are the charge and current densities, and E, D and H, B are the electric and magnetic pairs of field and induction , respectively. The dot indicates as usual the scalar product and the sign $\wedge$, the vector product.

A transformation from a reference frame S to a Galilei - transformed frame S' :

$$x' = x - v\,t \quad , \quad t' = t \qquad \text{(II,2)}$$

gives the following relationship between differential operators :

$$\begin{aligned} \nabla &= \nabla' , \\ \frac{\partial}{\partial t} &= \frac{\partial}{\partial t'} - v \cdot \nabla' . \end{aligned} \qquad \text{(II,2a)}$$

The application of (II,2a) into (II,1) and a comparison with Maxwell's equations, assumed valid in the frame S', lead to the transformation equations:

$$\rho' = \rho ,$$
$$\underset{\sim}{j}' = \underset{\sim}{j} - \rho \underset{\sim}{v} ,$$
$$\underset{\sim}{D}' = \underset{\sim}{D} ,$$
$$\underset{\sim}{E}' = \underset{\sim}{E} + \underset{\sim}{v} \wedge \underset{\sim}{B} , \quad \text{(II,3)}$$
$$\underset{\sim}{B}' = \underset{\sim}{B} ,$$
$$\underset{\sim}{H}' = \underset{\sim}{H} - \underset{\sim}{v} \wedge \underset{\sim}{D} .$$

For the potentials φ, $\underset{\sim}{A}$, given by :

$$\underset{\sim}{E} = -\underset{\sim}{\nabla}\varphi - \frac{\partial \underset{\sim}{A}}{\partial t} ,$$

$$\underset{\sim}{B} = \underset{\sim}{\nabla} \wedge \underset{\sim}{A}$$

one obtains

$$\underset{\sim}{A}' = A , \quad \varphi' = \varphi - \underset{\sim}{v} \cdot \underset{\sim}{A} . \quad \text{(II,4)}$$

On the other hand, as is well-known (4), the invariance of the phase of a radiation plane wave :

$$\underset{\sim}{A} = \underset{\sim}{A}_0 \cos(\underset{\sim}{k}.\underset{\sim}{x} - \omega t) \quad \text{(II,5)}$$

with angular frequency ω, wave vector $\underset{\sim}{k}$ and velocity c in the frame S, gives rise, under the Galilei transformations (II,2), to the following relationship between ω and ω' :

$$\omega' = \omega\left(1 - \frac{\underset{\sim}{v}.\underset{\sim}{k}}{ck}\right) \quad \text{(II,6)}$$

and between c and c':

$$c' = c\left(1 - \frac{\underset{\sim}{v}.\underset{\sim}{k}}{ck}\right) \quad \text{(II,6a)}$$

All experiments, began with the famous observations by Michelson and Marley, as is well-known, deny the validity of relations such as (II,3),(II,4),(II,6),(II,6a), based on an assumed invariance of electromagnetic laws under the Galilei group.

II.2) Einstein's relativity principle and the Lorentz and Poincaré Groups.

Lorentz and Poincaré discovered the groups of transformations under which the laws of electromagnetism are invariant, in agreement with experiment.

And the beautiful work developed by Einstein led to the discovery of the relativity principle valid for all laws of nature : these laws are the same everywhere and at all times , they do not depend of where nor when they are obtained; they are independent of the directions in space and of the state of motion of the reference frame, as long as this motion is uniform. As Wigner has said(2): ''Einstein articulated the postulates about the symmetry of space, that is, the equivalence of directions and of different points of space, eloquently''.

In elementary textbooks (5), the special Lorentz transformations which connect a reference frame S to another, S', in uniform translation with respect to S, with a constant velocity v, parallel to axis Ox (which coincides with O'x') are established. These formulae read :

$$\begin{aligned} x' &= \gamma(x - vt) , \\ y' &= y , \\ z &= z , \\ t' &= \gamma(t - \frac{v}{c^2} x) , \\ \gamma &= (1 - \beta^2)^{-1/2} , \qquad \beta = \frac{v}{c} \end{aligned} \qquad (II,7)$$

where c is the velocity of light in vacuum. If β^2 is negligible as compared to the unity , these formulae go over into those of Galilei. And it is now a simple exercise, by the use of the method pointed out in the preceding paragraph, to deduce the (experimentally verified) transformation laws of the electromagnetic field and of the frequency of a plane wave corresponding to the relations (II,7).

It follows from (II,7) that a length and a time interval between two events are not invariant quantities. The fact that a space distance between two simultaneous events for an observer S generates a time distance between the same events for another S', leads one to accept, as a natural generalisation, the fusion of ordinary space and time into a four-dimensional ''space-time'' linear vector space. Let us call $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$, the coordinates of a point of this space, referred to a given frame.

In general, the Lorentz group is the set of linear, homogeneous transformations which map space- time into space-time :

$$x'^{\alpha} = \ell^{\alpha}{}_{\beta}\, x^{\beta} \qquad (II,8)$$

and which preserves the scalar product of two four-vectors, as defined, the latter, by the bilinear form :

$$(x|y) = x^{\mu} y^{\nu} g_{\mu\nu} \tag{II,9}$$

where we are having in mind a summation over repeated indices from 0 to3; and :

$$g_{00} = - g_{11} = - g_{22} = - g_{33} = 1 \quad ; \qquad g_{\alpha\beta} = 0 \text{ for } \alpha \neq \beta \tag{II,10}$$

the condition :

$$(x'| y') = (x|y) \tag{II,11}$$

imposes the following restrictions on the Lorentz coefficients $\ell^{\alpha}{}_{\beta}$:

$$\ell^{\alpha}{}_{\mu}\, g_{\alpha\beta}\, \ell^{\beta}{}_{\nu} = g_{\mu\nu} \tag{II,12}$$

It is seen that the square of the norm of a vector :

$$(x|x) = (x^{0})^{2} - (\underset{\sim}{x})^{2} \tag{II,13}$$

is not positive definite and the condition $(x/x) = 0 \Rightarrow (x) = 0$ is not valid in such a space.

In a notation similar to that introduced in (I,21) - (I,32), we shall express a vector of space-time in terms of a basis $|\lambda_0), |\lambda_1), |\lambda_2), |\lambda_3)$:

$$|x) = x^{\alpha} |\lambda_{\alpha}) \tag{II,14}$$

A Lorentz transfomation changes this vector into :

$$|x') = x^{\alpha} |\lambda'_{\alpha}) \tag{II,15}$$

where $|\lambda'_{\alpha})$, $\alpha = 0, 1, 2, 3$, are the vectors of the new basis . We have :

$$|x') = x'^{\alpha} | \lambda_{\alpha}) \tag{II,15a}$$

Call $(y|$ a vector of the dual space

$$(y| = y^{\alpha} (\lambda_{\alpha}| \tag{II,16}$$

and denote by $g_{\mu\nu}$ the scalar product $(\lambda_\mu | \lambda_\nu)$:

$$(\lambda_\mu | \lambda_\nu) = g_{\mu\nu} = g_{\nu\mu} \qquad \text{(II,17)}$$

It follows from (II,15) and (II,15a) that :

$$x'^{\alpha} g_{\alpha\beta} = x^{\nu} (\lambda_\beta | \lambda'_\nu) \qquad \text{(II,18)}$$

If a new (contravariant) tensor $g^{\alpha\beta}$ is introduced by the equations :

$$g_{\alpha\eta} g^{\eta\beta} = \delta_{\alpha}^{\ \beta} = \delta^{\beta}_{\ .\alpha} \qquad \text{(II,19)}$$

where δ^{β}_{α} is the unit matrix :

$$\delta^{\beta}_{\alpha} = 1 \quad \text{for } \alpha = \beta$$
$$= 0 \quad \text{for } \alpha \neq \beta$$

one obtains from (II,18) and (II,19):

$$x'^{\mu} = x^{\alpha} (\lambda_\beta | \lambda'_\alpha) g^{\beta\mu} \qquad \text{(II,20)}$$

Let us call :

$$(\lambda^{\mu}| = (\lambda_\nu | g^{\nu\mu}$$

and

$$\ell^{\ \mu}_{\lambda\alpha} = (\lambda^{\mu} | \lambda'_\alpha) .$$

Formula (II,20) reduces to (II,8).

Usually one postulates the Lorentz metric tensor (II,17),(II,10), the scalar product (II,9) and the contravariant tensor $g^{\alpha\beta}$ (II,19). Clearly, one can also write (II,9) under the form :

$$(x | y) = x^{\alpha} y_{\alpha}$$

where

$$y_{\alpha} = g_{\alpha\beta} y^{\beta}$$

and the vector (II,14) can also be written :

$$| x) = x_\alpha | \lambda^\alpha)$$

In view of the identification of the numbers x^μ with real coordinates ($x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$), the Lorentz group matrix $\ell = \{\ell^\mu{}_\nu\}$ is formed of real numbers. It is well-known that an alternative way to represent the Lorentz transformations is obtained by replacing the $g_{\mu\nu}$ elements in (II,17) by the unit matrix elements $\delta_{\mu\nu}$ and at the same time identifying the coordinates x^μ with $x^4 = x_4 = ict$, $x^1 = x_1 = x$, $x^2 = x_2 = y$, $x^3 = x_3 = z$, the scalar product of two such vectors being defined as :

$$(x,y) = -(x | y) = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4$$
$$= \vec{x} \cdot \vec{y} - x_0 y_0 \ .$$

In this case the transformation parameters , represented by $a_{\mu\nu}$:

$$x'_\mu = a_{\mu\nu} x_\nu$$

are not real numbers (a_{jk} and a_{44} are real numbers, a_{4j} and a_{j4} are pure imaginary numbers, where $j,k = 1,2,3$).

A consequence of the equations (II,12) is that the determinant of the matrix $\{\ell^\mu\}$ is ± 1 since

$$\left[\det \{\ell^\mu{}_\nu\}\right]^2 = 1 \ .$$

The equations (II,12) give, for $\mu = \nu = 0$:

$$\ell^\alpha{}_0 \, g_{\alpha\beta} \, \ell^\beta{}_0 = 1$$

whence :

$$(\ell^0{}_0)^2 = 1 + \sum_{\alpha=0}^{3} (\ell^\alpha{}_0)^2 \geqslant 1$$

that is to say :

$$\ell^0{}_0 \geqslant 1 \quad \text{or} \quad \ell^0{}_0 \leqslant -1$$

One is thus led to distinguish four components of the Lorentz group L (the lower sign is that of the determinant, the upper sign indicates the direction of time flow):

1) the proper orthochronous component L_+^+ the elements of which, $\ell = \{\ell^\mu{}_\nu\}$ are such that :

$$\det \ell = +1 \quad , \quad \ell^o{}_o \geqslant 1 \ ;$$

2) the proper non-orthochronous component L_+^- for which

$$\det \ell = +1 \quad , \quad \ell^o{}_o \geqslant -1 \ ;$$

3) the improper orthochronous component L_-^+ :

$$\det \ell = -1 \quad , \quad \ell^o{}_o \geqslant 1 \ ;$$

4) the improper non-orthochronous component L_-^- :

$$\det \ell = -1 \quad , \quad \ell^o{}_o \leqslant -1$$

It is clear that the first component , L_+^+, is a subgroup of L, the <u>proper orthochronous group L_+^+</u> ; it contains the identity ($\ell^\mu{}_\nu = \delta^\mu{}_\nu$) and an element of this group can be obtained from any other element of L_+^+ by a continuous variation of the six parameters. The sets L_+^- , L_-^+ , L_-^- , above are obviously not groups since they do not contain the identity. The sum or union of these sets with L_+^+ are groups :

a) $L_+ = L_+^+ \cup L_+^-$ (proper Lorentz group);

b) $L^+ = L_+^+ \cup L_-^+$ (orthochronous group) ; (II,21)

c) $L_o = L_+^+ \cup L_-^-$ (orthochronous group).

The orthochronous proper Lorentz group L_+^+ is formed of elements which transform a positive time-like vector x :

$$(x \mid x) = (x^o)^2 - (\underset{\sim}{x})^2 > 0 \quad , \quad x^o > 0$$

into another positive time-like vector :

$$x'^\mu = \ell^\mu_\nu \ x^\nu \ ,$$
$$(x' \mid x') > 0 \quad , \quad x'^o > 0 \ .$$

A space-like surface σ is formed of points such that the distance between any two of them, x and y, in space-time is negative:

$$(x-y/x-y) = (x^o - y^o)^2 - (\vec{x} - \vec{y})^2 < 0$$

A particular space-like surface is a plane perpendicular to the time-axis. For any two points of σ , one is always outside of the light-cone of the other, the light-cone at the origin being the set of points x such that:

$$(x / x) = 0$$

Given the fact that the four-dimensional volume d^4x is invariant, since :

$$d^4x' = \left| \frac{\partial x'}{\partial x} \right| d^4x \qquad (II,22)$$

and

$$\left| \frac{\partial x'}{\partial x} \right| = \det \{ \ell^\mu_\nu \} = 1$$

an integral of the form $\int F(x)\, d^4x$ has the transformation properties of $F(x)$. The integral which generalises a three-dimensional sum is:

$$\int_\sigma f_\lambda(x)\, d\sigma^\lambda$$

where

$$d\sigma^\lambda = (dx^1\, dx^2\, dx^3,\ dx^2\, dx^3\, dx^o,\ dx^3\, dx^1\, dx^o,\ dx^1\, dx^2\, dx^o)$$

is a vector normal to each point of σ . The surface σ being space-like, $d\sigma^\lambda$ is a time-like vector.

In the special case when σ is a plane perpendicular to the time-axis one has :

$$\int_\sigma f_\lambda\, d\sigma^\lambda \rightarrow \int_V f_o\, d^3x$$

The generalisation of the Gauss theorem to space-time is :

$$\int_W \frac{\partial f^\lambda}{\partial x^\lambda}\, d^4x = \int_{\sigma_2} f^\lambda\, d\sigma_\lambda - \int_{\sigma_1} f^\lambda\, d\sigma_\lambda \qquad (II,23)$$

where W is a four dimensional region bounded by the space like surfaces σ_2 and σ_1.

To each of the Lorentz groups (II,21) is associated a Poincaré group, the corresponding inhomogeneous Lorentz group. Thus the proper orthochroneous Poincaré group P_+ is composed of the elements $\{a, \ell\}$ such that :

$$x'^\mu = a^\mu + \ell^\mu_\nu\, x^\nu$$

where

$$\det \ell = +1 \qquad , \quad \ell^{o}_{o} > 1.$$

If a Lorentz transformation is defined by six independent parameters ℓ^{μ}_{ν}, the corresponding Poincaré transformation is determined by ten parameters a^{μ}, ℓ^{μ}_{ν}.

Clearly, if :

$$x''^{\mu} = a^{\mu}_{1} + \ell^{\mu}_{1\nu}\, x'^{\nu} \qquad ,$$

$$x'^{\nu} = a^{\nu}_{2} + \ell^{\nu}_{2\lambda}\, x^{\lambda}$$

one gets :

$$x''^{\mu} = a^{\mu}_{1} + \ell^{\mu}_{1\nu}\, a^{\nu}_{2} + \ell^{\mu}_{1\nu}\, \ell^{\nu}_{2\lambda}\, x^{\lambda}$$

It is, thus, seen that the multiplication law for the Poincaré group is:

$$\{a_1, \ell_1\}\ \{a_2, \ell_2\} = \{a_1 + \ell_1\, a_2, \ell_1\ \ell_2\}$$

II.3) <u>Lorentz-covariant form of Maxwell's equations</u>.

The physical basis of the covariant form of Maxwell's equations is the fact that electric charge of a physical system is Lorentz-invariant. If $\rho(x)$ is the density of a charge distribution in a frame S we must have :

$$\rho(x)\, d^3x = \rho'(x')\, d^3x'$$

where the dash indicates the values measured by an observer of a frame S'. A comparison with the equation (II,22) shows that ρ transforms like a time coordinate. Thus

$$\underset{\sim}{j} = \rho \underset{\sim}{v} = \rho c \frac{d\underset{\sim}{x}}{dx^{o}}$$

transforms like $d\underset{\sim}{x}$. Therefore the four quantities :

$$j^{\mu} = (\mu_{o}\, c\rho\ ,\ \mu_{o}\underset{\sim}{j})$$

where

$$c^{2} = \frac{1}{\varepsilon_{o}\,\mu_{o}}$$

are the components of a four-vector :

$$j'^{\mu}(x') = \ell^{\mu}_{\nu}\, j^{\nu}(x)$$

The continuity equation :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0$$

is written :

$$\frac{\partial j^{\lambda}}{\partial x^{\lambda}} = 0$$

where the sum over a repeated index is understood.

