



**CBPF - CENTRO BRASILEIRO DE PESQUISAS FÍSICAS**

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**Rio de Janeiro**

**Notas de Física**

**CBPF-NF-004/10**

**December 2010**

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of Fierz and Schur Type**

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# Extended Supersymmetric Quantum Mechanics of Fierz and Schur Type

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December 29, 2010

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## Abstract

We discuss two independent constructions to introduce an  $N$ -extended Supersymmetric Quantum Mechanics. The first one makes use of the Fierz identities while the second one (divided into two subcases) makes use of the Schur lemma. The  $N$  supercharges  $Q_I$  are square roots of a free Hamiltonian  $H$  given by the tensor product of a  $D$ -dimensional Laplacian and a  $2d$ -dimensional identity matrix operator. We present the mutual relations among  $N$ ,  $D$  and  $d$ . The *mod* 8 Bott's periodicity of Clifford algebras is encoded, in the Fierz case, in the Radon-Hurwitz function and, in the Schur case, in an extra independent function.

CBPF-NF-004/10

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# 1 Introduction

Since its introduction in [1] the Supersymmetric Quantum Mechanics has found deep applications to mathematics, mathematical physics, nuclear physics and so on. An important line of research focused on its extension to  $N$  independent supercharges corresponding to the square roots of the Hamiltonian operator (see [2, 3, 4, 5, 6, 7, 8, 9]). The connection between Extended Supersymmetry and Clifford algebras was pointed out in [2, 6, 7]. In several works [2, 4, 5, 8, 9] the Fierz identities [10] were used to close the  $N$ -extended superalgebra; in this construction the Hamiltonian can depend on a Kähler (for  $N = 2$ ) or hyper-Kähler (for  $N = 4$ ) background.

In this letter we show that, besides the Fierz construction (the  $F$ -type supersymmetry) another extension of the Supersymmetric Quantum Mechanics (denoted as  $S$ -type supersymmetry) can be based on the Schur lemma (see [11]), expressing the most general algebra (real, almost complex or quaternionic) which commutes with a given set of Clifford gamma matrices .

We consider the free Hamiltonian given by the tensor product of a  $D$ -dimensional Laplacian and a  $2d$ -dimensional identity matrix. We write down the mutual relations among  $N$ ,  $D$  and  $d$ . The *mod* 8 Bott's periodicity of Clifford algebras is encoded, in the Fierz case, in the Radon-Hurwitz  $G(r)$  function (see [6]) defined, for  $r = 1, 2, \dots, 8$ , by

$r$	1	2	3	4	5	6	7	8
$G(r)$	1	2	4	4	8	8	8	8

(1)

In the Schur case, besides the Radon-Hurwitz function, another function encodes the Bott's periodicity. It will be denoted as " $K(r)$ " and defined, for  $r = 0, 1, 2, \dots, 7$ , by

$r$	0	1	2	3	4	5	6	7
$K(r)$	1	2	4	4	4	2	1	1

(2)

$K(r)$  gives us, see [12, 13], the real (1), almost complex (2) and quaternionic (4) properties of the  $Cl(p, q)$  Clifford algebra according to  $p - q = r + 2 \text{ mod } 8$ .

The Schur case is divided into two subcases. It depends on two mutually commuting sets of gamma matrices. According to which set is picked up to express the Schur (real, almost complex, quaternionic) property we end up with either the subcase  $S1$  ( $D$  can be arbitrarily large, but  $N$  is at most  $N \leq 4$ ) or the subcase  $S2$  ( $N$  can be arbitrarily large, but  $D$  is at most  $\leq 3$ ).

We work with Euclidean gamma matrices. Their vectorial indices are raised and lowered with the identity matrix. Similarly, their spinorial indices are raised and lowered in terms of a charge conjugation matrix that we can choose to be the identity. This means that, as a practical rule, we do not need to bother about the position (up or down) of the indices.

## 2 $N$ -Extended Supersymmetric Quantum Mechanics for a free Hamiltonian

The free Schrödinger equation for a  $D$ -dimensional non-relativistic particle reads as

$$i\hbar \frac{d}{dt} \Psi(t, \vec{x}) = H \Psi(t, \vec{x}), \quad (3)$$

where  $H$  is the free Hamiltonian, given by a tensor product of a  $D$ -dimensional Laplacian  $\vec{\nabla}^2 = \partial_{x_1}^2 + \dots + \partial_{x_D}^2$  and a  $2d$ -dimensional identity matrix operator  $\mathbf{1}_{2d}$ :