Define a quantity $H^{\mu\nu}$ in the following way :

$$H^{ok} = - H^{ko} = \mu_o c\, D^k,$$

$$H^{23} = - H^{32} = \mu_o H_x \ , \ H^{31} = - H^{13} = \mu_o H_y \ , \ H^{12} = - H^{21} = \mu_o H_z$$

where D^k, $k = 1,2,3$ are the components of the electric displacement vector and H_x, H_y, H_z those of the magnetic field. The equations :

$$\nabla \cdot D = \rho \qquad , \qquad \nabla \wedge H - \frac{\partial D}{\partial t} = j$$

can be synthetized into

$$\frac{\partial H^{\mu\nu}}{\partial x^{\nu}} = j^{\mu}$$

The vector caracter of j^{μ} leads to $H^{\mu\nu}$ being the components of a tensor :

$$H'^{\mu\nu}(x') = \ell^{\mu}{}_{\alpha}\, \ell^{\nu}{}_{\beta} H^{\alpha\beta}(x) \quad .$$

In the same way, the equations :

$$\nabla \cdot B = 0 \qquad , \qquad \nabla \wedge E + \frac{\partial B}{\partial t} = 0$$

result from the equations :

$$\frac{\partial F^{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial F^{\lambda\mu}}{\partial x^{\nu}} + \frac{\partial F^{\nu\lambda}}{\partial x^{\mu}} = 0$$

where $F^{\mu\nu}$ is a tensor defined by :

$$F^{ko} = - F^{ok} = - c\, \varepsilon_o\, \mu_o\, E^k \ ,$$

$$F^{23} = - F^{32} = B_x \ , \ F^{31} = - F^{13} = B_y \ , \ F^{12} = - F^{21} = B_z$$

Finally, if one sets :

$$\mathcal{A}_k = A_k \quad , \quad \mathcal{A}_o = c\,\varepsilon_o\,\mu_o\,\phi$$

where ϕ is the so called scalar potential, A_k the vector-potential, the equations which define these :

$$\underset{\sim}{B} = \underset{\sim}{\nabla} \wedge \underset{\sim}{A} \quad , \qquad \underset{\sim}{E} = -\underset{\sim}{\nabla}\phi - \frac{\partial \underset{\sim}{A}}{\partial t}$$

can be written :

$$F^{\mu\nu} = \frac{\partial \mathcal{A}^{\nu}}{\partial x_{\mu}} - \frac{\partial \mathcal{A}^{\mu}}{\partial x^{\nu}}$$

The equations for the four-vector $\mathcal{A}^{\mu}$, are, when $\varepsilon = \mu = 1$:

$$\Box\, \mathcal{A}^{\mu} = j^{\mu} \quad , \quad \frac{\partial \mathcal{A}^{\mu}}{\partial x^{\mu}} = 0 \qquad \text{(II,24)}$$

where

$$\Box = g^{\mu\nu} \frac{\partial}{\partial x^{\mu}} \frac{\partial}{\partial x^{\nu}} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \underset{\sim}{\nabla}^2 \qquad \text{(II,24a)}$$

is the scalar D'Alembertian operator :

$$\Box' = \Box$$

II.4) <u>The Lagrangean formalism for the electromagnetic field.</u>

The equations of a free particle are :

$$\frac{dp^{\mu}}{ds} = 0$$

where :

$$ds^2 = dz^{\mu}\, dz_{\mu}$$

and :

$$p^k = \frac{m_o}{(1-\beta^2)^{1/2}} \frac{dz^k}{dt} \quad , \qquad p^o = \frac{m_o c}{(1-\beta^2)^{1/2}}$$

They are deduced from an action principle :

$$\delta S = 0$$

the action function being:

$$S = \int L \, dt ,$$

$$L = - m_o c^2 (1-\beta^2)^{1/2}$$

How are Lagrange's equations to be related to a field ? One searches for a Lagrange function a scalar constructed with the electromagnetic potential a^μ and its first space-time derivatives :

$$L = L\left(A^\mu(\underset{\sim}{x},t) , \frac{\partial A^\mu}{\partial x^\nu}(\underset{\sim}{x}, t), x\right)$$

If L is assumed to be invariant with respect to the proper orthochronous Poincaré group, it does not depend explicitely on x :

$$L = L(A^\mu(\underset{\sim}{x},t), A^\mu{}_{,\nu}) \quad ; A^\mu{}_{,\nu} = \frac{\partial A^\mu}{\partial x^\nu} \quad .$$

By analogy with equation (I,4) one defines the total Lagrangean:

$$\overline{L} = \int L \, d^3x \tag{II,25}$$

and introduces the action function S as :

$$S = \int_{t_1}^{t_2} \overline{L} \, dt = \frac{1}{c} \int_W L \, d^4x \; .$$

A field distribution A^μ is determined by the solution of the inhomogeneous wave equation (II,24) and appropriate boundary conditions. The action principle states that out of all possible distributions, the physical field is the one which makes S stationary. If, therefore, the set of possible distributions is defined by a parameter α :

$$A^\mu = A^\mu(x; \alpha) \quad \text{where} \quad x = (t,\underset{\sim}{x})$$

and (see (I,6)) :

$$\delta A^\mu = \left(\frac{\partial A^\mu}{\partial \alpha}\right)_o d\alpha$$

one has

$$\delta S = \frac{1}{c} \int_W \left\{ \frac{\partial L}{\partial A^\mu} \delta A^\mu + \frac{\partial L}{\partial A^\mu{}_{,\nu}} \delta A^\mu{}_{,\nu} \right\} d^4x =$$

$$= \frac{1}{c} \int_W \left\{ \frac{\partial L}{\partial A^\mu} \delta A^\mu + \frac{\partial L}{\partial A^\mu{}_{,\nu}} \frac{\partial}{\partial x^\nu} \delta A^\mu \right\} d^4x$$

and a partial integration gives :

$$\delta S = \frac{1}{c}\int_{\sigma} \frac{\partial L}{\partial A^{\mu}{}_{,\nu}} \delta A^{\mu} \, d\sigma^{\nu} + \frac{1}{c}\int_{W} \left\{ \frac{\partial L}{\partial A^{\mu}} - \frac{\partial}{\partial x^{\nu}} \frac{\partial L}{\partial A^{\mu}{}_{,\nu}} \right\} \delta A^{\mu} \, d^4x$$

where σ is the boundary of the four-dimensional region W. The postulate :

$$\delta S = 0$$

under the assumption that $\delta A^{\mu} = 0$ on σ, gives rise to the field Lagrange's equations :

$$\frac{\partial L}{\partial A^{\mu}} - \frac{\partial}{\partial x^{\nu}} \frac{\partial L}{\partial A^{\mu}{}_{,\nu}} = 0 \quad . \qquad \text{(II,26)}$$

In general, the Lagrange function is a sum of three parts : the particles' Lagrangean, the Lagrangean of the free fields and the interaction between fields and particles

$$L = L_{part.} + L_f + L_{int.}$$

The complete action of a point particle in interaction with an electromagnetic field is :

$$S = - m_o c \int ds + \frac{1}{2} \frac{1}{c} \int A^{\mu}{}_{,\nu} A_{\mu}{}^{,\nu} d^4x - \frac{1}{c} \int j_{\mu} A^{\mu} d^4x$$

where

$$j^{\mu}(x) = e c \int_{-\infty}^{\infty} \frac{dz^{\mu}}{ds} \delta(x-z(s)) \, ds$$

The interaction term is therefore :

$$S_{int} = - e \int A^{\mu}(z) \frac{dz_{\mu}}{ds} \, ds$$

The equations of motion of this system, obtained by variation of the particle's world-line and of the field are, respectively :

$$m_o c \frac{d}{ds} \left(\frac{dz^{\mu}}{ds}\right) = e F^{\mu\nu} \frac{dz_{\nu}}{ds} \quad ,$$

$$\Box A^{\mu} = e c \int_{-\infty}^{\infty} \frac{dz^{\mu}}{ds} \delta(x-z(s)) \, ds.$$

II. 5) Canonical form of the field equations.

The total Lagrangean $\overline{L}$,(II,25) is a functional of A_μ, $\dot{A}_\mu$:

$$\overline{L} = \int L(A^\mu(\underset{\sim}{x},t)\ ,\ A^\mu{}_{,\nu}(\underset{\sim}{x},t))\ d^3x = \overline{L}[A_\mu, \dot{A}_\mu] \ .$$

where $A_\lambda^{o\mu} = c\, A^\mu{}_{,o}$.

The functional derivative of $\overline{L}$ with respect to A^μ is defined by the following relationship :

$$\frac{\delta \overline{L}}{\delta A^\mu} = \frac{\partial L}{\partial A^\mu} - \frac{\partial}{\partial x^k} \frac{\partial L}{\partial A^\mu{}_{,k}}$$

with a summation over the index k from 1 to 3. On the other hand, the functional derivative of $\overline{L}$ with respect to $\dot{A}^\mu$ is

$$\frac{\delta \overline{L}}{\delta \dot{A}^\mu} = \frac{\partial L}{\partial \dot{A}^\mu}$$

These definitions are suggested by a comparison of the variation of $\overline{L}$:

$$\delta\, \overline{L} = \int \left(\frac{\delta \overline{L}}{\delta A^\mu}\, \delta A^\mu + \frac{\delta \overline{L}}{\delta \dot{A}^\mu}\, \delta \dot{A}^\mu \right) d^3x$$

(which is a generalisation of the differential of a function :

$$d\,\varphi = \sum_k \frac{\partial \varphi}{\partial x^k}\, dx^k$$

the three space-coordinates playing the role of continuous indices)

with :

$$\delta\, \overline{L} = \int \delta L\ d^3x = \int \left\{ \left(\frac{\partial L}{\partial A^\mu} - \frac{\partial}{\partial x^k} \frac{\partial L}{\partial A^\mu{}_{,k}} \right) \delta A^\mu + \frac{\partial L}{\partial \dot{A}^\mu}\, \delta \dot{A}^\mu \right\} d^3x$$

up to a term $\int \frac{\partial}{\partial x^k} \left(\frac{\partial L}{\partial A^\mu{}_{,k}}\, \delta A^\mu \right) d^3x$ which vanishes on the boundary of the integration domain.

The Lagrange's equations (II,26) can be written in the following form:

$$\frac{\delta \overline{L}}{\delta A^\mu(x)} - \frac{\partial}{\partial t} \frac{\delta L}{\delta \dot{A}^\mu(x)} = 0$$

similar to (I,9).

The canonical conjugate momentum is defined by :

$$\pi_\mu(x) = \frac{\delta \bar{L}}{\delta \dot{A}^\mu(x)}$$

and the Hamiltonian density, by :

$$H = \pi_\mu \dot{A}^\mu - L$$

which is a function of A^μ, $A^\mu_{,k}$, π_μ. The total Hamiltonian is the functional:

$$\bar{H} = \int H \, d^3x = \bar{H}[A^\mu, \pi_\mu] .$$

It follows from:

$$\delta \bar{H} = \int \left(\frac{\delta \bar{H}}{\delta A^\mu} \delta A^\mu + \frac{\delta \bar{H}}{\delta \pi_\mu} \delta \pi_\mu \right) d^3x$$

and

$$\delta \bar{H} = \int \left\{ \pi_\mu \delta \dot{A}^\mu + \dot{A}^\mu \delta \pi_\mu - \frac{\partial L}{\partial A^\mu} \delta A^\mu - \frac{\partial L}{\partial A^\mu_{,k}} \delta A^\mu_{,k} - \frac{\partial L}{\partial \dot{A}^\mu} \delta \dot{A}_\mu \right\} d^3x$$

$$= \int \left\{ \dot{A}^\mu \delta \pi_\mu - \dot{\pi}_\mu \delta A^\mu \right\} d^3x$$

that :

$$\dot{A}^\mu(x) = \frac{\delta \bar{H}}{\delta \pi_\mu(x)} , \quad \dot{\pi}_\mu(x) = - \frac{\delta \bar{H}}{\delta A^\mu(x)} , \quad x = (ct, \underset{\sim}{x}) \qquad \text{(II,27)}$$

which are the field canonical equations.

II.6) <u>The field's Poisson brackets. Constants of motion.</u>

Let $F[A^\mu, \pi_\mu]$ and $G[A^\mu, \pi_\mu]$ be two functionals of $A^\mu(x)$ and $\pi_\mu(x)$. The Poisson bracket ,as defined in (I,15), is now extended to this case by the form :

$$\{F, G\} = \int \left(\frac{\delta F}{\delta A^\mu(x)} \frac{\delta G}{\delta \pi_\mu(x)} - \frac{\delta F}{\delta \pi_\mu(x)} \frac{\delta G}{\delta A^\mu(x)} \right) d^3x \qquad \text{(II,28)}$$

We can write :

$$A^\mu(\underset{\sim}{x}, t) = \int A^\lambda(\underset{\sim}{y}, t) \, \delta^\mu_\lambda \, \delta(\underset{\sim}{y} - \underset{\sim}{x}) \, d^3y$$

whence :

$$\frac{\delta A^\mu(\underset{\sim}{x}, t)}{\delta A^\lambda(\underset{\sim}{y}, t)} = \delta^\mu_\lambda \, \delta(\underset{\sim}{x} - \underset{\sim}{y})$$

By definition, $A_\mu(\underset{\sim}{x},t)$ and $\pi_\lambda(\underset{\sim}{y},t)$, at the same instant t, are independent variables :

$$\frac{\delta A^\mu(\underset{\sim}{x},t)}{\delta\ \pi_\lambda(\underset{\sim}{y},t)} = 0 \qquad (II,29)$$

Therefore:

$$\left\{A^\mu(x),\ \overline{H}\right\} = \frac{\delta\overline{H}}{\delta\pi_\mu(x)} ,$$

$$\left\{\pi_\mu(x),\ \overline{H}\right\} = -\frac{\delta\overline{H}}{\delta A^\mu(x)}$$

These equations and (II,27) lead to the Hamilton-Heisenberg classical equations:

$$\dot{A}^\mu(x) = \left\{A^\mu(x),\ \overline{H}\right\} ,$$

$$\dot{\pi}^\mu(x) = \left\{\pi_\mu(x),\ \overline{H}\right\} .$$

The fundamental Poisson brackets of the field variables at the same instant are

$$\begin{aligned} \left\{A^\mu(\underset{\sim}{x},t),\ A^\lambda(\underset{\sim}{y},t)\right\} &= 0, \\ \left\{\pi_\mu(\underset{\sim}{x},t),\ \pi_\lambda(\underset{\sim}{y},t)\right\} &= 0 \qquad (II,30) \\ \left\{A^\mu(\underset{\sim}{x},t),\ \pi_\lambda(\underset{\sim}{y},t)\right\} &= \delta^\mu_{\ \lambda}\,\delta(\underset{\sim}{x}-\underset{\sim}{y}) . \end{aligned}$$

The generator of an infinitesimal canonical transformation, with parameter ε ,

$$\begin{aligned} A^\mu \rightarrow A'^\mu(x) &= A^\mu(x) + \delta\ A^\mu(x) , \\ \pi_\mu \rightarrow \pi'_\mu(x) &= \pi_\mu(x) + \delta\ \pi_\mu(x) , \qquad (II,31) \\ \delta\ A^\mu &= \varepsilon_{(k)} \left(\frac{\partial A_\mu}{\partial\varepsilon_{(k)}}\right)_0 , \\ \delta\ \pi_\mu &= \varepsilon_{(k)} \left(\frac{\partial\pi_\mu}{\partial\varepsilon_{(k)}}\right)_0 \end{aligned}$$

is a functional $\overline{U}\left[A^\mu,\ \pi_\mu\right]$ such that (see (I,49)):

$$\delta\ A^\mu = \varepsilon_{(k)} \frac{\delta\ \overline{U}_{(k)}}{\delta\ \pi_\mu} , \qquad \delta\ \pi_\mu = -\varepsilon_{(k)} \frac{\delta\ \overline{U}_{(k)}}{\delta\ A^\mu} \qquad (II,32)$$

The variation of a quantity $F[A^\mu, \pi_\mu]$ which is a functional of the field variables will be given by the equation :

$$\delta F = \int (\frac{\delta F}{\delta A^\mu} \delta A^\mu + \frac{\delta F}{\delta \pi_\mu} \delta \pi_\mu) d^3x = \varepsilon_{(k)} \int (\frac{\delta F}{\delta A^\mu} \frac{\delta \overline{U}_{(k)}}{\delta \pi_\mu} - \frac{\delta F}{\delta \pi_\mu} \frac{\delta \overline{U}_{(k)}}{\delta A^\mu}) d^3x$$

hence

$$\delta F = \varepsilon_{(k)} \{ F , \overline{U}_{(k)} \} . \qquad (II,33)$$

On the other hand, if G is a functional of A^μ and π_μ and a function of t:

$$G = G [A^\mu, \pi_\mu ; t]$$

one has :

$$\frac{dG}{dt} = \frac{\partial G}{\partial t} + \int d^3x (\frac{\delta G}{\delta A^\mu} \dot{A}^\mu + \frac{\delta G}{\delta \pi_\mu} \dot{\pi}_\mu)$$

or

$$\frac{dG}{dt} = \frac{\partial G}{\partial t} + \{ G , \overline{H} \} \qquad (II,34)$$

It follows from (II,33) and (II,34) that every infinitesimal canonical transformation which leaves the Hamiltonian invariant, determines a constant of motion, the generator of the transformation (see paragr. I,9) .

We shall see latter ((II,42),(II,46)) the expression of the generators of translations and rotations, respectively the energy momentum vector and the angular momentum tensor of a field.

II.7) Free-field Poisson brackets.

The Poisson brackets written down in (II,28) refer to field variables taken at the same time, i.e., at different points of a plane perpendicular to the time axis. As such a plane is not an invariant surface, it is of interest to know the Poisson brackets between two fields taken at any two points in space-time. To derive these we need to use the free-field equation.

We have, by definition :

$$\{ A^\mu(x), A^\nu(x') \} = \int d^3y \left[\frac{\delta A^\mu(\underset{\sim}{x},t)}{\delta A^\lambda(\underset{\sim}{y},t)} \frac{\delta A^\nu(\underset{\sim}{x}',t')}{\delta \pi_\lambda(\underset{\sim}{y},t)} - \frac{\delta A^\mu(\underset{\sim}{x},t)}{\delta \pi_\lambda(\underset{\sim}{y},t)} \frac{\delta A^\nu(\underset{\sim}{x}',t')}{\delta A^\lambda(\underset{\sim}{y},t)} \right]$$

where we have taken the time of the independent variables $A^{\lambda}(\underset{\sim}{y},t), \pi_{\lambda}(\underset{\sim}{y},t)$ equal to that of $A^{\mu}(\underset{\sim}{x},t)$. In virtue of the equation (II,29), this reduces to:

$$\left\{A^{\mu}(x),\ A^{\nu}(x')\right\} = \int d^3y\ \frac{\delta A^{\mu}(\underset{\sim}{x},t)}{\delta A^{\lambda}(\underset{\sim}{y},t)}\ \frac{\delta A^{\nu}(\underset{\sim}{x}',t')}{\delta \pi_{\lambda}(\underset{\sim}{y},t)}$$

Develop $A^{\nu}(\underset{\sim}{x}',t')$ in series around t,

$$A^{\nu}(\underset{\sim}{x}',t') = A^{\nu}(\underset{\sim}{x}',t) + (t'-t)\left(\frac{\partial A^{\nu}(\underset{\sim}{x}',t')}{\partial t'}\right)_{t'=t} + \dots$$

On the other hand, for a free field :

$$L = \frac{1}{2}\ g_{\mu\alpha}\, g^{\lambda\nu} A^{\mu}{}_{,\lambda} A^{\alpha}{}_{,\nu} \qquad ;$$

$$\pi_{\nu} = \frac{1}{2}\ \frac{\partial L}{\partial A^{\nu}{}_{,0}} = \frac{1}{c^2}\ g_{\nu\alpha}\ \dot{A}^{\alpha} \qquad ;$$

$$g^{\mu\nu}\pi_{\nu} = \frac{1}{c^2}\ \dot{A}^{\mu} \quad ;$$

$$\left(\frac{\partial^2 A^{\nu}(\underset{\sim}{x}',t')}{\partial t'^2}\right)_{t'=t} = c^2\ \underset{\sim}{\nabla}'^2\ A^{\nu}(\underset{\sim}{x}',t)$$

therefore :

$$\left\{A^{\mu}(\underset{\sim}{x},t),\ A^{\nu}(\underset{\sim}{x}',t')\right\} = \int d^3y\ \delta^{\mu}_{\lambda}\ \delta(\underset{\sim}{x}-\underset{\sim}{y})\left[\frac{\delta A^{\nu}(\underset{\sim}{x}',t)}{\delta\pi_{\lambda}(\underset{\sim}{y},t)} + (t'-t)\ c^2\ \frac{\delta\pi^{\nu}(\underset{\sim}{x}',t)}{\delta\pi_{\lambda}(\underset{\sim}{y},t)} + \right.$$

$$\left. + \frac{1}{2!}\ c^2(t'-t)^2\ \underset{\sim}{\nabla}'^2\ \frac{\delta A^{\nu}(\underset{\sim}{x}',t)}{\delta\pi_{\lambda}(\underset{\sim}{y},t)} + \frac{1}{3!}\ c^4(t'-t)^3\ \nabla'^2\ \frac{\delta \pi^{\nu}(\underset{\sim}{x}',t)}{\delta\pi_{\lambda}(\underset{\sim}{y},t)} + \dots\right]$$

that is

$$\left\{A^{\mu}(x),\ A^{\nu}(x')\right\} = c^2\ g^{\mu\nu}\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\,(t'-t)^{2n+1}\left[c^2\ \nabla'^2\right]^n\ \delta(\underset{\sim}{x}'-\underset{\sim}{x})$$

The Fourier representation:

$$\delta(\underset{\sim}{x}'-\underset{\sim}{x}) = \frac{1}{(2\pi)^3}\int d^3\ k\ e^{i\underset{\sim}{k}\cdot(\underset{\sim}{x}'-\underset{\sim}{x})}$$

leads to:

$$\left\{ A^{\mu}(x), A^{\nu}(x') \right\} = c\, g^{\mu\nu}\, D(x'-x) \qquad (II,35)$$

where the Jordan-Pauli function is defined by:

$$D(x-x') = -D(x'-x) = \frac{1}{(2\pi)^3} \int d^3 k\, e^{i\underset{\sim}{k}.(\underset{\sim}{x}'-\underset{\sim}{x})} \frac{1}{k_o} \sin\left[k_o(x'_o - x_o)\right]$$

It vanishes if x and x' are on a space-like surface σ and fulfills the conditions:

$$\square\, D(x) = 0 \;;$$

$$\int_{\sigma} \frac{\partial D}{\partial x^{\mu}}\, d\sigma^{\mu} = 1 \;;$$

$$D(x) = 0 \text{ if } (x|x) < 0$$

If the field $A^{\mu}(x)$ and its first derivatives $(\partial A^{\mu})/(\partial x^{\nu})$ are given on a space-like surface, the field will be determined at any point y of space-time by the relation p:

$$A^{\mu}(y) = \int_{\sigma} \left[D(y-x) \frac{\partial A^{\mu}(x)}{\partial x^{\nu}} - A^{\mu}(x) \frac{\partial}{\partial x^{\nu}} D(y-x) \right] d\sigma^{\nu}$$

II.8) Noether's theorem.

We shall now prove an important theorem due to E.Noether, which allows one to construct the constants of motion associated to the symmetry groups of a field theory. This theorem may be stated as follows :

To every continuous transformation of the (geometrical or dynamical) coordinates of a field , $\psi^{[\alpha]}$, for which the field transformation law is known, a certain combination of $\psi^{[\alpha]}$ and its first derivatives $\psi^{[\alpha]}_{,\lambda}$, is associated which is covariant and conserved in time, if the transformation leaves the action invariant.

let :

$$x^{\mu} \rightarrow x'^{\mu} = x^{\mu} + \delta x^{\mu} \qquad (II,36)$$

be an infinitesimal geometric transformation of the coordinates in space-time defined by a set of parameters $\omega^{[\nu]}$, such that :

$$\delta x^{\mu} = f^{\mu}_{[\nu]} \omega^{[\nu]} \qquad (II,37)$$

In the case of a space-time translation, one has:

$$\omega^{[\nu]} = \varepsilon^{\nu} \quad ; \quad f^{\mu}_{[\nu]} = \delta^{\mu}{}_{\nu} \tag{II,38}$$

For an infinitesimal Lorentz-transformation:

$$\omega^{[\nu]} = \varepsilon^{\nu\lambda} = -\varepsilon^{\lambda\nu} ;$$

$$f^{\mu}_{[\nu]} \equiv f^{\mu}_{\nu\lambda} = -f^{\mu}_{\lambda\nu} = \frac{1}{2} (\delta^{\mu}_{\nu} \, g_{\lambda\alpha} - \delta^{\mu}{}_{\lambda} \, g_{\nu\alpha}) \, x^{\alpha}$$

The mappings (II,36) induce a transformation of the field variables :

$$\psi^{[\alpha]}(x) \rightarrow \psi'^{[\alpha]}(x') = \psi^{[\alpha]}(x) + \bar{\delta}\psi^{[\alpha]}(x)$$

The variation $\bar{\delta}\psi^{[\alpha]}(x)$ results from a change in the form of the field function and from a change of x into x'; we shall assume it proportional to the infinitesimal parameters $\omega^{[\beta]}$:

$$\bar{\delta}\psi^{[\alpha]}(x) = \mathcal{F}^{[\alpha]}_{[\beta]}(x) \, \omega^{[\beta]} \tag{II,39}$$

The variations of the fields we considered previously (see (II,31)), are variations in form only;

$$\delta\psi^{[\alpha]}(x) = \psi'^{[\alpha]}(x) - \psi^{[\alpha]}(x),$$

i.e., the original and transformed fields are taken at the same point x.

One can express $\delta\psi^{[\alpha]}$ in terms of the parameters $\omega^{[\nu]}$ and of $\mathcal{F}^{[\alpha]}_{[\beta]}(x)\,\omega^{[\beta]}$. Indeed :

$$\delta\psi^{[\alpha]}(x) = \psi'^{[\alpha]}(x') - \psi^{[\alpha]}(x) - (\psi'^{[\alpha]}(x') - \psi'^{[\alpha]}(x)) =$$

$$= (\mathcal{F}^{[\alpha]}_{[\beta]}(x) - \psi^{[\alpha]}_{,\lambda}(x) \, f^{\lambda}_{[\beta]}) \, \omega^{[\beta]} \tag{II,40}$$

The action is a functional of the field and of the domain of integration R :

$$S[\psi^{[\alpha]} ; R] = \int_R L \, d^4x$$

where

$$L = L(x; \psi^{[\alpha]}(x), \psi^{[\alpha]}_{,\lambda}(x))$$

The variation of the action is :

$$\bar{\delta} S = S\left[\psi'^{[\alpha]} ; R'\right] - S\left[\psi^{[\alpha]} ; R\right]$$

and is equal to

$$\bar{\delta} S = S\left[\psi'^{[\alpha]} ; R'\right] - S\left[\psi^{[\alpha]} ; R'\right] + S\left[\psi^{[\alpha]} ; R'\right] - S\left[\psi^{[\alpha]} ; R\right]$$

that is

$$\bar{\delta} S = \int_{R'} L(x', \psi'^{[\alpha]}(x'), \psi'^{[\alpha]}_{,\lambda}(x'))d^4x' - \int_{R'} L(x', \psi^{[\alpha]}(x'), \psi^{[\alpha]}_{,\lambda}(x'))d^4x'$$

$$+ \int_{R'} L(x', \psi^{[\alpha]}(x'), \psi^{[\alpha]}_{,\lambda}(x'))d^4x' - \int_R L(x, \psi^{[\alpha]}(x), \psi^{[\alpha]}_{,\lambda}(x))\, d^4x$$

where R' is the transformed domain of integration resulting from (II,36).

We thus have :

$$\bar{\delta} S = \int_{R'} \delta L\, d^4x' + \int_{R'} L(x', \psi^{[\alpha]}(x'), \psi^{[\alpha]}_{,\lambda}(x'))d^4x' - \int_R L(x, \psi^{[\alpha]}(x), \psi^{[\alpha]}_{,\lambda}(x))d^4x$$

hence , if one keeps only terms in first order in the ω's :

$$\bar{\delta} S = \int_R d^4x \left(\frac{\partial L}{\partial \psi^{[\alpha]}} \delta\psi^{[\alpha]} + \frac{\partial L}{\partial \psi^{[\alpha]}_{,\lambda}} \delta\psi^{[\alpha]}_{,\lambda} + \frac{\partial L}{\partial x^\lambda} \delta x^\lambda + L \frac{\partial}{\partial x^\lambda}(\delta x^\lambda)\right)$$

Here we have taken into account that :

$$d^4x' = \left|\frac{\partial x}{\partial x'}\right| d^4x = \left(1 + \frac{\partial}{\partial x^\lambda}(\delta x^\lambda)\right) d^4x$$

Now the field equations :

$$\frac{\partial L}{\partial \psi^{[\alpha]}} - \frac{\partial}{\partial x^\lambda} \frac{\partial L}{\partial \psi^{[\alpha]}_{,\lambda}} = 0$$

allows us to write $\bar{\delta}$ S under the following form :

$$\bar{\delta} S = \int \frac{\partial}{\partial x^\lambda}\left[\frac{\partial L}{\partial \psi^{[\alpha]}_{,\lambda}} \left(\mathcal{F}^{[\alpha]}_{[\beta]}(x) - \psi^{[\alpha]}_{,\nu} f^{\nu}_{[\beta]}\right) + L f^{\lambda}_{[\beta]}\right]\omega^{[\beta]}\, d^4x .$$

It is now seen that the postulate of invariance of the action under the transformations in question :

$$\bar{\delta} S = 0$$

for any infinitesimal ω's , leads to the existence of the set of quantities , the Noether tensor $N^{\lambda}_{[\beta]}$:

$$N^{\lambda}_{[\beta]} = \frac{\partial L}{\partial \psi^{[\alpha]}_{,\lambda}} \left(\psi^{[\alpha]}_{,\nu} f^{\nu}_{[\beta]} - \mathcal{F}^{[\alpha]}_{[\beta]} \right) - L f^{\lambda}_{[\beta]} \qquad \text{(II,41)}$$

the divergence of which vanishes :

$$\frac{\partial N^{\lambda}_{[\beta]}}{\partial x^{\lambda}} = 0 \qquad \text{(II,41a)}$$

Therefore, if this equation is integrated over a four-dimensional region between two space-like surfaces σ_1 and σ_2 , and if the field quantities which occur in $N^{\lambda}_{[\beta]}$ vanish at infinite space-like distances, we can write, according to Gauss theorem (II,23)

$$\int d^4x \frac{\partial N^{\lambda}_{[\beta]}}{\partial x^{\lambda}} = \int_{\sigma_2} d\sigma_{\lambda} N^{\lambda}_{[\beta]} - \int_{\sigma_1} d\sigma_{\lambda} N^{\lambda}_{[\beta]} = 0$$

This means that the quantity

$$S_{[\beta]} = \int_{\sigma} d\sigma_{\lambda} N^{\lambda}_{[\beta]} \qquad \text{(II,41b)}$$

does not depend on the space-like surface σ . If this is a plane perpendicular to the time axis, then

$$S_{[\beta]} = \int d^3x \, N^{o}_{[\beta]}$$

does not depend on time, it is a conserved quantity :

$$\frac{d S_{[\beta]}}{dx^{o}} = 0 .$$

Clearly, the Noether tensor $N^{\lambda}_{[\beta]}$ is defined up to a term which contains the divergence of a certain antisymmetric tensor. Indeed, if such a tensor is :

$$t^{\lambda\nu}_{[\beta]}(x) = - t^{\nu\lambda}_{[\beta]}(x)$$

then the two quantities $N^{\lambda}_{[\beta]}$ and $N'^{\lambda}_{[\beta]}$:

$$N'^{\lambda}_{[\beta]} = N^{\lambda}_{[\beta]} + \frac{\partial}{\partial x^{\nu}} t^{\lambda\nu}_{[\beta]}$$

will give rise to the same conserved objects, since :

$$\frac{\partial N'^{\lambda}_{[\beta]}}{\partial x^{\lambda}} = \frac{\partial N^{\lambda}_{[\beta]}}{\partial x^{\lambda}}$$

II.9) The field energy-momentum vector.

Suppose now that the transformations (II,36) are translations in space-time by the vector ε^{ν} :

$$\omega^{[\nu]} = \varepsilon^{\nu} \quad ; \qquad f^{\mu}_{[\nu]} = \delta^{\mu}_{\nu} \quad .$$

In this case, the field variables are invariant :

$$\psi'^{[\alpha]}(x') = \psi^{[\alpha]}(x) \ ,$$

therefore :

$$\mathcal{F}^{[\alpha]}_{[\beta]}(x) \equiv 0 \ .$$

The Noether quantity is the energy-momentum tensor :

$$N^{\alpha}_{[\beta]} \equiv T^{\alpha}_{\ \beta} = \frac{\partial L}{\partial \psi^{[\nu]}_{,\alpha}} \psi^{[\nu]}_{,\beta} - L\, \delta^{\alpha}_{\beta} \qquad \text{(II,42)}$$

The field energy-momentum four-vector is therefore :

$$P^{\lambda} = g^{\lambda\beta} \int d\sigma_{\alpha}\, T^{\alpha}_{\ \beta} = g^{\lambda\beta} \int d^3x\, T^{o}_{\ \beta} \qquad \text{(II,42a)}$$

hence, for the Hamiltonian :

$$\overline{H} = P^{o} = \int d^3x \left(\frac{\partial L}{\partial \psi^{[\nu]}_{,o}} \psi^{[\nu]}_{,o} - L\right) = \int d^3x (\pi_{[\nu]} \dot{\psi}^{[\nu]} - L) \qquad \text{(II,42b)}$$

and for the momentum:

$$P^{k} = g^{k\ell} \int d^3x\, T^{o}_{\ell} = c \int d^3x\, \pi_{[\nu]} \psi^{[\nu],k} = - P_{k} \quad .$$

If one regards the field equations under the form (II,32), the generators for a space-time translations are $\overline{U}^{\lambda} = - P^{\lambda}$.

II.10) The field angular momentum tensor.

When the infinitesimal transformations of the coordinates are homogeneous Lorentz transformations :

$$x'^{\alpha} = \ell^{\alpha}{}_{\beta} x^{\beta}$$

one has :

$$\ell^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} + \varepsilon^{\alpha}{}_{\beta}$$

where the parameters $\varepsilon^{\alpha}_{\beta}$ are to be retained only to first order. We may write:

$$x'^{\alpha} = x^{\alpha} + \varepsilon_{\alpha\beta}\, x^{\beta}$$

where

$$\varepsilon_{\alpha\beta} = g_{\alpha\lambda}\, \varepsilon^{\lambda}{}_{\beta}$$

and, in view of (II,12) :

$$\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$$

We can still write :

$$x'^{\alpha} = x^{\alpha} + \delta x^{\alpha} \quad ,$$

$$\delta x^{\alpha} = \frac{1}{2} f^{\alpha}_{\lambda\nu} \varepsilon^{\lambda\nu} \quad , \qquad \text{(II,43)}$$

$$f^{\alpha}_{\lambda\nu} = - f^{\alpha}_{\nu\lambda} = (\delta^{\alpha}_{\lambda} g_{\nu\eta} - \delta^{\alpha}_{\nu} g_{\lambda\eta})\, x^{\eta}$$

Now

$$\psi'^{[\alpha]}(x') = \psi^{[\alpha]}(x) + \frac{1}{2} \mathcal{F}^{[\alpha]}{}_{\lambda\nu}(x)\, \varepsilon^{\lambda\nu}$$

and we shall set :

$$\mathcal{F}^{[\alpha]}_{\lambda\nu}(x) = M^{[\alpha]}_{[\beta]\lambda\nu}\, \psi^{[\beta]}(x) \qquad \text{(II,44)}$$

where

$$M^{[\alpha]}_{[\beta]\lambda\nu} = - M^{[\alpha]}_{[\beta]\nu\lambda}$$

The Noether quantity is thus the following :

$$N^{\lambda}_{\mu\nu} = \frac{\partial L}{\partial \psi^{[\alpha]}_{,\lambda}} \left(\psi^{[\alpha]}_{,\eta} f^{\eta}_{\mu\nu} - \mathcal{F}^{[\alpha]}_{\mu\nu} \right) - L\, f^{\lambda}_{\mu\nu}$$

that is, in view of equations (II,41),(II,42) and (II,43):

$$N^{\lambda}_{\mu\nu} = T^{\lambda}_{\mu}\, x_{\nu} - T^{\lambda}_{\nu}\, x_{\mu} - \frac{\partial L}{\partial \psi^{[\alpha]}} M^{[\alpha]}_{[\beta]\mu\nu} \psi^{[\beta]}$$

This is the angular momentum tensor density; the first two terms constitute its orbital part, the last term the spin part. The angular momentum tensor is :

$$J_{\mu\nu} = \int d\sigma_\lambda \; N^\lambda_{\mu\nu} = \int d^3 x \, N^o_{\mu\nu} \qquad \text{(II,45)}$$

The generator of the canonical transformation corresponding to a spatial rotation is :

$$U_{k\ell} = - J_{k\ell} \qquad \text{(II,46)}$$

The reader will verify that the conservation of the tensor $N_{\lambda\mu\nu}$ requires a symmetric energy momentum tensor : $T_{\mu\nu} = T_{\nu\mu}$.

II.11) The Lorentz geometrical nature of fields.

The study of the representations of the Poincaré group determines the geometrical nature of the wave fields, the equations of which are invariant under the proper Poincaré group(6). The fields must belong to a representation space of this group and they can only be scalars, spinors, vectors and spinors or tensors of higher rank.