$$H = -\frac{\hbar^2}{2m} \vec{\nabla}^2 \otimes \mathbf{1}_{2d}. \quad (4)$$

This free dynamical system provides a realization of the  $N$ -extended Supersymmetric Quantum Mechanics if we can construct  $N$  independent supercharges  $Q_I$ ,  $I = 0, 1, \dots, N-1$ , which are the square roots of the Hamiltonian  $H$  and satisfy the graded algebra (the  $N$ -Extended Supersymmetry Algebra)

$$\begin{aligned} \{Q_I, Q_J\} &= \frac{1}{2} \delta_{IJ} H, \\ [H, Q_I] &= 0, \\ \{N_F, Q_I\} &= 0, \\ [N_F, H] &= 0, \\ \{N_F, N_F\} &= 2 \cdot \mathbf{1}_{2d}. \end{aligned} \quad (5)$$

Besides  $Q_I$  and  $H$ , the Extended Supersymmetry Algebra admits as a generator the diagonal fermion-number operator  $N_F = \mathbf{1}_d \oplus (-\mathbf{1}_d)$ .  $N_F$  possesses  $d$  eigenvalues  $+1$  corresponding to the even states (the “bosons”) and  $d$  eigenvalues  $-1$  corresponding to the odd states (the “fermions”).

## 3 The $F$ -type realization of supersymmetry

The  $F$ -type  $N$ -Extended Supersymmetry is introduced, for the free particle, through the positions

$$\begin{aligned} Q_0 &= \frac{\hbar}{2\sqrt{m}} (\Gamma^\mu)_{ab} \partial_\mu, \\ Q_i &= \frac{\hbar}{\sqrt{2m}} (\gamma^i)^{\mu\nu} (\Gamma_\mu)_{ab} \partial_\nu, \end{aligned} \quad (6)$$

where  $i = 1, 2, \dots, N-1$  and  $a, b = 1, 2, \dots, 2d$ .

The construction makes use of the irreducible  $D$ -dimensional representation of the  $Cl(0, N-1)$  Clifford algebra with generators  $\gamma^i$  and of the  $2d$ -dimensional Weyl-type

representation of the  $Cl(0, D)$  Euclidean Clifford algebra generated by  $\Gamma_\mu$ . We have\*

$$\begin{aligned} \{\gamma^i, \gamma^j\} &= -2\delta^{ij}\mathbf{1}, & i, j &= 1, \dots, N-1, \\ \{\Gamma^\mu, \Gamma^\nu\} &= -2\delta^{\mu\nu}\mathbf{1} & \mu, \nu &= 1, \dots, D. \end{aligned} \quad (7)$$

Given  $D$ , the minimal value  $d$ , corresponding to the irreducible representation of the Clifford algebra, is unambiguously fixed and the maximal value  $N_{max}$  of the supersymmetric extension can be computed. Expressing, for  $D \geq 1$ ,

$$D = 8k + r, \quad (8)$$

with  $r = 0, 1, 2, \dots, 7$ , we have, for the minimal  $d$ ,

$$d = \frac{1}{2}G(r+1) \cdot 16^k \quad (9)$$

and, for  $N_{max}$ ,

$$N_{max} = 8k + G(r+1). \quad (10)$$

One should note the appearance of the Radon-Hurwitz function.

We present, up to  $D \leq 16$ , the table

$D$	$d$	$N_{max}$
1	1	2
2	2	4
3	2	4
4	4	8
5	4	8
6	4	8
7	4	8
8	8	9
9	16	10
10	32	12
11	32	12
12	64	16
13	64	16
14	64	16
15	64	16
16	128	17

(11)


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\*The closure of the (5) algebra can be easily proven by making use of several Fierz identities. As an example, the vanishing of the anticommutator between  $Q_i$  and  $Q_j$ , for  $i \neq j$ , requires, in particular, the vanishing of the term  $[\Gamma_\mu, \Gamma_\rho]\partial_\nu\partial_\sigma B_{\mu\rho\nu\sigma}^{ij}$ , where  $B_{\mu\rho\nu\sigma}^{ij} = \gamma_{\mu\nu}^i\gamma_{\rho\sigma}^j + \gamma_{\mu\nu}^j\gamma_{\rho\sigma}^i$ . The term above vanishes due to the fact that  $(B_{\mu\rho\nu\sigma}^{ij})^- = B_{\mu\rho\nu\sigma}^{ij} - B_{\rho\mu\nu\sigma}^{ij}$  is antisymmetric in the exchange  $\nu \leftrightarrow \sigma$ . Indeed,  $(B_{\mu\rho\nu\sigma}^{ij})^- = -(B_{\mu\rho\sigma\nu}^{ij})^-$ .