Here is a list of the simplest fields (where we take c = 1) :

1) Complex scalar field : $\phi(x), \phi^*(x)$: $\phi'(x') = \phi(x)$

Free field equation :

$$(\Box + K^2)\,\phi(x) = 0 \quad , \quad (\Box + K^2)\,\phi^*(x) = 0 \, .$$

Lagrangean :

$$L = -\,(K^2 \phi^* \phi - g^{\mu\nu} \phi^*_{,\mu} \phi_{,\nu}).$$

Energy momentum tensor :

$$T^{\mu\nu} = \frac{\partial L}{\partial \psi^{[\lambda]}_{,\mu}} \psi^{[\lambda],\nu} + \frac{\partial L}{\partial \psi^{[\lambda]*}_{,\mu}} \psi^{*[\lambda],\nu} - L \; g^{\mu\nu}$$

$$= g^{\mu\alpha} g^{\nu\beta} (\phi^*_{,\alpha} \phi_{,\beta} + \phi^*_{,\beta} \phi_{,\alpha}) - L \, g^{\mu\nu}$$

Hamiltonian and momentum :

$$\pi^* = \frac{\partial L}{\partial \dot\phi} = \dot\phi^*$$

$$\overline{H} = \int d^3x (\pi^* \pi + (\nabla \phi^* . \nabla \phi) + K^2 \phi^* \phi),$$

$$P^k = \int d^3x (\pi^* \phi_{,k} + \pi \phi^*_{,k}) = \int d^3x \; T_{ok} \, .$$

Angular momentum (only orbital since $M^{[\alpha]}_{[\beta]\lambda\nu} = 0$)

$$L_{k\ell} = \int d^3x \left[\pi^*(\phi_{,k}\, x_\ell - \phi_{,\ell}\, x_k) + \pi(\phi^*_{,k}\, x_\ell - \phi^*_{,\ell} x_k)\right]$$

$$= \int d^3x\, (T_{ok}\, x_\ell - T_{o\ell}\, x_k)$$

The two scalars ϕ, ϕ^* are equivalent to the couple of real fields ϕ_1, ϕ_2 such that :

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 - i\,\phi_2)\,, \qquad \text{(II,47)}$$

$$\phi^* = \frac{1}{\sqrt{2}}(\phi_1 + i\,\phi_2)\,.$$

and one can describe the quantities associated to a single real field by expressing the above ones in terms of ϕ_1, ϕ_2 and dropping the terms in ϕ_2 .

2) Real vector field , $\phi^\mu(x)$:

$$\phi'^\mu(x') = \ell^\mu_{\ \nu}\phi^\nu(x).$$

Free-field equation :

$$(\Box + K^2)\phi^\mu(x) = 0\ ;\quad \phi^\mu_{\ ,\mu} = 0$$

Lagrangean :

$$L = -\frac{1}{2}\, g_{\alpha\beta}\,(K^2\,\phi^\alpha\,\phi^\beta - g^{\mu\lambda}\,\phi^\alpha_{,\mu}\,\phi^\beta_{,\lambda}) \qquad \text{(II,48)}$$

In view of the condition $\phi^\mu_{,\mu} = 0$, one can write (L is defined up to a four-divergence) :

$$L = -\frac{1}{2}\, g_{\alpha\beta}\,(K^2\,\phi^\alpha\,\phi^\beta - g^{\mu\lambda}\,F^\alpha_\mu\,F^\beta_\lambda)$$

where :

$$F^\alpha_{,\mu} = \frac{\partial\phi_\mu}{\partial x_\alpha} - \frac{\partial\phi^\alpha}{\partial x^\mu}$$

Energy-momentum tensor :

$$T^\alpha_\beta = g_{\nu\eta}\, g^{\alpha\lambda}\,\phi^\eta_{,\lambda}\,\phi^\nu_{,\beta} - L\,\delta^\alpha_\beta$$

Hamiltonian and momentum :

$$\pi_\alpha = \frac{\partial L}{\partial\phi^\alpha_{,o}} = g_{\alpha\beta}\,\phi^\beta_{,o} = \phi_{\alpha,o}$$

$$H = \frac{1}{2}\int d^3x(\pi_\alpha \pi^\alpha + \phi^\alpha_{,k} + \phi_{\alpha,k} + K^2 \phi_\alpha \phi^\alpha)$$

$$P^k = \int d^3x\, \pi_\alpha \phi^{\alpha,k} = -P_k$$

Angular momentum :

$$J_{\mu\nu} = \int d^3x\; N^o_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu},$$

$$L_{\mu\nu} = (T^o_\mu x_\nu - T^o_\nu x_\mu)\, d^3x = \int\left[\pi_\alpha(\phi^\alpha_{,\mu} x_\nu - \phi^\alpha_{,\nu} x_\mu) - L(\delta^o_\mu x_\nu - \delta^o_\nu x_\mu)\right] d^3x,$$

$$S_{\mu\nu} = \int (\phi_\mu \pi_\nu - \phi_\nu \pi_\mu)\, d^3x$$

3) Dirac's spinor field $\psi(x) = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$. We take $\hbar = 1$, $c = 1$.

Free-field equation of motion :

$$(i\gamma^\alpha \frac{\partial}{\partial x^\alpha} - K)\psi(x) = 0$$

where γ^o, γ^1, γ^2, γ^3 are four-by-four matrices satisfying

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu}.$$

The adjoint field $\bar{\psi}$ is defined by

$$\bar{\psi}(x) = \psi^+(x)\,\gamma^o$$

and ψ^+ is the hermitian conjugate of ψ: $\psi^+ = (\psi^*_1, \psi^*_2, \psi^*_3, \psi^*_4)$. Its equation is :

$$i\frac{\partial\bar{\psi}}{\partial x^\mu}\gamma^\mu + K\bar{\psi} = 0, \qquad (\gamma^\mu)^+ = \gamma^o \gamma^\mu \gamma^o$$

The Poincaré transformed of $\psi(x)$

$$\psi'(x') = D(\ell)\,\psi(x)$$

where :

$$\ell^\mu_\nu D(\ell)\,\gamma^\nu\, D^{-1}(\ell) = \gamma^\mu \qquad \text{(II,49)}$$

For an infinitesimal transformation one has :

$$\ell^\mu_\nu = \delta^\mu_\nu + \varepsilon^\mu_\nu,$$

$$D_{\alpha\beta}(\varepsilon) = \delta_{\alpha\beta} + \frac{1}{2} M^{\lambda\nu}_{\alpha\beta} \varepsilon_{\lambda\nu} \qquad \text{(II,50)}$$

It follows from the equations (II,49) and (II,50) that :

$$i\left[M^{\lambda\nu}, \gamma^{\eta}\right] = \gamma^{\nu} g^{\lambda\eta} - \gamma^{\lambda} g^{\nu\eta}$$

which is satisfied by :

$$M^{\lambda\nu} = \frac{1}{4}(\gamma^{\lambda}\gamma^{\nu} - \gamma^{\nu}\gamma^{\lambda})$$

The Lagrangean is

$$L = \frac{i}{2}\left(\bar{\Psi}\gamma^{\mu}\frac{\partial\Psi}{\partial x^{\mu}} - \frac{\partial\bar{\Psi}}{\partial x^{\mu}}\gamma^{\mu}\Psi(x)\right) - K\,\bar{\Psi}\Psi$$

The energy-momentum tensor has the form :

$$T^{\lambda\nu} = \frac{i}{2} g^{\alpha\lambda}\left(\bar{\Psi}\gamma^{\nu}\frac{\partial\Psi}{\partial x^{\alpha}} - \frac{\partial\bar{\Psi}}{\partial x^{\alpha}}\gamma^{\nu}\Psi\right)$$

and the spin tensor :

$$S^{\nu;\alpha\beta} = -\frac{\partial L}{\partial\Psi_{\lambda,\nu}} M_{\lambda\eta;\alpha\beta}\Psi_{\eta} - \bar{\Psi}_{\lambda} M_{\lambda\eta;\alpha\beta}\frac{\partial L}{\partial\bar{\Psi}_{\eta,\nu}}$$

$$= \frac{1}{4}\bar{\Psi}(x)\left[\gamma^{\nu}\sigma^{\alpha\beta} + \sigma^{\alpha\beta}\gamma^{\nu}\right]\Psi(x)$$

where

$$\sigma^{\alpha\beta} = \frac{1}{2i}(\gamma^{\alpha}\gamma^{\beta} - \gamma^{\beta}\gamma^{\alpha}).$$

II.12) The current-vector.

When the wave fields are complex functions, the Lagrangean - which must be real (hermitian in the quantized version of the theory) - must depend only on terms of the form $\Psi^{*[\alpha]}\Psi_{[\alpha]}$, $g^{\mu\lambda}\,\Psi^{*[\alpha]}_{,\lambda}\Psi_{[\alpha],\mu}$. It will therefore be invariant if the field is multiplied by an arbitrary phase factor (gauge transformations of the first kind) :

$$\Psi'^{[\alpha]}(x) = e^{i\eta}\Psi^{[\alpha]}(x)\,,\quad \Psi^{*'[\alpha]}(x) = e^{-i\eta}\,\Psi^{*[\alpha]}(x)$$

where η is a real number.

This constitutes an example of a non-geometrical group of transformations, the importance of which gives support to the second definition of the invariance principle in paragraph I,4).

For η infinitesimal we have, up to terms of first order in η :

$$\psi'^{[\alpha]}(x) = (1 + i\eta)\, \psi^{[\alpha]}(x)$$

$$\psi^{*'[\alpha]}(x) = (1 - i\eta)\, \psi^{*[\alpha]}(x)$$

As the coordinates x do not change, we have in (II,41):

$$f^{\mu}{}_{[\nu]} = 0$$

However (see (II,40)) :

$$\mathcal{F}^{[\alpha]}{}_{[\beta]} \equiv i\,\psi^{[\alpha]} \quad , \quad \omega^{[\beta]} = \eta$$

The conserved Noether's quantity is the current-vector :

$$N^{\nu} \equiv j^{\nu} = i \left(\frac{\partial L}{\partial \psi^{*[\alpha]}_{,\nu}} \psi^{*[\alpha]} - \frac{\partial L}{\partial \psi^{[\alpha]}_{,\nu}} \psi^{[\alpha]} \right),$$

i.e. such that : $\dfrac{\partial j^{\nu}}{\partial x^{\nu}} = 0$.

The charge of the field is :

$$Q = \int d\sigma_{\nu}\; j^{\nu} = \int d^3x\, j^0 .$$

Thus the current-vector of a complex scalar field is :

$$j^{\nu} = i\, g^{\nu\lambda} \left(\phi^{*}\, \phi_{,\lambda} - \phi^{*}{}_{,\lambda}\, \phi \right) , \qquad \text{(II,51)}$$

that of a spinor field :

$$j^{\nu} = \bar{\psi}\, \gamma^{\nu} \psi \qquad \text{(II,52)}$$

The interaction lagrangean with an electromagnetic field is :

$$L_{int} = - e\, \bar{\psi}\, \gamma^{\nu}\, \psi A_{\nu}(x).$$

which is to be added to the free electromagnetic and spinor field Lagrangeans.

Another non-geometric mapping is the gauge transformation of the second kind :

$$A'^{\mu}(x) = A^{\mu}(x) + \frac{\partial \Lambda(x)}{\partial x_{\mu}} ,$$

$$\psi'(x) = e^{ie\Lambda(x)}\, \psi(x) ,$$

$$\bar{\psi}'(x) = e^{-ie\Lambda(x)}\, \bar{\psi}(x)$$

where $\Lambda(x)$ is a scalar function, solution of the wave equation $\Box\Lambda = 0$.

Other quantities of the general form of Q where the current is of the form (II,52), are the baryon number B, the lepton numbers L_μ and L_e. They are, perhaps, related to interactions of the corresponding spinors with hypothetical vector fields in the same way as the electric charge is revealed in interaction with the electromagnetic field. The conservation of the baryon number and of the lepton number, although, empirically established, are however not yet fully understood. In quantum theory, these charges Q, B, L_μ, L_e have integral numbers .

II. 13) Outline of the transition into quantum theory.

The correspondence principle states that, given in classical theory, its quantum transcription will be achieved by the substitution of (hermitian) linear operators for (observable) classical variables and of commutators divided by $i\hbar$ for Poisson brackets between such variables :

$$\{A , B\} \rightarrow \frac{1}{i\hbar} [A, B] = \frac{1}{i\hbar}(AB - BA)$$

Thus the position of the coordinates and momenta of particles A,B,... are hermitian linear operators which satisfy the commutation rules (see (I,18)):

$$[x_{jA} , x_{kB}] = [p_{jA} , p_{kB}] = 0 ,$$

$$[x_{jA} , x_{kB}] = i\hbar\, \delta_{jk}\, \delta_{AB}$$

The canonical equations of a system of particles, with a hamiltonian operator H(x,p), are (see (I,17)):

$$i\hbar\, \dot{x}_{jA} = [x_{jA} , H] , \quad i\hbar\, \dot{p}_{jA} = [p_{jA} , H]$$

and the time evolution of a physical variable F(x,p,t) is defined by the equation :

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{1}{i\hbar} [F , H] . \qquad \text{(II,53)}$$

It is to be noted that x and p do not depend explicitely on t.

These equations give the description of our system of point-
in the Heisenberg picture. The operators are here represented by matrices and one is interested in the search for transformations which will change the Hamiltonian matrix into diagonal form. The diagonal elements of H are the energy eigenvalues of our system, its only possible observable energy values. Operators which commute among themselves can all be diagonalised by a given transformation. The physical definition of the state of a system is given through the maximum number of commuting observables each of which can thus have a given value in the state in question.

Such operators act on a Hilbert space, a vector space with complex numbers as scalars and a scalar product defined by :

$$\langle \Phi | \Psi \rangle = \langle \Psi | \Phi \rangle^*$$

$$\langle \Phi | \Sigma c_i \Psi_i \rangle = \Sigma c_i \langle \Phi | \Psi_i \rangle$$

$$\langle \Phi | \Phi \rangle \geq 0 \ ,$$

$\langle \Phi | \Phi \rangle = 0$ implies $|\Phi\rangle = 0$. Such a vector space must be complete : if $|\Psi_i\rangle$, $i = 1, 2, \ldots$, is a Cauchy sequence of state vectors, i.e., such that given $\varepsilon > 0$ there is an integral number n for which

$$(\langle \Psi_i - \Psi_j | \Psi_i - \Psi_j \rangle)^{1/2} < \varepsilon \ , \quad i,j \geq n$$

then the sequence has a limit $|\Psi\rangle$:

$$\lim_{j \to \infty} (\langle \Psi_j - \Psi | \Psi_j - \Psi \rangle)^{1/2} = 0 \ .$$

If instead of the Heisenberg equations (II,53), we regard the time evotion of the system as described by that of a state vector $|\Psi\rangle$, the fundamental equation is the Schrödinger's equation :

$$i \hbar \frac{\partial |\Psi\rangle}{\partial t} = H |\Psi\rangle$$

This is the Schrödinger picture according to which the operators do not depend explicitly on time.

Given an operator A of which all the eigenvalues a and eigenfunctions $|a\rangle$ are known :

$$A|a\rangle = a|a\rangle \quad , \qquad \langle a | a' \rangle = \delta(a-a'),$$

one may develop $|\Psi\rangle$ into this complete set of eigenfunctions :

$$|\Psi\rangle = \int d\,a\ \psi(a,t)\,|a\rangle$$

where

$$\psi(a,t) = \langle a|\Psi\rangle$$

The Schrödinger equation will read in this case, in terms of the amplitudes $\psi(a,t)$:

$$i\hbar\,\frac{\partial\psi(a,t)}{\partial t} = \int d\,b\,\langle a|H|b\rangle\,\psi(b,t).$$

The usual formulation of one-particle quantum mechanics in $\mathbf{x}$ - space is based on the equation :

$$i\hbar\,\frac{\partial}{\partial t}\psi(\underset{\sim}{x},t) = \int d^3y\,\langle \underset{\sim}{x}|H|y\rangle\,\psi(\underset{\sim}{y},t)$$

where, for local potentials :

$$\langle \underset{\sim}{x}|H|\underset{\sim}{y}\rangle = H(\underset{\sim}{y})\,\delta(\underset{\sim}{x}-\underset{\sim}{y}).$$

Our system may, however, be more complicated. In the case of a system of photons, this will be described in the Heisenberg picture by operators $A^\mu(x)$ such that (see (II,30)):

$$\left[A^\mu(\underset{\sim}{x},t),\ A^\lambda(\underset{\sim}{y},t)\right] = \left[\pi_\mu(\underset{\sim}{x},t),\ \pi_\lambda(\underset{\sim}{y},t)\right] = 0$$

$$\left[A^\mu(\underset{\sim}{x},t),\ \pi_\lambda(\underset{\sim}{y},t)\right] = i\hbar\ \delta^\mu{}_\lambda\,\delta(\underset{\sim}{x}-\underset{\sim}{y}) \qquad \text{(II,54)}$$

and the free-field covariant commutation rules are (see (II,35)):

$$\left[A^\mu(x),\ A^\lambda(y)\right] = i\hbar c\ g^{\mu\lambda}\,D(x-y). \qquad \text{(II,54a)}$$

The Lorentz supplementary condition, however, cannot be transcribed in a simple way into quantum theory. If the operator relation $\frac{\partial A^\mu}{\partial x^\mu} = 0$ were true , one would have $\frac{\partial \pi^\lambda}{\partial x^\lambda} = 0$ and this would be in conflict with the commutators (II,54; 54a). The correction transcription of the Lorentz supplementary condition in quantum electrodynamics is to consider the positive frequency part, $A^{\mu(+)}$ of A^μ, according to the Fourier representation (λ is the polarisation):

$$A^\mu(x) = \frac{1}{(2\pi)^{3/2}}\int\frac{d^3k}{(2k^0)^{1/2}}\sum_\lambda e^\mu(\lambda,\underset{\sim}{k})\left[a(\lambda,\underset{\sim}{k})\,e^{-ikx} + a^+(\lambda,\underset{\sim}{k})\,e^{ikx}\right] =$$

$$= A^{\mu(+)} + A^{\mu(-)}$$

and select those state vectors $|\Psi\rangle$ restricted by the condition :

$$\frac{\partial A^{\mu(+)}}{\partial x^{\mu}} |\Psi\rangle = 0.$$

Another example of a quantum system is a pion in interaction with a photon. As pions have a strong interaction with nucleons, the Lagrangean of pions and photons must include the terms relative to the nucleons field. The corresponding Lagrangean is thus (weak interactions being neglected) :

$$L = L_{\pi} + L_N + L_{rad} + L_{int} \tag{II,55}$$

where :

$$L_{\pi} = \frac{1}{2} (\mu^2 \varphi_j^2 - g^{\alpha\beta} \varphi_{j,\alpha} \varphi_{j,\beta})$$

is the free pion-field Lagrangean described by three real (hermitian) fields φ_j , j = 1,2,3;

$$L_N = \frac{1}{2} (\bar{\Psi}_N \gamma^{\mu} \frac{\partial \Psi_N}{\partial x^{\mu}} - \frac{\partial \bar{\Psi}_N}{\partial x^{\mu}} \gamma^{\mu} \Psi_N) - K \bar{\Psi}_N \Psi_N$$

is the free nucleon-field Lagrangean where $\Psi_N = \begin{pmatrix} \Psi_p \\ \Psi_n \end{pmatrix}$ Ψ_p and Ψ_n are the four-component spinors of the proton and neutron respectively; L_{rad} is given by (II,48) with $K = 0$ and $\phi^{\mu} = A^{\mu}$;

$$L_{int} = g \bar{\Psi}_N \gamma^5 \tau_j \Psi_N \varphi_j + e \bar{\Psi}_N \gamma_{\mu} \frac{1+\tau_3}{2} \Psi_N A^{\mu} + e j_{\mu}^{\pi} A^{\mu}$$

comprises the nucleon-pion interactions, with :

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad ;$$

the nucleon-photon interaction and the pion photon interaction where j_{μ}^{π} is given in equation (II,51) and (II,47). The total electric current of this system which obeys Noether's theorem, is :

$$j^{\mu} = i g^{\mu\lambda}(\varphi^{*} \varphi_{,\lambda} - \varphi^{*}_{,\lambda}\varphi) + \bar{\Psi}_N \gamma^{\mu} \frac{1+\tau_3}{2} \Psi_N \tag{II,56}$$

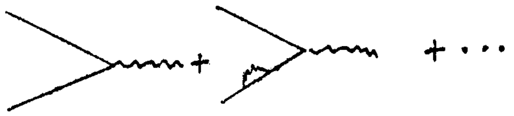
The nucleon field-all spin 1/2 fields- is however quantized according to anticommutators, as raquired by the exclusion principle .

II. 14) <u>Conserved current and equality of charges of elementary particles.</u>

An illustration of the importance of conserved Noether's quantities is furnished by the relationship between conserved current and the equality of charges of elementary particles. The charge of an electron is revealed by its interaction with a photon. If weak interactions are neglected, the electron charge is given by the matrix element :

$$Q_e = \int \langle e_{out} | \psi^+ \psi | e_{in} \rangle \, d^3x$$

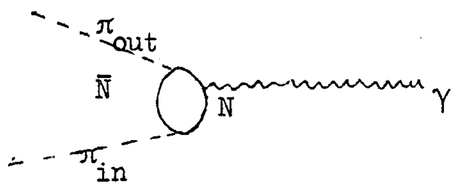
where $| e_{in} \rangle$ and $| e_{out} \rangle$ are the incident and outgoing electron in interaction with a photon :



For an incoming and outgoing pion, the charge will be :

$$Q_\pi = \int \langle \pi_{out} | j_o^\pi | \pi_{in} \rangle \, d^3x + \int \langle \pi_{out} | \psi_N^+ \psi_N | \pi_{in} \rangle \, d^3x$$

The first term is of the type of the above one and we may assume equal to it, the second term might give additional contributions and make $Q_\pi \neq Q_e$, as illustrated by graphs such as the following one :



However, the solutions ψ_N and φ to the equations derived from the Lagrangean (II,55) must be such that in the remote past they describe free nucleons and pions :

$$\lim_{t \to -\infty} (\Box + \mu^2) \varphi = 0$$

$$\lim_{t \to -\infty} \left(i \gamma^\lambda \frac{\partial \psi_N}{\partial x^\lambda} - K \psi_N \right) = 0$$

Therefore, in the remote past, $\int \Psi_N^+ \Psi_N \, d^3x$ = number of nucleons - number of antinucleons ; since there is only one pion in the distant past one has :

$$\lim_{t \to -\infty} \langle \pi_{out} | \int \Psi_N^+ \Psi_N \, d^3x \, | \pi_{in} \rangle = 0$$

and

$$\lim_{t \to -\infty} Q_\pi(t) = \langle \pi_{out} | \int j_o^\pi \, d^3x \, | \pi_{in} \rangle = Q_\pi^o$$

which we assume equal to the electron's charge .

In view of the total current conservation, (II,56), one has :

$$\frac{d\, Q_\pi(t)}{dt} = 0$$

hence $Q_\pi = Q_\pi^o = Q_e$.

II. 15) Symmetry operations in quantum theory.

To canonical transformations in classical physics, there correspond unitary transformations in quantum theory : given a state vector $|\Psi\rangle$, one can define a new state vector $|\Psi'\rangle$ after the physical variables of the system have undergone a certain transformation. It is postulated that there exists a linear operator U which transforms $|\Psi\rangle$ into $|\Psi'\rangle$

$$|\Psi'\rangle = U |\Psi\rangle.$$

The transformation U in Hilbert space is a symmetry operation if the physical laws obtained from $|\Psi'\rangle$ are the same as those obtained from $|\Psi\rangle$. Let R be the transformation of physical variables which induces the mapping U of the Hilbert space into itself. We shall write $U = U(R)$. If the set of all possible transformations R forms a group, the set of operations U(R) will also form a group - and is called a representation (an infinite dimensional one) - of $\{R\}$ if :

a) the unity in Hilbert space corresponds to the unity in the set $\{R\}$;

b) to the product $R_2 R_1$, $R_1, R_2 \in \{R\}$, there corresponds the product $U(R_2)\, U(R_1)$ up to a phase factor $\omega(R_2, R_1)$:

$$U(1) = I$$

$$U(R_2)\, U(R_1) = \omega(R_2, R_1)\, U(R_2 R_1)$$

The symmetry operation U is unitary if one postulates the conservation of the scalar product :

$$\langle \bar{\Psi}' | \bar{\Psi}' \rangle = \langle \bar{\Psi} | U^+ U | \bar{\Psi} \rangle = \langle \bar{\Psi} | \bar{\Psi} \rangle \qquad \text{(II,57)}$$

which implies :

$$U^+ U = I.$$

In general, however, what must be postulated is the probability conservation :

$$|\langle \bar{\Psi}' | \bar{\Psi}' \rangle|^2 = |\langle \bar{\Psi} | \bar{\Psi} \rangle|^2 \qquad \text{(II,58)}$$

U is <u>unitary</u> if this is satisfied by the equality of the amplitudes (II,57). The equation (II,58) may, however, be still satisfied if :

$$\langle \Psi' | \Psi' \rangle = \langle \bar{\Psi} | \bar{\Psi} \rangle^*$$

in which case U is called <u>anti-unitery</u> (this is the case of time reversal; in classical mechanics the corresponding transformation is anti-canonical in the sense that it changes the sign of the Poisson bracket $\{x,p\}$).

How do the operators which describe physical variables transform when $|\bar{\Psi}\rangle \rightarrow |\Psi'\rangle = U|\bar{\Psi}\rangle$? Two main types of transformations may be defined. The Schrödinger type assumes that the operators $O^{[\alpha]}(x)$ do not change :

$$O^{[\alpha]'}(x) = O^{[\alpha]}(x)$$

$$|\bar{\Psi}'\rangle_S = U|\bar{\Psi}\rangle$$

so that :

$$\langle \bar{\Psi}' | O^{[\alpha]'}(x) | \bar{\Psi}' \rangle_S = \langle \bar{\Psi} | U^+ O^{[\alpha]}(x) U | \bar{\Psi} \rangle .$$

The Heisenberg type of transformation assumes invariance of the state vectors:

$$|\bar{\Psi}'\rangle_H = |\bar{\Psi}\rangle$$

so that

$$\langle \bar{\Psi}' | O^{[\alpha]'}(x) | \bar{\Psi}' \rangle_H = \langle \bar{\Psi} | O^{[\alpha]'}(x) | \bar{\Psi} \rangle$$

The equivalence between both types of transformation is assured by the equality :

$$\langle \bar{\Psi}' | O^{[\alpha]'}(x) | \bar{\Psi}' \rangle_H = \langle \bar{\Psi}' | O^{[\alpha]'}(x) | \bar{\Psi}' \rangle_S$$

hence :

$$\langle \bar{\Psi} | O^{[\alpha]'}(x) | \bar{\Psi} \rangle = \langle \bar{\Psi} | U^+ O^{[\alpha]}(x) U | \bar{\Psi} \rangle .$$

The operators transform therefore in the following way :

$$O^{[\alpha]'}(x) = U^{+} O^{[\alpha]}(x) U . \qquad (II,59)$$

If U is a continuous operator, we shall write, for an infinitesimal transformation with parameters $\omega^{[\nu]}$ (see paragraph II,8):

$$U = I - \frac{i}{\hbar} U_{[\nu]} \omega^{[\nu]} \qquad (II,60)$$

and shall call the hermitian operators $U_{[\nu]}$ the generators of the group. The equations (II,59) and (II,60) give rise to the equation :

$$\delta O(x) = O^{[\alpha]'}(x) - O^{[\alpha]}(x) = \frac{1}{i\hbar}[O^{[\alpha]}(x), U_{[\nu]}]\omega^{[\nu]}$$

which corresponds to the classical relationship (II,33). Therefore, if one sets, as in (II,39) :

$$\bar{\delta} O^{[\alpha]}(x) = O^{[\alpha]'}(x') - O^{[\alpha]}(x) = \mathcal{F}^{[\alpha]}_{[\beta]}(x)\, \omega^{[\beta]}$$

one gets :

$$\delta O^{[\alpha]}(x) = (\mathcal{F}^{[\alpha]}_{[\beta]}(x) - O^{[\alpha]}_{,\lambda} f^{\lambda}_{[\beta]})\, \omega^{[\beta]}$$

hence :

$$\mathcal{F}^{[\alpha]}_{[\beta]}(x) - O^{[\alpha]}_{,\lambda}(x) f^{\lambda}_{[\beta]} = \frac{1}{i\hbar} [O^{[\alpha]}(x), U_{[\beta]}]$$

where $f^{\lambda}_{[\beta]}$ is given by relation (II,37).

The reader will find the form of these equations corresponding to space-time translation, $U_{[\nu]} \to P_{\nu}$, to a homogeneous Lorentz transformation, $U_{[\nu]} \to J_{\lambda\nu}$ and to a gauge transformation of the second kind , $U_{[\nu]} \to -Q\sqrt{\frac{\hbar}{c}}$. Thus, in an infinitesimal translation along the axis of abcissae by the amount a :

$$x' = x + a \qquad (II,61)$$

one should have, according to (II,59) and (II,60) :

$$x' = U^{+} x U = (1 + \frac{i}{\hbar} a\, p_x)\, x\, (1 - \frac{i}{\hbar} a\, p_x) \qquad (II,62)$$

the comparison of (II,61) and (II,62) leads to the commutation relation between x and p_x :

$$[x, p_x] = i\hbar$$

If one performs a rotation by an infinitesimal angle φ around the z axis:

$$x' = x - \varphi y \ , \quad y' = y + \varphi x \ , \quad z' = z$$

or

$$\underset{\sim}{x}' = R(\varphi)\underset{\sim}{x} \ , \quad R(\varphi) = I + \begin{pmatrix} 0 & -\varphi & 0 \\ \varphi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

one must have :

$$x' = U^{+} x U = (I + \frac{i}{\hbar} \varphi L_z) \, x (I - \frac{i}{\hbar} \varphi L_z) = x - \varphi y$$

$$y' = U^{+} y U = (I + \frac{i}{\hbar} \varphi L_z) \, y (I - \frac{i}{\hbar} \varphi L_z) = y + \varphi x$$

hence

$$[x, L_z] = i \hbar y \ ,$$

$$[y, L_z] = i \hbar x$$

If the rotation is around an axis $\underset{\sim}{n}$ by an angle α , one has :

$$U(\alpha, \underset{\sim}{n}) = I - \frac{i}{\hbar} \alpha(\underset{\sim}{n} \cdot \underset{\sim}{L})$$

and the consideration of two successive rotations around two axis in different order gives rise to the angular momentum commutation rules :

$$\underset{\sim}{L} \times \underset{\sim}{L} = i \hbar \underset{\sim}{L}$$

The equations which connect a field $O^{[\alpha]}(x)$ with its charge Q :

$$\frac{1}{\sqrt{\hbar c}} \left[O^{[\alpha]}(x) \, , Q \right] = O^{[\alpha]}(x) \ ,$$

$$\frac{1}{\sqrt{\hbar c}} \left[O^{[\alpha]+}(x), Q \right] = - O^{[\alpha]+}(x)$$

follow from the identification of

$$O^{[\alpha]'}(x) = e^{i\varepsilon} O^{[\alpha]}(x) \simeq (1 + i \varepsilon) \, O^{[\alpha]}(x)$$

with :

$$O^{[\alpha]'}(x) = U^{+}(\varepsilon) \, O^{[\alpha]}(x) \, U(\varepsilon) = (I - \frac{i}{\sqrt{\hbar c}} Q \varepsilon) \, O^{[\alpha]}(x) (I + \frac{i}{\sqrt{\hbar c}} Q \varepsilon)$$

The charge operator is the generator of the gauge group.

-:-:-:-

III

THE PROBLEM OF THE ENERGY-MOMENTUM TENSOR CONSERVATION IN THE RELATIVISTIC THEORY OF GRAVITATION

-:-:-

III.1) Introduction.

In field theories in a flat space ,i.e., in a space with a Lorentz metric tensor (II,10), conserved quantities $S_{[\beta]}(x_o)$ (see parag.II,8):

$$\frac{d\, S_{[\beta]}}{dx^o} = 0$$

are those derived from Noether's tensors $N^{\lambda}_{[\beta]}$

$$S_{[\beta]} = \int d\sigma_{\lambda}\, N^{\lambda}_{[\beta]}(x)$$

which obey the divergence equation :

$$\frac{\partial N^{\lambda}_{[\beta]}}{\partial x^{\lambda}} = N^{\lambda}{}_{[\beta],\lambda} = 0$$

In the relativistic theory of gravitation, such conservation laws are not covariant. In the presence of a gravitational field, $g^{\mu\nu}(x)$, the covariant divergence of a tensor contains, besides the usual four-divergence, additional terms which depend on the gravitational field derivatives. Einstein's equations are established in such a way that the covariant divergence of the energy-momentum tensor - the source of the field - vanish, as a generalisation of the equation (II,41a) for this tensor. It follows from this that one cannot build up a conserved energy-momentum vector.

In this section, we shall present of brief review of Einstein's theory and then examine, in a simple fashion, this question. It will be seen that objects can be constructed which are conserved but these objects are not tensors nor unique.

III.2) The search for relativistic gravitational field equations.

The starting point of a relativistic theory of gravitation is the search for covariant equations which generalise Poisson's equation for the Newton gravitational field $V(\underset{\sim}{x})$:

$$\nabla^2 V(\underset{\sim}{x}) = - G\, \rho_m(\underset{\sim}{x}) \, . \qquad \text{(III,1)}$$

G is the gravitational coupling constant, $\rho_m(\underset{\sim}{x})$ is the mass density, source of V. This may be regarded as the static limit of the equation :

$$\Box V(x) = G\, \rho_m(x) \qquad \text{(III,2)}$$

where $\Box$ is the D'Alembertian operator (II,24a) and $x = (x^0, \underset{\sim}{x})$. Now, however, unlike the electric charge, the mass is not Lorentz invariant and thus $\rho_m(x)$ cannot be regarded as the time component of a four-vector. In view of the equivalence between mass and energy, we see from equations (II,42a) and (II,42b) that the mass density is the zero-zero component of the energy momentum tensor. The problem reduces then to find , from simple physical arguments, a tensor $B_{\mu\nu}$ which be a function of the gravitational field and its first and second derivatives, and equate it to $f.T_{\mu\nu}$:

$$B_{\mu\nu}(x) = f\, T_{\mu\nu}(x) \qquad \text{(III,3)}$$

where f is a coupling constant. The equation (III,3) must go over into equation (III,2) in the approximation for weak fields.

Einstein's beautiful theory of gravitation identifies this field with the metric of the Riemannian space-time geometry. In its construction, Einstein was intuitively guided by two principles:

1) the equivalence principle

2) the postulate of covariance of natural laws (not only under the Poincaré transformation group but) under continuous one-to-one coordinate transformations :

$$x'^{\mu} = f^{\mu}(x^0, x^1, x^2, x^3) \qquad \mu = 0,1,2,3 \qquad \text{(III,4)}$$

with continuous first derivatives $\dfrac{\partial x'^{\mu}}{\partial x^{\lambda}}$ and non-vanishing jacobian .

The so-called apparent forces are those - like the inertial, centrifugal and Coriolis forces - which are proportional to inertial mass and which can be transformed away by a proper choice of the coordinate system. Thus, as well-known, if the equations of a point-particle have the form:

$$m \ddot{\underset{\sim}{x}} = \underset{\sim}{F} \qquad \text{(III,5)}$$

in an inertial frame S, the transition into a frame S' which rotates around the z-axis of S with constant angular velocity ω :

$$\begin{aligned} x' &= x \cos \omega t + y \sin \omega t \\ y' &= -x \sin \omega t + y \cos \omega t \\ z' &= z \end{aligned} \qquad \text{(III,6)}$$

or

$$\begin{aligned} x &= x' \cos \omega t - y' \sin \omega t \\ y &= x' \sin \omega t + y' \cos \omega t \end{aligned}$$

leads equations (III,5) into assuming the form :

$$\begin{aligned} m(\ddot{x}' - 2\,\omega\,\dot{y}' - \omega^2 x') &= F'_x , \\ m(\ddot{y}' + 2\,\omega\,\dot{x}' + \omega^2 y') &= F'_y \\ m\,\ddot{z}' &= F'_z \end{aligned} \qquad \text{(III,7)}$$

where

$$\begin{aligned} F'_x &= F_x \cos \omega t + F_y \sin \omega t , \\ F'_y &= -F_x \sin \omega t + F_y \cos \omega t ; \\ F'_z &= F_z . \end{aligned}$$

The terms proportional to $\dot{\underset{\sim}{x}}'$ - the Coriolis force - and to $\underset{\sim}{x}'$ - the centrifugal force - can thus be transformed away by a proper choice of the coordinate system, namely S.