## 4 The $S$ -type realization of supersymmetry

The  $S$ -type  $N$ -Extended Supersymmetry for the free particle is introduced through the generators  $(\gamma^i)_{\alpha\beta}, (\Gamma^m)_{\alpha\beta}$  ( $\alpha, \beta = 1, \dots, d$ ), which satisfy the set of equations

$$\begin{aligned} \{\gamma^i, \gamma^j\} &= -2\delta^{ij}\mathbf{1}, & i, j &= 1, \dots, N-1, \\ \{\Gamma^m, \Gamma^n\} &= -2\delta^{mn}\mathbf{1}, & m, n &= 1, \dots, D, \\ [\gamma^i, \Gamma^m] &= 0. \end{aligned} \tag{12}$$

We have

$$\begin{aligned} Q_0 &= \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & \frac{1}{m}\partial_n\Gamma^n \\ \partial_n\Gamma^n & 0 \end{pmatrix}, \\ Q_i &= \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & \frac{1}{m}\gamma_i\partial_n\Gamma^n \\ -\gamma_i\partial_n\Gamma^n & 0 \end{pmatrix}. \end{aligned} \tag{13}$$

In the subcase  $S1$  at first we fix  $D$  and compute the maximal number of  $\gamma^i$  matrices admitted by the Schur lemma. Conversely, in the subcase  $S2$  we fix at first  $N$  and compute the maximal number of  $\Gamma^n$  matrices admitted by the Schur lemma. The subcases  $S1$  and  $S2$  coincide for  $D = d = 1, N = 2$  and for  $D = 3, d = 4, N = 4$ .

### 4.1 The $S1$ -subcase

Let us fix  $D \geq 1$  given by

$$D = 8k + r, \tag{14}$$

with  $r = 0, 1, 2, \dots, 7$ . The minimal  $d$  corresponding to the irreducible representation of the Clifford algebra is

$$d = \frac{1}{2}G(r+1) \cdot 16^k. \tag{15}$$

The maximal value  $N_{max}$  of the extended supersymmetry is given by the  $K(r)$  function:

$$N_{max} = K(r). \tag{16}$$

Up to  $D \leq 16$  we have the table

$D$	$d$	$N_{max}$
1	1	2
2	2	4
3	2	4
4	4	4
5	4	2
6	4	1
7	4	1
8	8	1
9	16	2
10	32	4
11	32	4
12	64	4
13	64	2
14	64	1
15	64	1
16	128	1

(17)

## 4.2 The $S_2$ -subcase

Let us fix the number of extended supersymmetries being given by

$$N = 8k + r + 1, \quad (18)$$

with  $r = 0, 1, 2, \dots, 7$ . The minimal  $d$  is expressed by

$$d = \frac{1}{2}G(r+1) \cdot 16^k, \quad (19)$$

while the maximal value  $D_{max}$  is obtained through

$$D_{max} = K(r) - 1. \quad (20)$$

Up to  $N \leq 14$  we have the table (the missing cases correspond to a Hamiltonian with no Laplacian, namely  $D = 0$ ):

$N$	$d$	$D_{max}$
2	1	1
3	2	3
4	2	3
5	4	3
6	4	1
10	16	1
11	32	3
12	32	3
13	64	3
14	64	1

(21)

## 5 Conclusions

In this paper we investigated two independent (for a generic value  $N$ ) types of  $N$ -Extended Supersymmetric Quantum Mechanics associated with a free Hamiltonian (given by the tensor product of a  $D$ -dimensional Laplacian and a  $2d$ -dimensional identity matrix operator) and based on Clifford algebras. We investigated the mutual relations among  $N$ ,  $D$  and  $d$  and proved that, in the construction based on the Fierz identities ( $F$ -type supersymmetry), the Bott's periodicity of the Clifford algebra is encoded in the Radon-Hurwitz function. For the construction based on the Schur lemma ( $S$ -type supersymmetry) the Bott's periodicity is also encoded in the extra function (2).

The  $S$ -type supersymmetry is split into two subcases ( $S1$  and  $S2$ ). For a given  $N > 4$ ,  $d$  is the same for both  $F$ -type and  $S2$ -type supersymmetry. On the other hand, in the Fierz case the  $D$ -dimensional Laplacian admits  $D \geq 4$  while, in the  $S2$ -case, the Laplacian admits at most  $D = 3$ .

It is outside the scope of this letter and left for future investigations to analyze the consistency conditions for  $S$ -type supersymmetry for a generic Hamiltonian. In the Fierz case the constraints require a non-trivial Kähler ( $N = 2$ ) or hyper-Kähler ( $N = 4$ ) background.

## Acknowledgments

This work was supported by Edital Universal CNPq Proc. 472903/2008-0.

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