Now the famous Einstein's elevator experiment led him to state that the gravitational force is, at least locally, equivalent to an accelerated reference frame and can, therefore, be regarded as an apparent force. The elevator experiment is this : an observer enclosed in a box verifies that all objects inside the box have a downward acceleration, independent of their mass. His interpretation of this fact is either a) that there is a source of a gravitational field at the bottom, which attracts all objects and communicates them the observed acceleration; or b) that the box is accelerated upwards and the inertia of all objects inside it gives them the observed downward acceleration. The two interpretations are equivalent because the inertial mass of any body is equal to its gravitational mass, as has been found experimentally. Einstein postulated this equality and the full equivalence between a homogeneous gravitational field and an accelerated frame of reference. This equivalence principle has the immediate consequence that any energy signal, such as a light ray, travelling across a gravitational field, is deflected by it.

Now, the space-time internal ds^2, which in an inertial frame has the form

$$ds^2 = dx^o - (\underset{\sim}{dx})^2 = g^{(o)}_{\mu\nu}\, dx^\mu\, dx^\nu \qquad \text{(III,8)}$$

where $g^{(o)}_{\mu\nu}$ is the Lorentz metric tensor (II,10), vests the more general form:

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu\, dx^\nu \quad ; \quad g_{\mu\nu}(x) = g_{\nu\mu}(x) \qquad \text{(III,9)}$$

when it refers to a non-inertial frame of reference. Thus in the case of the rotation (III,6), one has :

$$ds^2 = \left[1 - \frac{\omega^2}{c^2}(x'^2 + y'^2)\right](dx^o)^2 - (\underset{\sim}{dx'})^2 - 2\frac{\omega}{c}\left[y'\, dx' + x' dy'\right] dx^o$$

The heuristic considerations of the equivalence principle led Einstein to postulate that the metric tensor $g_{\mu\nu}(x)$ is identical with the gravitational field and that the description of all physical processes produced by this field is to be given by the Riemannian geometry of the four-dimensional space-time continuum. This is the essential postulate in Einstein's theory. As the equation of motion in any theory can be manipulated into a covariant form, it is the dynamical meaning of $g_{\mu\nu}(x)$ which is important and shows that the postulated invariance of natural laws with respect to general coordinate transformations is a dynamical - not geometric - invariance (see Fock, ref.[7]).

In the paragraphs III.3) - 9) the principal notions involved in the establishment of Einstein'gravitational field equations will be briefly recalled.

III.3) Tensors in a Riemannian space

Let us then consider a four-dimensional space in which the neighborhood of each point x has an interval defined by the Riemannian metric of the form (III,9); the transformation laws (III,4) and their inverse establish a one-to-one correspondence between two different maps of each of these regions.

A contravariant vector $F^\alpha(x)$ is a set of four functions associated to each such point which transforms, under the correspondence (III,4) as :

$$F'^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\beta}} F^{\beta}(x) \qquad \text{(III,10)}$$

where the sum over repeated indices is, as before, understood, and $\frac{\partial x'^{\alpha}}{\partial x^{\beta}}$ are the derivatives of the functions (III,4). The law (III,10) is a natural extension of the transformation law for coordinate differentials :

$$dx'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} dx^{\beta}.$$

which are linear in the dx^{β}.

The notion of <u>covariant vector</u> results from the differentiation of an invariant function :

$$\varphi'(x') = \varphi(x)$$

One has :

$$\frac{\partial \varphi'(x')}{\partial x'^{\alpha}} = \frac{\partial \varphi(x)}{\partial x^{\beta}} \frac{\partial x^{\beta}}{\partial x'^{\alpha}}$$

The derivatives $\frac{\partial \varphi(x)}{\partial x^{\beta}}$ are said to form a covariant vector. This entity is, in general, a set $K_{\alpha}(x)$ such that, under the transformations (III,4), transform as :

$$K'_{\alpha}(x') = K_{\beta}(x) \frac{\partial x^{\beta}}{\partial x'^{\alpha}} \qquad \text{(III,11)}$$

In general, a <u>tensor</u> with contravariant and covariant indices, $T^{\mu\nu\ldots}_{\alpha\beta\ldots}(x)$, is defined by the equations :

$$T'^{\mu'\nu'\ldots}_{\alpha'\beta'\ldots}(x') = \frac{\partial x'^{\mu'}}{\partial x^{\mu}} \frac{\partial x'^{\nu'}}{\partial x^{\nu}} \ldots T^{\mu\nu\ldots}_{\alpha\beta\ldots}(x) \frac{\partial x^{\alpha}}{\partial x'^{\alpha'}} \frac{\partial x^{\beta}}{\partial x'^{\beta'}} \ldots \qquad \text{(III,12)}$$

which is an extension of the transformation of direct products of vectors :
$A^{\mu}(x)\ A^{\nu}(x)\ldots B_{\alpha}(x)\ B_{\beta}(x)\ldots$
A <u>tensor density</u> of weight n transforms like (III,12) with an extra-factor J^{n} on the right-hand side, where J is the Jacobian of the mappings (III,4).

A contraction of a tensor with μ upper indices and ℓ lower indices is the sum over one lower and one upper indice and it is another tensor with μ-1 upper and ℓ-1 lower indices. The scalar product of two vectors is the contraction of their one-upper-one-lower-index tensor product and is invariant:

$$A'^{\mu'}(x')\, B'_{\mu'}(x') = \frac{\partial x'^{\mu'}}{\partial x^{\mu}} A^{\mu}(x)\, B_{\alpha}(x) \frac{\partial x^{\alpha}}{\partial x'^{\mu'}} = \delta^{\alpha}_{\mu} A^{\mu}(x)\, B_{\alpha}(x)$$

$$= A^{\mu}(x)\, B_{\mu}(x) \,.$$

δ^{μ}_{ν} is a tensor :

$$\delta'^{\mu'}_{\alpha'} \equiv \frac{\partial x'^{\mu'}}{\partial x'^{\alpha'}} = \frac{\partial x'^{\mu'}}{\partial x^{\mu}} \cdot \delta^{\mu}_{\alpha} \frac{\partial x^{\alpha}}{\partial x'^{\alpha'}}$$

The symmetry properties of tensors refer only to indices of the same floor and are invariant with respect to (III,4).

III.4) Parallel displacement.

An immediate consequance of the definitions (III,10),(III,11) is that the derivative of a vector (tensor) is not a tensor (of higher rank), except if the transformations (III,4) are linear (Lorentz metric). Indeed, according to (III,10) :

$$\frac{\partial F'^{\mu}(x')}{\partial x'^{\alpha}} = \frac{\partial}{\partial x'^{\alpha}} \left(\frac{\partial x'^{\mu}}{\partial x^{\beta}} F^{\beta}(x)\right) = \frac{\partial}{\partial x^{\lambda}} \left(\frac{\partial x'^{\mu}}{\partial x^{\beta}} F^{\beta}(x)\right) \frac{\partial x^{\lambda}}{\partial x'^{\alpha}}$$

$$= \frac{\partial x'^{\mu}}{\partial x^{\beta}} \cdot \frac{\partial F^{\beta}}{\partial x^{\lambda}} \frac{\partial x^{\lambda}}{\partial x'^{\alpha}} + \frac{\partial x^{\lambda}}{\partial x'^{\alpha}} F^{\beta}(x) \frac{\partial^2 x'^{\mu}}{\partial x^{\lambda} \partial x^{\beta}} \qquad \text{(III,13)}$$

The last term is the obstacle for the derivative of $F^{\mu}(x)$ being a tensor - it vanishes when the transformations (III,4) are linear.

It is an essential assumption in what follows that it is always possible to find, in every point of space-time, a frame referred to which the neighborhood of this point is describable by the Lorentz (minkowskian) geometry. In such a locally inertial frame, the parallel displacement of a vector to another point of the neighborhood does not change the vector components and the scalar product of this vector and any other is invariant.

For a general reference frame , the notion of parallel displacement of a vector $F^{\alpha}(x)$ from the point $M(x)$ to another of its neighborhood, $M'(x + dx)$ must leave the scalar product of F^{α} with an arbitrary vector, invariant. If δF^{α} is the change in F^{α} due to such a parallel displacement, we shall set it as a bilinear function of $F^{\alpha}(x)$ and of dx^{α}:

$$\delta F^{\mu}(x) = -\Gamma^{\mu}_{\alpha\beta}(x)\, F^{\alpha}(x)\, dx^{\beta} \qquad \text{(III,14)}$$

The Γ's are called Christoffel symbols. It can be shown that they are symmetric i.e., that in every point of the space-time continuum they can be made to obey the equality :

$$\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha} \qquad \text{(III,15)}$$

The prooof can be given by choosing dx^{μ} as the vector F^{μ} and by choosing new coordinates x'^{μ} locally minkowskian :

$$\delta(dx'^{\alpha}) = 0 \qquad \text{(III,16)}$$

In fact, in view of this choice and of the definition :

$$dx^{\mu} = \frac{\partial x^{\mu}}{\partial x'^{\nu}} dx'^{\nu}$$

one obtains :

$$\delta(dx^{\mu}) = \frac{\partial^2 x^{\mu}}{\partial x'^{\nu} \partial x'^{\lambda}} \frac{\partial x'^{\nu}}{\partial x^{\alpha}} \frac{\partial x'^{\lambda}}{\partial x^{\beta}} dx^{\alpha} dx^{\beta}$$

which, compared with

$$\delta(dx^{\mu}) = -\Gamma^{\mu}_{\alpha\beta} dx^{\alpha} dx^{\beta}$$

gives

$$\Gamma^{\mu}_{\alpha\beta}(x) = -\frac{\partial^2 x^{\mu}}{\partial x'^{\nu} \partial x'^{\lambda}} \frac{\partial x'^{\nu}}{\partial x^{\alpha}} \frac{\partial x'^{\lambda}}{\partial x^{\beta}}$$

which makes the equality (III,15) obvious. The essential point here is that the choice of locally minkowskian frame is possible everywhere and that the equality (III,15) can thus always be satisfied.

The Christoffel symbols - which, as it will be shown, do not form a tensor - define an affine connexion around the point x.

Given the metric tensor $g_{\alpha\beta}(x)$ (and the reader will easily show that this is indeed a tensor) , one can associate a contravariant tensor $g^{\mu\nu}(x)$ such that for all x :

$$g^{\mu\nu}(x)\, g_{\nu\alpha}(x) = \delta^{\mu}_{\alpha} \qquad \text{(III,17)}$$

Its components are given by :

$$g^{\mu\nu}(x) = \frac{\Delta^{\mu\nu}}{g} \qquad \text{(III,18)}$$

where g is the determinant of the $g_{\alpha\beta}$'s and $\Delta^{\alpha\beta}$ is the cofactor of element $g_{\alpha\beta}$. The development of g according to the element of a line α_o :

$$g = \sum_{\beta} g_{\alpha_o\beta} \, \Delta^{\alpha_o\beta}$$

gives

$$\frac{\partial g}{\partial g_{\alpha\beta}} = \Delta^{\alpha\beta}$$

since $\Delta^{\alpha_o\beta}$ does not contain $g_{\alpha_o\beta}$. Therefore :

$$g^{\mu\nu}(x) = \frac{1}{g}\,\frac{\partial g}{\partial g_{\mu\nu}}$$

From these relations and (III,17) one obtains :

$$dg = \frac{\partial g}{\partial g_{\mu\nu}}\, dg_{\mu\nu} = \Delta^{\mu\nu}\, dg_{\mu\nu} = g\, g^{\mu\nu}\, dg_{\mu\nu} = - g\, g_{\mu\nu}\, dg^{\mu\nu} \;;$$

$$\frac{\partial g}{\partial x^{\alpha}} = g\, g^{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} = - g\, g_{\mu\nu}\,\frac{\partial g^{\mu\nu}}{\partial x^{\alpha}} \qquad \text{(III,19)}$$

The tensors $g_{\mu\nu}$ and $g^{\mu\nu}$ are instrumental in the transition between contravariant and covariant vectors :

$$F^{\mu}(x) = g^{\mu\nu}(x)\, F_{\nu}(x)\;,$$

$$F_{\alpha}(x) = g_{\alpha\beta}(x)\, F^{\beta}(x)$$

and for the raising and lowering of indices.

III.5) Transformation laws of the Christoffel symbols.

The Christoffel symbols can be expressed in terms of the metric tensor and its first derivatives. By definition , given a vector $F^{\mu}(x)$, the parallel displacement is such that :

$$\delta(F^{\alpha}(x)\, F_{\alpha}(x)) = 0$$

and this means that the following equation holds :

$$g_{\alpha\beta}(x + dx)\, \overline{F}^{\alpha}(x + dx)\, \overline{F}^{\beta}(x + dx) - g_{\alpha\beta}(x)\, F^{\alpha}(x)\, F^{\beta}(x) = 0$$

where $\overline{F}^{\alpha}(x + dx)$ is the vector obtained from $F^{\alpha}(x)$ by parallel displacement along dx^{λ} :

$$\overline{F}^{\alpha}(x + dx) = F^{\alpha}(x) - \Gamma^{\alpha}_{\mu\nu} F^{\mu}(x)\, dx^{\nu}$$

From this relation and from (up to terms in dx) :

$$g_{\alpha\beta}(x + dx) = g_{\alpha\beta}(x) + \frac{\partial g_{\alpha\beta}}{\partial x^{\lambda}}\, dx^{\lambda}$$

one gets :

$$\frac{\partial g_{\alpha\beta}}{\partial x^{\lambda}} - g_{\alpha\eta}\Gamma^{\eta}_{\beta\lambda} - g_{\eta\beta}\Gamma^{\eta}_{\alpha\lambda} = 0 \qquad \text{(III,20)}$$

Let us use the symmetry of the Γ's with respect to their lower indices. We have (interchange α and λ then β and λ):

$$\frac{\partial g_{\lambda\beta}}{\partial x^{\alpha}} - g_{\lambda\eta}\Gamma^{\eta}_{\beta\alpha} - g_{\eta\beta}\Gamma^{\eta}_{\alpha\lambda} = 0 ,$$

$$\frac{\partial g_{\alpha\lambda}}{\partial x^{\beta}} - g_{\alpha\eta}\Gamma^{\eta}_{\beta\lambda} - g_{\eta\lambda}\Gamma^{\eta}_{\alpha\beta} = 0 .$$

The sum of the last two equations gives :

$$\frac{\partial g_{\lambda\beta}}{\partial x^{\alpha}} + \frac{\partial g_{\alpha\lambda}}{\partial x^{\beta}} - g_{\alpha\eta}\Gamma^{\eta}_{\beta\lambda} - g_{\eta\beta}\Gamma^{\eta}_{\alpha\lambda} - 2\, g_{\lambda\eta}\Gamma^{\eta}_{\alpha\beta} = 0$$

whence , in view of (III,20) :

$$\frac{\partial g_{\lambda\beta}}{\partial x^{\alpha}} + \frac{\partial g_{\alpha\lambda}}{\partial x^{\beta}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\lambda}} - 2\, g_{\lambda\eta}\Gamma^{\eta}_{\alpha\beta} = 0$$

Therefore (see (III,17)) :

$$\Gamma^{\nu}_{\alpha\beta}(x) = \frac{1}{2}\, g^{\nu\lambda}(x) \left(\frac{\partial g_{\lambda\beta}}{\partial x^{\alpha}} + \frac{\partial g_{\alpha\lambda}}{\partial x^{\beta}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\lambda}} \right) \qquad \text{(III,21)}$$

which is searched-for expression of the Christoffel symbols.

The corresponding all-lower-indices symbols are :

$$\Gamma_{\xi,\alpha\beta} = g_{\xi\nu}\Gamma^{\nu}_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial g_{\xi\beta}}{\partial x^{\alpha}} + \frac{\partial g_{\alpha\xi}}{\partial x^{\beta}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\xi}}\right) \qquad \text{(III,22)}$$

From these expressions and from the transformation laws of the tensors $g_{\alpha\beta}$, the reader will be able to obtain the transformation formulae for the Γ's

$$\Gamma'^{\nu}_{\alpha\beta}(x') = \frac{\partial x'^{\nu}}{\partial x^{\lambda}}\Gamma^{\lambda}_{\eta\varepsilon}(x)\frac{\partial x^{\eta}}{\partial x'^{\alpha}}\frac{\partial x^{\varepsilon}}{\partial x'^{\beta}} - \frac{\partial^2 x'^{\nu}}{\partial x^{\eta}\partial x^{\varepsilon}}\frac{\partial x^{\eta}}{\partial x'^{\alpha}}\frac{\partial x^{\varepsilon}}{\partial x'^{\beta}} \qquad \text{(III,23)}$$

<u>The Christoffel symbols</u> - because of the last term at the right hand side - <u>are not a tensor</u>.

If the tensor vanishes at a point in a given coordinate system, it will clearly vanish at the transformed point in all coordinate systems. For the Christoffel symbols, on the contrary, it is always possible to choose a coordinate system (up to a linear transformation) referred to which $\Gamma^{\nu}_{\alpha\beta}$ vanishes locally. Indeed, let x^{μ}_{o} be a point in a given frame for which $\Gamma^{\nu}_{\alpha\beta}(x_o) \neq 0$. If one carries out the following transformation :

$$x'^{\nu} = x^{\nu} - x^{\nu}_{o} + \frac{1}{2}\Gamma^{\nu}_{\eta\varepsilon}(x_o)(x^{\eta} - x^{\eta}_{o})(x^{\varepsilon} - x^{\varepsilon}_{o})$$

one gets :

$$\frac{\partial x'^{\nu}}{\partial x^{\lambda}} = \delta^{\nu}_{\lambda} + \Gamma^{\nu}_{\lambda\varepsilon}(x_o)(x^{\varepsilon} - x^{\varepsilon}_{o}) \quad ; \quad \delta^{\nu}_{\lambda} = \frac{\partial x^{\nu}}{\partial x'^{\lambda}} + \Gamma^{\nu}_{\eta\varepsilon}(x_o)(x^{\eta} - x^{\eta}_{o})\frac{\partial x^{\varepsilon}}{\partial x'^{\lambda}}$$

and

$$\frac{\partial^2 x'^{\nu}}{\partial x^{\eta}\partial x^{\varepsilon}} = \Gamma^{\nu}_{\eta\varepsilon}(x_o)$$

therefore, since :

$$\left(\frac{\partial x'^{\nu}}{\partial x^{\lambda}}\right)_o = \delta^{\nu}_{\lambda} \quad ; \quad \left(\frac{\partial x^{\eta}}{\partial x'^{\alpha}}\right)_o = \delta^{\eta}_{\alpha}$$

the transformed of $\Gamma^{\nu}_{\alpha\beta}(x_o)$, according to (III,23) is :

$$\Gamma'_{\alpha\beta} = \delta^{\nu}_{\lambda}\,\Gamma^{\lambda}_{\eta\varepsilon}(x_o)\,\delta^{\eta}_{\alpha}\,\delta^{\varepsilon}_{\beta} - \Gamma^{\nu}_{\eta\varepsilon}(x_o)\,\delta^{\eta}_{\alpha}\,\delta^{\varepsilon}_{\beta} = 0 \qquad \text{(III;23a)}$$

What is the meaning of this system, the so-called geodesic syste[m]

In a Riemannian space it is natural to define a geodesic line as the one for which the variation of the arc lenght between two points vanishes :

$$\delta \int_{s_o}^{s_1} (g_{\mu\nu}(z) \frac{dz^\mu}{ds} \frac{dz^\nu}{ds})^{1/2} ds = 0$$

where s is the parameter of the curve $z^\mu = z^\mu(s)$.

The equation thus obtained is :

$$\frac{du^\mu}{ds} + \Gamma^{\mu}_{\alpha\beta}(z)\, u^\alpha u^\beta = 0$$

where $u^\alpha = \frac{dz^\alpha}{ds}$. Einstein interpreted this as the equation of a particle in a gravitational field; the particle thus describes a geodesic in space-time the structure of which is Riemannian, due to this field. Therefore the force acting on the particle is $- m_o c^2 \Gamma^{\mu}_{\alpha\beta}(z)\, u^\alpha u^\beta$. The local vanishing of Γ in the geodesic system means that one changes into a new frame (the free-falling Einstein's elevator) within which the gravitational force has been locally transformed away. Only if the gravitational field is uniform can it be transformed away completely. In general, it will only be possible to transform it away in the neighborhood of a point in space-time.

III.6) Covariant derivatives.

Let $F^\alpha(x)$ be a vector field. At a neighboring point x + dx, this vector will be :

$$F^\alpha(x + dx) = F^\alpha(x) + d\, F^\alpha(x) \qquad \text{(III;24)}$$

At this point the vector obtained by parallel displacement of F^α from the point x is :

$$\bar{F}^{\alpha}(x + dx) = F^\alpha(x) - \Gamma^{\alpha}_{\mu\nu} F^\mu(x)\, dx^\nu \qquad \text{(III,24a)}$$

The difference between the two vectors (III,24) and (III,24a) is again a vector since they are taken at the same point.

$$F^\alpha(x + dx) - \bar{F}(x + dx) = d\, F^\alpha(x) + \Gamma^{\alpha}_{\mu\nu}(x)\, F^\mu(x)\, dx^\nu =$$

$$= (\frac{\partial F^\alpha}{\partial x^\nu} + \Gamma^{\alpha}_{\mu\nu} F^\mu)\, dx^\nu$$

The covariant derivative of a vector, by definition

$$F^{\alpha}_{\ ;\lambda} = F^{\alpha}_{\ ,\lambda} + \Gamma^{\alpha}_{\mu\nu} F^{\mu} \qquad \text{(III,25)}$$

is a tensor.

From the observation that the scalar product $F^{\alpha}(x)\ G_{\alpha}(x)$ is a scalar and that, therefore, its derivatives form a vector $A_{\lambda}(x)$:

$$A_{\lambda}(x) = F^{\alpha}_{\ ,\lambda}(x)\ G_{\alpha}(x) + F^{\alpha}(x)\ G_{\alpha,\lambda}(x)$$

one obtains the new vector :

$$A_{\lambda}(x) - F^{\alpha}_{\ ;\lambda}(x)\ G_{\alpha}(x) = F^{\mu}(x)\left[G_{\mu,\lambda}(x) - \Gamma^{\alpha}_{\mu\lambda}(x)\ G_{\alpha}(x)\right]$$

hence the covariant derivative of a covariant vector is the tensor :

$$G_{\mu;\lambda}(x) = G_{\mu,\lambda}(x) - \Gamma^{\alpha}_{\mu\lambda}(x)\ G_{\alpha}(x).$$

Again, given a second-rank covariant tensor $T_{\alpha\beta}(x)$ and two arbitrary vectors $A^{\alpha}(x)$, $B^{\beta}(x)$, at the same point, the product $T_{\alpha\beta}(x)\ A^{\alpha}(x)\ B^{\beta}(x)$ is a scalar, therefore :

$$\delta(T_{\alpha\beta}(x)\ A^{\alpha}(x)\ B^{\beta}(x)) = 0$$

whence, since A^{α} and B^{β} are arbitrary, in view of (III,14) :

$$\delta\ T_{\alpha\beta}(x) = (\Gamma^{\lambda}_{\alpha\eta} T_{\lambda\beta} + \Gamma^{\lambda}_{\beta\eta} T_{\alpha\lambda})\ dx^{\eta}$$

Now the difference between the two tensors :

$$T_{\alpha\beta}(x + dx) - \overline{T}_{\alpha\beta}(x + dx) = d\ T_{\alpha\beta}(x) - \delta\ T_{\alpha\beta}(x)$$

is again a tensor. The corresponding covariant derivative is thus :

$$T_{\alpha\beta;\eta} = T_{\alpha\beta,\eta} - \Gamma^{\lambda}_{\alpha\eta} T_{\lambda\beta} - \Gamma^{\lambda}_{\beta\eta} T_{\alpha\lambda}$$

By similar procedures one finds :

$$T^{\alpha\beta}_{\ \ ;\eta} = T^{\alpha\beta}_{\ \ ,\eta} + \Gamma^{\alpha}_{\lambda\eta} T^{\lambda\beta} + \Gamma^{\beta}_{\lambda\eta} T^{\alpha\lambda}$$

and, in general :

$$T^{\mu\nu\ldots}_{\alpha\beta\ldots;\lambda} = T^{\mu\nu\ldots}_{\alpha\beta\ldots;\lambda} + \Gamma^{\mu}_{a\lambda} T^{a\nu\ldots}_{\alpha\beta\ldots} + \Gamma^{\nu}_{a\lambda} T^{\mu a\ldots}_{\alpha\beta\ldots} + \ldots - \Gamma^{m}_{\alpha\lambda} T^{\mu\nu\ldots}_{m\beta\ldots} - \Gamma^{m}_{\beta\lambda} T^{\mu\nu\ldots}_{\alpha m\ldots} - \ldots \qquad \text{(III,26)}$$

The covariant differentiation depends therefore on the rank of the tensor which is going to be applied to. Thus we can write :

$$F^{\alpha}_{\ ;\lambda} = D^{\alpha}_{m\lambda} F^{m}(x)$$

where

$$D^{\alpha}_{m\lambda} = \frac{\partial}{\partial x^{\lambda}} \delta^{\alpha}_{m} + \Gamma^{\alpha}_{m\lambda} \qquad \text{(III,27a)}$$

Also :

$$T^{\alpha\beta}_{\ \ ;\lambda} = D^{\alpha\beta}_{mn\lambda} T^{mn}(x)$$

with

$$D^{\alpha\beta}_{mn\lambda} = \frac{\partial}{\partial x^{\lambda}} \delta^{\alpha}_{m} \delta^{\beta}_{n} + \Gamma^{\alpha}_{m\lambda} \delta^{\beta}_{n} + \Gamma^{\beta}_{n\lambda} \delta^{\alpha}_{m} \qquad \text{(III,27b)}$$

and so on.

Clearly the covariant divergence of a vector is the operator :

$$D^{\alpha}_{m\alpha} = \frac{\partial}{\partial x^{m}} + \Gamma^{\alpha}_{m\alpha}$$

to be applied to F^{m} and summed over m.

The covariant derivative of the fundamental metric tensor vanishes identically (see (III,20)):

$$g^{\alpha\beta}_{\ \ ;\lambda} = g^{\alpha\beta}_{\ \ ,\lambda} + \Gamma^{\alpha}_{\lambda\eta} g^{\eta\beta} + \Gamma^{\beta}_{\lambda\eta} g^{\alpha\eta} = 0$$

$$g_{\alpha\beta;\lambda} = g_{\alpha\beta,\lambda} - \Gamma^{\eta}_{\alpha\lambda} g_{\eta\beta} - \Gamma^{\eta}_{\beta\lambda} g_{\alpha} = 0 \qquad \text{(III,28)}$$

From this one is able to establish that :

$$\Gamma^{\alpha}_{m\alpha} = \frac{\partial}{\partial x^{m}} \log \sqrt{-g}$$

where $g = \det(g_{\alpha\beta})$, and hence :

$$F^{\alpha}_{\ ;\alpha} = \left(\frac{\partial}{\partial x^{m}} + \frac{\partial \log \sqrt{-g}}{\partial x^{m}} \right) F^{m}(x) = \frac{1}{\sqrt{-g}} (F^{m}(x) \sqrt{-g})_{,m} \qquad \text{(III,29)}$$

In the presence of a gravitational field the charge is :

$$Q = \int \sqrt{-g}\; j^{\lambda}(x)\, d\,\sigma_{\lambda}.$$

III.7) The Riemann tensor.

A space is flat when it is possible to carry out a coordinate transformation such that the metric tensor be identical to the Lorentz metric tensor (II,10) everywhere in this space. In it one can make a parallel displacement of any vector throughout and get a constant vector at all points. The ordinary derivative operator is then a vector and two of these operators commute. In a curved space these properties are not valid.

Let us consider the covariant derivative (III,25) of a vector in a curved space, which is a tensor T^{μ}_{α} :

$$T^{\mu}_{\alpha}(x) \equiv F^{\mu}{}_{;\alpha}(x) = F^{\mu}{}_{,\alpha}(x) + \Gamma^{\mu}_{\alpha m}(x)\, F^{m}(x)$$

If we calculate the covariant derivative of this tensor we have (according to (III,26)):

$$F^{\mu}{}_{;\alpha;\lambda} = T^{\mu}_{\alpha;\lambda} = T^{\mu}_{\alpha,\lambda} + \Gamma^{\mu}_{a\lambda}\, T^{a}_{\alpha} - \Gamma^{m}_{\alpha\lambda}\, T^{\mu}_{m} =$$

$$= F^{\mu}{}_{,\alpha,\lambda} + (\Gamma^{\mu}_{\alpha\eta})_{,\lambda}\, F^{\eta} + \Gamma^{\mu}_{\alpha\eta}\, F^{\eta}{}_{,\lambda} + \Gamma^{\mu}_{a\lambda}\, F^{a}{}_{,\alpha} - \Gamma^{m}_{\alpha\lambda}\, F^{\mu}{}_{,m} +$$

$$+ \Gamma^{\mu}_{a\lambda}\, \Gamma^{a}_{\alpha\eta}\, F^{\eta} - \Gamma^{m}_{\alpha\lambda}\, \Gamma^{\mu}_{m\eta}\, F^{\eta}$$

Let us now exchange the indices α and λ and subtract the expression thus obtained from the above one :

$$F^{\mu}{}_{;\alpha;\lambda} - F^{\mu}{}_{;\lambda;\alpha} = \left[(\Gamma^{\mu}_{\alpha\eta})_{,\lambda} - (\Gamma^{\mu}_{\lambda\eta})_{,\alpha} + \Gamma^{\mu}_{a\lambda}\Gamma^{a}_{\alpha\eta} - \Gamma^{\mu}_{a\alpha}\Gamma^{a}_{\lambda\eta} \right] F^{\eta}$$

In terms of the operators introduced previously, (III,27), we have :

$$(D^{\mu m}_{a\alpha\lambda} - D^{\mu m}_{a\lambda\alpha})\, D^{a}_{\eta m} = R^{\mu}_{\eta\alpha\lambda}$$

where

$$R^{\mu}_{\eta\alpha\lambda} = (\Gamma^{\mu}_{\alpha\eta})_{,\lambda} - (\Gamma^{\mu}_{\lambda\eta})_{,\alpha} + \Gamma^{\mu}_{a\lambda}\, \Gamma^{a}_{\alpha\eta} - \Gamma^{\mu}_{a\alpha}\, \Gamma^{a}_{\lambda\eta} \qquad \text{(III,30)}$$

is the Riemann curvature tensor.

A flat space by definition is such that :

$$R^{\mu}_{\eta\alpha\lambda}(x) = 0$$

everywhere. Clearly a space with a Lorentz metric has vanishing Γ-symbols and is therefore flat.

We list the following properties of the Riemann tensor :

$$R^{\mu}_{\eta\alpha\lambda} = - R^{\mu}_{\eta\lambda\alpha} \quad ;$$

$$R_{\mu\alpha\beta\gamma} = g_{\mu\eta} R^{\eta}_{\alpha\beta\gamma} = - R_{\alpha\mu\beta\gamma} = - R_{\mu\alpha\gamma\beta} = R_{\beta\gamma\mu\alpha} \quad ;$$

$$R_{1023} + R_{2031} + R_{3012} = 0$$

Out of the $4^4 = 256$ components of $R_{\mu\alpha\beta\gamma}$ only 20 are independent, in view of these symmetry properties (as R is antisymmetric in β and γ, these indices, for fixed μ, α, give rise to only 6 independent components instead of 16; similarly for the indices μ,α, for fixed β,γ; hence one would have 36 components; the symmetry with respect to the intercharge of $\mu\ \alpha$ with $\beta\ \gamma$ reduces these to 21)

III.8) Bianchi identities and the Ricci-Einstein tensor.

In a geodesic coordinate system, the covariant derivative of $R^{\mu}_{\alpha\beta\gamma}$ reduces to :

$$R^{\mu}_{\alpha\beta\gamma;\nu} = (\Gamma^{\mu}_{\alpha\beta})_{,\gamma,\nu} - (\Gamma^{\mu}_{\alpha\gamma})_{,\beta,\nu}$$

Therefore in this system the following relationship holds :

$$R^{\mu}_{\alpha\beta\gamma;\nu} + R^{\mu}_{\alpha\nu\beta;\gamma} + R^{\mu}_{\alpha\gamma\nu;\beta} = 0 \qquad \text{(III,31)}$$

These are Bianchi's identities : since the left-hand side is a tensor, the components of which vanish in the geodesic system, the relations hold in any coordinate system.

From the Riemann tensor one can form by contraction a second rank tensor:

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} \qquad \text{(III,32)}$$

This is the only tensor to be constructed from the Riemann tensor since :

$$R^{\alpha}_{\alpha\mu\nu} = g^{\beta\lambda} R_{\beta\lambda\mu\nu} = g^{\lambda\beta} R_{\lambda\beta\mu\nu} = - g^{\beta\lambda} R_{\beta\lambda\mu\nu} = 0$$

and

$$R^{\alpha}_{\mu\nu\alpha} = - R^{\alpha}_{\mu\alpha\nu} = - R_{\mu\nu} \qquad \text{(III,32a)}$$

The explicit form of $R_{\mu\nu}$ is, according to (III,30),(III,32) :

$$R_{\mu\nu} = (\Gamma^{\alpha}_{\alpha\mu})_{,\nu} - (\Gamma^{\alpha}_{\mu\nu})_{,\alpha} + \Gamma^{\alpha}_{a\nu}\Gamma^{a}_{\alpha\mu} - \Gamma^{\alpha}_{a\alpha}\Gamma^{a}_{\mu\nu} \qquad \text{(III,33)}$$

and it is seen that it is symmetric :

$$R_{\mu\nu} = R_{\nu\mu} .$$

The <u>scalar curvature</u> is now defined as the contraction of R^{μ}_{ν} :

$$R = g^{\mu\nu} R_{\mu\nu} \qquad \text{(III,34)}$$

The Ricci-Einstein tensor, G^{μ}_{α}, is one the covariant of which vanishes :

$$G^{\mu}_{\alpha;\mu} = 0 \qquad \text{(III,35)}$$

From the Bianchi identities (III,31) we get :

$$R^{\mu\alpha}{}_{\beta\gamma;\nu} + R^{\mu\alpha}{}_{\nu\beta;\gamma} + R^{\mu\alpha}{}_{\gamma\nu;\beta} = 0$$

and the contraction of the indices μ and β gives :

$$R^{\mu\alpha}{}_{\mu\gamma;\nu} + R^{\mu\alpha}{}_{\nu\mu;\gamma} + R^{\mu\alpha}{}_{\gamma\nu;\mu} = 0$$

Now in view of the definition (III,32) and the symmetry properties (III,32a) one has, by contraction of α and γ :

$$R_{;\nu} - R^{\alpha}_{\nu;\alpha} - R^{\mu}_{\nu;\mu} = 0$$

that is :

$$R_{;\nu} = 2 R^{\alpha}_{\nu;\alpha}$$

The Einstein-Ricci tensor , which satisfies the equation (III,3) is therefore :

$$G^{\alpha}_{\beta} = R^{\alpha}_{\beta} - \frac{1}{2} \delta^{\alpha}_{\beta} R$$

or

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \qquad \text{(III,36)}$$

III.9) Einstein's gravitational field equations.

It is seen, in view of the definitions (III,36),(III,34),(III,33) and the expression (III,21), that $G_{\alpha\beta}$ contains, besides non-linear combinations of $g_{\mu\nu}$ and $g_{\mu\nu,\lambda}$ second derivatives of the metric tensor (or gravitational potential field) $g_{\mu\nu}(x)$ (linearly). If the energy momentum tensor of matter, $T_{\mu\nu}$, must obey the covariant equation which generalises equation (II,41a) into a Riemannian space :

$$T^{\mu}_{\nu;\mu} = 0$$

then, in view of the relation (III,35) it is plausible to postulate that the gravitational field equations are of the form :

$$G^{\alpha}_{\beta} = K \; T^{\alpha}_{\beta} \qquad \text{(III,36a)}$$

where K is a coupling constant. This was the postulate proposed by Einstein in 1915 after several years of attempts at discovering the relativistic equations which may be regarded as a generalisation of Poisson's equation (III,1).

The reader is referred to the literature[7] if he is interested in the mathematical structure of Einstein's equations and their consequences. In the particular case of a time-independent and weak gravitational field (i.e. differing very little from the Lorentz metric tensor) the identification with Poisson's equation is achieved if one sets :

$$K = \frac{8\pi G}{c^2}$$

The energy-momentum tensor $T_{\mu\nu}$ contains the contribution of all matter and fields except the gravitational field. The equations of these fields, however, take into account the gravitational field, in the differentiation operators. Thus, the equations of a gravitational field produced by an electromagnetic radiation field (in the absence of charges) are :

$$G_{\mu\nu} = K \; T_{\mu\nu} \quad , \qquad \text{(III,37)}$$

where

$$T_{\mu\nu} = F_{\mu m} F^{m}_{\nu} + \frac{1}{4} g_{\mu\nu} F_{mn} F^{mn}$$

is the energy-momentum tensor of the radiation field determined by the equations :

$$(\sqrt{-g}\ F^{\mu\nu})_{,\nu} = 0 ,$$

$$F_{\mu\nu,\lambda} + F_{\lambda\mu,\nu} + F_{\nu\lambda,\mu} = 0 , \qquad F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$$

In this case, since $T_{\mu\nu}$ is traceless

$$T \equiv T_{\mu}^{\ \mu} = 0 ,$$

Einstein's equations, which have the equivalent form :

$$R_{\mu\nu} = K (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T),$$

reduce to

$$R_{\mu\nu} = K\ T_{\mu\nu} .$$

III.10) The gravitational field energy-momentum pseudo-tensor. Einstein's variational principle.

The extension of the equation (II,41a) to a Riemannian space is :

$$T^{\mu\nu}_{\ \ ;\nu} = 0$$

and this is imposed by Einstein's equations. We have, according to (III,26):

$$T^{\mu\nu}_{\ \ ,\nu} + \Gamma^{\mu}_{\lambda\eta} T^{\lambda\eta} + \Gamma^{\eta}_{\lambda\eta} T^{\mu\lambda} = 0$$

or

$$T^{\nu}_{\alpha,\nu} + \Gamma^{\eta}_{\lambda\eta} T_{\alpha}^{\ \lambda} - \Gamma^{\lambda}_{\alpha\eta} T_{\lambda}^{\ \eta} = 0 \qquad \text{(III,38)}$$

Therefore we cannot define a conserved quantity of the form (II,41b). The meaning of this relationship is the following : the tensor $T_{\mu\nu}$, source of the gravitational field, cannot be conserved alone; the gravitational field interacts with itself, since it gives rise to an additional energy which, in turn, contributes to this field. It is thus not unexpected that only a combination of the original $T_{\mu\nu}$ and of gravitational field quantities can be conserved.

The question then arises whether it is possible to separate this conserved quantity into two parts, one associated to the source, the other to the gravitational field. We are going to show, following mainly Adler et al[(7)], that it is possible to transform the equation (III,38) into one of the following type:

$$(\sqrt{-g}\, T_{\alpha}^{\ \nu} + \sqrt{-g}\ t_{\alpha}^{\cdot\nu})_{,\nu} = 0 \qquad \text{(III,39)}$$

which is of the usual form (II,41) and which allows us to construct a conserved object :

$$P_{\alpha} = \int d\sigma_{\nu} \sqrt{-g}\, (T_{\alpha}^{\ \nu} + t_{\alpha}^{\ \nu}). \qquad \text{(III,40)}$$

This object is however, not a vector since it turns out that the quantity $t_{\alpha}^{\ \nu}$ is not a tensor. Therefore the meaning attached to such an object, which depends on the coordinate system, is not quite clear.

From the expression (III,21) for the Christoffel symbols, one gets :

$$\Gamma^{\eta}_{\lambda\eta} = \frac{1}{2}\, g^{\nu\beta} \frac{\partial g_{\nu\beta}}{\partial x^{\lambda}}$$

since :

$$g^{\nu\beta}\left(\frac{\partial g_{\lambda\beta}}{\partial x^{\nu}} - \frac{\partial g_{\lambda\nu}}{\partial x^{\beta}}\right) = g^{\nu\beta}\frac{\partial g_{\lambda\beta}}{\partial x^{\nu}} - g^{\beta\nu}\frac{\partial g_{\lambda\beta}}{\partial x^{\nu}} = 0 .$$

Therefore, according to (III,19) :

$$\Gamma^{\eta}_{\lambda\eta} = \frac{1}{2g}\frac{\partial g}{\partial x^{\lambda}} = (\log\sqrt{-g})_{,\lambda} = \frac{(\sqrt{-g})_{,\lambda}}{\sqrt{-g}} \qquad \text{(III,41)}$$

On the other hand, as the tensor $T^{\mu\nu}$ is symmetric, we may write :

$$\Gamma^{\lambda}_{\alpha\eta} T_{\lambda}^{\ \eta} = \frac{1}{2}(\Gamma_{\lambda,\alpha\eta} + \Gamma_{\eta,\alpha\lambda})\, T^{\lambda\eta}$$

so, in view of equation (III,22):

$$\Gamma^{\lambda}_{\alpha\eta} T_{\lambda}^{\ \eta} = \frac{1}{2}\frac{\partial g_{\lambda\eta}}{\partial x^{\alpha}}\, T^{\lambda\eta}$$

Thus the equation (III,38) becomes :

$$T_{\alpha,\nu}^{\nu} + \frac{(\sqrt{-g})_{,\lambda}}{\sqrt{-g}}\, T_{\alpha}^{\cdot\lambda} - \frac{1}{2}\, g_{\lambda\eta,\alpha} T^{\lambda\eta} = 0$$

or :

$$\frac{1}{\sqrt{-g}}(T_\alpha^\nu \sqrt{-g})_{,\nu} - \frac{1}{2} g_{\lambda\eta,\alpha} T^{\lambda\eta} = 0$$

Einstein's equations allow us to write :

$$(T_\alpha^\nu \sqrt{-g})_{,\nu} - \frac{1}{2} g_{\lambda\eta,\alpha}\sqrt{-g} \frac{1}{K}(R^{\lambda\eta} - \frac{1}{2} g^{\lambda\eta} R) = 0$$

It is seen that the problem of establishing equation (III,39) reduces to finding a quantity t_α^ν such that :

$$(\sqrt{-g}\, t_\alpha^\nu)_{,\nu} = -\frac{1}{2} g_{\lambda\eta,\alpha} \sqrt{-g} \frac{1}{K} G^{\lambda\eta} \qquad \text{(III,42)}$$

Let us then consider the Einstein's gravitational action S_g:

$$S_g = \int_\rho R \sqrt{-g}\, d^4x \ , \qquad R = g^{\mu\nu} R_{\mu\nu}, \qquad \text{(III,43)}$$

where the integration is taken over a certain domain ρ of four-dimensional space. R, the scalar curvature, is invariant; $\sqrt{-g}\, d^4x$ is another invariant, hence S_g is a scalar functional.

The variation of S_g with respect to a variation of the field $g_{\mu\nu}$ such that

$$\delta g_{\mu\nu} = \delta g_{\mu\nu,\lambda} = 0 \quad \text{on the boundary of } \rho$$

is :

$$\delta S_g = \int \left[\sqrt{-g}\; g^{\mu\nu} \delta R_{\mu\nu} + g^{\mu\nu} R_{\mu\nu} \delta(\sqrt{-g}) + R_{\mu\nu}\sqrt{-g}\; \delta g^{\mu\nu}\right] d^4x \ .$$

In a geodesic coordinate system, according to (III,33),(III,23a) :

$$\delta R_{\mu\nu} = \delta(\Gamma^\alpha_{\alpha\mu})_{,\nu} - \delta(\Gamma^\alpha_{\mu\nu})_{,\alpha}$$

In this system, the ordinary derivative coïncides with the covariant derivative, hence :

$$\delta R_{\mu\nu} = \delta(\Gamma^\alpha_{\alpha\mu})_{;\nu} - \delta(\Gamma^\alpha_{\mu\nu})_{;\alpha}$$

Since both sides of this equation are tensors (the reader will show that although $\Gamma^\lambda_{\alpha\beta}$ is not a tensor, $\delta\Gamma^\lambda_{\alpha\beta}$ <u>is a tensor</u>) the equation is generally valid.

On the other hand, the relations (III,17),(III,18) lead us to write:

$$\delta(\sqrt{-g}) = -\frac{1}{2}(\sqrt{-g})\, g_{\mu\nu}\delta g^{\mu\nu},$$

Therefore:

$$\delta S_g = \int_\rho \sqrt{-g}\left[\delta(\Gamma^\alpha_{\alpha\mu})_{;\nu} - \delta(\Gamma^\alpha_{\mu\nu})_{;\alpha}\right] g^{\mu\nu} d^4x +$$

$$+ \int_\rho G_{\mu\nu}\sqrt{-g}\;\delta g^{\mu\nu} d^4x .$$

The first integral can be transformed into an integrral over the boundary of ρ. Indeed, let us set :

$$g^{\mu\nu}\left[\delta(\Gamma^\alpha_{\alpha\mu})_{;\nu} - \delta(\Gamma^\alpha_{\mu\nu})_{;\alpha}\right] = (g^{\mu\nu}\delta\Gamma^\alpha_{\alpha\mu})_{;\nu} - (g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu})_{;\alpha}$$

since $g^{\mu\nu}_{\ \ ;\alpha} = 0$. Now, as $\delta\Gamma^\lambda_{\alpha\beta}$ is a tensor we can write :

$$(g^{\mu\nu}\delta\Gamma^\alpha_{\alpha\mu})_{;\nu} - (g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu})_{;\alpha} = A^\nu_{\ ;\nu} - B^\alpha_{\ ;\alpha}$$

where A^ν and B^ν are two vectors. Therefore, according to the divergence formula (III,29):

$$\int_\rho \sqrt{-g}\left[\delta(\Gamma^\alpha_{\alpha\mu})_{;\nu} - \delta(\Gamma^\alpha_{\mu\nu})_{;\alpha}\right] g^{\mu\nu} d^4x = \int_{\text{boundary}} \sqrt{-g}\left[A^\nu - B^\nu\right] d\sigma_\nu .$$

A^ν and B^ν, however, vanish on the boundary of ρ because $\delta g^{\mu\nu}$ and $\delta\Gamma^\lambda_{\alpha\beta}$ vanish there. We are thus left with :

$$\delta S_g = \int_\rho G_{\mu\nu}\sqrt{-g}\,\delta g^{\mu\nu} d^4x$$

whence :

$$\frac{\delta S_g}{\delta g^{\mu\nu}} = \sqrt{-g}\; G_{\mu\nu} \qquad \text{(III,44)}$$

This result shows that if one can construct a lagragean density $L_m\sqrt{-g}$ depending upon the gravitational field $g_{\mu\nu}$, the matter and other fields, and such that the variation of the action :

$$S_m = \int_\rho L_m \sqrt{-g}\; d^4x \qquad \text{(III,45)}$$

be equal to

$$\delta S_m = -K\int_\rho T_{\mu\nu}\sqrt{-g}\;\delta g^{\mu\nu} d^4x ,$$

then <u>Einstein's equations</u> (III,36a) <u>are deducible from a variational principle</u>:

$$\delta S = 0$$

where

$$S = S_g + S_m$$

according to (III,43) and (III,45) and $\delta g^{\mu\nu} = 0$, $\delta g^{\mu\nu}_{,\lambda} = 0$ over the boundary of ρ.

Let us now go back to the construction of an object t^{ν}_{α} satisfying equation (III,39). We first split the scalar density $R\sqrt{-g}$, equ.(III,34), into a sum of terms which contain first derivatives of $g_{\mu\nu}$ at most and another term which is the sum of divergences. The idea is, by transforming away these divergences, to reduce $R\sqrt{-g}$ to another form $\mathcal{L}$ which contains at most first derivatives in $g_{\mu\nu}$. This is so because only then will the Lagrange equations yield field equations containing second derivatives of $g_{\mu\nu}(x)$ at most, as postulated by Einstein. In view of the identity:

$$\sqrt{-g}\; g^{\mu\nu}\left[(\Gamma^{\alpha}_{\alpha\mu})_{,\nu} - (\Gamma^{\alpha}_{\mu\nu})_{,\alpha}\right] = (\sqrt{-g}\; g^{\mu\nu}\Gamma^{\alpha}_{\alpha\mu})_{,\nu} - (\sqrt{-g}\; g^{\mu\nu}\Gamma^{\alpha}_{\mu\nu})_{,\alpha} +$$

$$+ (\sqrt{-g}\; g^{\mu\nu})_{,\alpha}\Gamma^{\alpha}_{\mu\nu} - (\sqrt{-g}\; g^{\mu\nu})_{,\nu}\Gamma^{\alpha}_{\alpha\mu}$$

One may write , in view of (III,33) and (III,34):

$$R\sqrt{-g} = (\sqrt{-g}\; g^{\mu\nu}\Gamma^{\alpha}_{\alpha\mu})_{,\nu} - (\sqrt{-g}\; g^{\mu\nu}\Gamma^{\alpha}_{\mu\nu})_{,\alpha} + \mathcal{L}$$

where

$$\mathcal{L} = \sqrt{-g}\; g^{\mu\nu}\left[\Gamma^{\alpha}_{a\nu}\Gamma^{a}_{\alpha\mu} - \Gamma^{\alpha}_{a\alpha}\;\Gamma^{a}_{\mu\nu}\right] + (\sqrt{-g}\, g^{\mu\nu})_{,\alpha}\Gamma^{\alpha}_{\mu\nu} - (\sqrt{-g}\, g^{\mu\nu})_{,\nu}\Gamma^{\alpha}_{\alpha\mu}$$

can be transformed into (see (III,28) and (III,41)):

$$\mathcal{L} = \sqrt{-g}\; g^{\mu\nu}\left[\Gamma^{\alpha}_{\mu\nu}\Gamma^{m}_{m\alpha} - \Gamma^{m}_{\alpha\mu}\;\Gamma^{\alpha}_{m\nu}\right].$$

Therefore the action S_g,(III,43), transforms into :

$$S_g = \int_{\rho}\mathcal{L}\; d^4x + \int_{\text{bound.}}\sqrt{-g}\, g^{\mu\nu}\,\Gamma^{\alpha}_{\alpha\mu}\;\; d\sigma_{\nu} - \int_{\text{bound.}}\sqrt{-g}\, g^{\mu\nu}\Gamma^{\alpha}_{\mu\nu}\, d\sigma_{\alpha}$$

Thus, for variations such that $\delta g_{\mu\nu}$ and $\delta g_{\mu\nu,\lambda}$ vanish at the boundary, the variational derivative of S_g will be equal to that of the first term above :

$$E = \int_\rho \mathcal{L}\, d^4x \ ,$$

$$\frac{\delta S_g}{\delta g_{\mu\nu}} = \frac{\partial E}{\delta g_{\mu\nu}} \qquad \text{(III,46)}$$

The important point here is that $\mathcal{L}$ is not a scalar density.

Now :

$$\delta E = \int_\rho \Big[\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} \delta g^{\mu\nu} + \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}_{,\lambda}} \delta g^{\mu\nu}_{,\lambda} \Big] d^4x =$$

$$= \int_\rho \Big[\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \Big(\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}_{,\lambda}} \Big)_{,\lambda} \Big] \delta g^{\mu\nu}\, d^4x + \int_{\text{bound.}} \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}_{,\lambda}} \delta g^{\mu\nu} d^4x$$

and the last integral vanishes if $\delta g^{\mu\nu} = 0$ at the boundary. Therefore, in view of (III,44) and (III,46):

$$\sqrt{-g}\; G_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \Big(\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}_{,\lambda}} \Big)_{,\lambda}$$

Let us multiply both sides of this equation by $g^{\mu\nu}_{,\alpha}$ and remark that :

$$\mathcal{L}_{,\alpha} = \frac{\partial \mathcal{L}}{\partial g^{mn}} g^{mn}_{,\alpha} + \frac{\partial \mathcal{L}}{\partial g^{mn}_{,\eta}} g^{mn}_{,\eta,\alpha}$$

to get

$$g^{\mu\nu}_{,\alpha} \sqrt{-g}\; G_{\mu\nu} = \mathcal{L}_{,\alpha} - \Big(\frac{\partial \mathcal{L}}{\partial g^{mn}_{,\lambda}} g^{mn}_{,\alpha} \Big)_{,\lambda} = \Big(\mathcal{L} \delta^\lambda_\alpha - \frac{\partial \mathcal{L}}{\partial g^{mn}_{,\lambda}} g^{mn}_{,\alpha} \Big)_{,\lambda} \qquad \text{(III,47)}$$

Clearly, from the definition :

$$g^{\mu\nu} g_{\mu\eta} = \delta^\nu_\eta$$

one has

$$g^{\mu\nu}_{,\alpha} g_{\mu\eta} + g^{\mu\nu} g_{\mu\eta,\alpha} = 0$$

hence

$$\sqrt{-g}\, g^{\mu\nu}_{,\alpha} G_{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}_{,\alpha} g_{\mu m} g_{\nu n} G^{mn} = \sqrt{-g}\, g_{\mu m,\alpha} G^{\mu m} \qquad \text{(III,47a)}$$

Therefore, if this equation is compared with equations (III,42),(III,47),(III,47a) then one may define the quantity t^ν_α by means of the relation :

$$\sqrt{-g}\; t^{\nu}_\alpha = \frac{1}{2K} \Big(\mathcal{L}\, \delta^\nu_\alpha - \frac{\partial \mathcal{L}}{\partial g^{mn}_{,\nu}} g^{mn}_{,\alpha} \Big)$$

and this quantity - the so-called energy-momentum pseudo-tensor of the gravitational field - satisfies the conservation law (III,39).

Since $\mathcal{L}$ is not a scalar density, it follows that $t_\alpha{}^\nu$ is not a tensor. One thus understands that although it is possible to construct the object P_α given in (III,40), which is conserved, this object is not generally a four-vector and depends on the coordinate system.

Other quantities have been constructed by different authors (Landau - Lifshitz, Möller, etc.) and the object $t_\alpha{}^\nu$ is thus not uniquely defined.

Such difficulties still remain in an alternative field theory of gravitation(8) the basic equations of which are of the form :

$$\Box\, \varphi_{\mu\nu} = K(T_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} T^{\alpha}{}_{\alpha})$$

the symmetric tensor $\varphi_{\mu\nu}$ describing this field in a flat space. As shown by Thirring one is led in such a theory to 'renormalize' the metric and introduce a metric $g_{\mu\nu}(x)$ which depends on the field $\varphi_{\mu\nu}(x)$.

The fact that the gravitational field has an universal interaction not only with all other fields but also with itself leads to the non-linear effects which are essentially responsible for the difficulty in the construction of an energy-momentum tensor associated to the gravitational field. On the other hand, the fact that it is always possible, in Einstein's theory , to choose a geodesic system in an arbitrary point of space-time, means that locally the gravitational field is transformed away and that , therefore, at this point the energy and momentum of this field can be made to vanish. Therefore, the energy-momentum object cannot be a tensor and has to depend on the coordinate system.

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